FATOU AND LITTLEWOOD THEOREMS FOR NON-INTEGRABLE KERNELS

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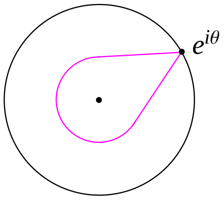
1. Fatou Theorem & Littlewood Theorem

Fatou(1906) [5] proved the following:

Theorem (Fatou Theorem)

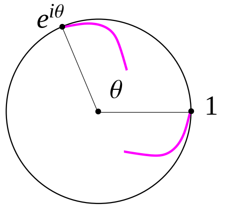
Let f be a bounded analytic function on the unit disk $U = \{|z| < 1\}$ in $\mathbb{C}$. Then f has non-tangential limit at a.e. $e^{i\theta} \in \partial U$.



Littlewood (1927) [9, 10] proved the sharpness of non-tangential approach.

Theorem (Littlewood Theorem)

Let $\gamma \subset U$ be a tangential curve at 1 and let γ_θ be the rotation. Then there exists a bounded analytic function f on U such that the limit of f along γ_θ does not exists for a.e. $e^{i\theta} \in \partial U$.



Many generalizations of Fatou Theorem

- Hardy space H^p
- Harmonic functions
- Local Fatou theorem

- Invariant harmonic functions. Korányi (1969) [8]
- Square root of the Poisson kernel. Sjögren (1983)
- Non non-tangential convergence. Nagel-Stein (1984) [13]
- Harmonic functions on trees
- Symmetric spaces

Rather few works for Littlewood Theorem

- Zygmund (1949) [21]
- Lohwater-Piranian (1957) [11]
- Hakim-Sibony (1983) [6]
- HA (1990) [1, 2]
- Salvatori-Vignati (1997) [17]
- Di Biase (1998) [4]
- Hirata (2003) [7].

Motivation

Fatou Theorem and Littlewood Theorem should go hand in hand.

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2. Fatou & Littlewood Theorems for harmonic functions on $\mathbb{R}^{n+1}_+$

Let $\Psi(x) = (1 + |x|^2)^{-(n+1)/2}$ for $x \in \mathbb{R}^n$ and put $\Psi_t(x) = \frac{1}{t^n}\Psi(\frac{x}{t})$ for $t > 0$. Then $\Psi_t * 1 = c_n$ and

$$\frac{\Psi_t * f(x)}{\Psi_t * 1} = \frac{1}{c_n}\int_{\mathbb{R}^n} \frac{tf(y)dy}{(|x-y|^2 + t^2)^{(n+1)/2}}$$

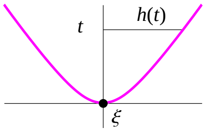
is the Poisson integral $Pf(x,t)$ for the half space $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$.

- $A > 0$.
- $f \lesssim g \iff f \le Ag$.
- $f \sim g \iff f \lesssim g, g \lesssim f$.

Define the approach region for $h(t) > 0$ by

$$\mathscr{A}_h(\xi) = \{(x,t) : |x - \xi| < h(t)\} \quad \text{for } \xi \in \mathbb{R}^n.$$

If $h(t) \sim t$, then $\mathscr{A}_h(\xi)$ is nontangential.



Theorem A (Fatou Theorem)

Let $1 \le p \le \infty$. If $f \in L^p(\mathbb{R}^n)$, then $Pf(x,t)$ has nontangential limit $f(\xi)$ at a.e. $\xi \in \mathbb{R}^n$.

Theorem B (Littlewood Theorem)

If $\limsup_{t\to 0} h(t)/t = \infty$, then ${}^\exists f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that

$$\lim_{\substack{t\to 0\\ (x,t)\in\mathcal{A}_h(\xi)}} Pf(x,t) \text{ fails to exist at every } \xi \in \mathbb{R}^n.$$

If γ is a tangential curve in $\mathbb{R}^{n+1}_+$ ending at $\partial\mathbb{R}^{n+1}_+$, then ${}^\exists f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that

$$\lim_{\substack{t\to 0\\ (x,t)\in\gamma+\xi}} Pf(x,t) \text{ fails to exist at every } \xi \in \mathbb{R}^n.$$

- Theorem A: Fatou(1906) [5].
- Theorem B: a.e. non-existence: Littlewood (1927) [9, 10]; everywhere non-existence: HA (1990) [1, 2].
- Higher integrability of f does not improve the admissible tangency.

3. Non-integrable Kernel

Sjögren(1983) [18, 19, 20] gave extensions of the Fatou theorem for fractional Poisson integrals. Let

$$P(z,\zeta) = \frac{1}{2\pi}\frac{1-|z|^2}{|z-\zeta|^2}$$

be the Poisson kernel for the unit disk U. Then the classical Poisson integral

$$Pf(z) = \int_{\partial U} P(z,e^{i\theta}) f(e^{i\theta}) d\theta$$

is, of course, harmonic, i.e., $\Delta Pf = 0$. Consider the fractional integral, or the λ-Poisson integral

$$u = P_\lambda f(z) = \int_{\partial U} P(z,e^{i\theta})^{\lambda+1/2} f(e^{i\theta}) d\theta.$$

Then, with the invariant or hyperbolic Laplacian

$$\widetilde{\Delta} = \frac{1}{4}(1-|z|^2)^2\Delta,$$

u enjoys

$$\widetilde{\Delta} u = (\lambda^2 - \frac{1}{4})u.$$

Study the boundary behavior of the normalization

$$\mathcal{P}_\lambda f(z) = \frac{P_\lambda f(z)}{P_\lambda 1(z)}.$$

If $\lambda > 0$, then the Fatou theorem holds for $\mathcal{P}_\lambda f$.

Theorem

If $f \in L^1(\partial U)$, then $\mathcal{P}_\lambda f(z)$ has nontangential limit $f(e^{i\theta})$ at a.e. $e^{i\theta} \in \partial U$.

Sjögren(1983, 1997) [18, 19, 20], Rönning(1997) [14, 15, 16] observed, if $\lambda = 0$, then suddenly tangential limits appear.

Theorem

Suppose $f \in L^p(\partial U)$ with $1 \le p \le \infty$. Then $\mathcal{P}_0 f(z)$ has limit $f(e^{i\theta})$ along $\mathcal{A}_h(e^{i\theta})$ at a.e. $e^{i\theta} \in \partial U$, where

$$h(t) \lesssim \begin{cases} t(\log 1/t)^p & \text{if } 1 \le p < \infty, \\ t^{1-\varepsilon} \quad \text{for } {}^\forall\varepsilon > 0 & \text{if } p = \infty. \end{cases}$$

How should we understand the tangential nature?

The tangential nature is caused by the non-integrability of the kernel.

$$P(z,\zeta)^{1/2} = \sqrt{\frac{1}{2\pi}\frac{1-|z|^2}{|z-\zeta|^2}} \sim \frac{1}{|z-\zeta|}.$$

Half space version: Brundin (2002) [3], Mizuta-Shimomura (2002) [12]. Define $(P_0 f)(x,t)$ by

$$\int_{\mathbb{R}^n} \left[\frac{t}{c_n(|x-y|^2+t^2)^{(n+1)/2}}\right]^{n/(n+1)} f(y)dy.$$

$(P_0 1)(x,t) \equiv \infty$ (non-integrable).

Fix a bounded open set $\Omega \subset \mathbb{R}^n$ and regard $(P_0\chi_\Omega)(x,t)$ as a substitute of $(P_0 1)(x,t)$. Study the normalization $(P_0 f)(x,t)/(P_0\chi_\Omega)(x,t)$.

Theorem C

Let $1 \le p \le \infty$. Suppose, for small $t > 0$,

(1) $h(t) \lesssim t(\log 1/t)^{p/n}$ if $1 \le p < \infty$,

(2) $h(t) \lesssim t^{1-\varepsilon}$ for ${}^\forall\varepsilon > 0$ if $p = \infty$.

If $f \in L^p(\mathbb{R}^n)$, then

$$\lim_{\substack{t\to 0\\ (x,t)\in\mathcal{A}_h(\xi)}} \frac{(P_0 f)(x,t)}{(P_0\chi_\Omega)(x,t)} = f(\xi) \quad \text{for a.e. } \xi \in \Omega.$$

- For the critical power $n/(n+1)$, certain tangential limits exist.
- Possible tangency depends on the Lebesgue exponent p for which $f \in L^p(\mathbb{R}^n)$.

The tangential nature in Theorem C is caused by the non-integrability of the kernel.
Let $\Phi(x) = \Psi(x)^{n/(n+1)} = (1+|x|^2)^{-n/2}$. Then

$$\frac{(P_0 f)(x,t)}{(P_0\chi_\Omega)(x,t)} = \frac{\Phi_t * f(x)}{\Phi_t * \chi_\Omega(x)}.$$

Obs. $\Phi \notin L^1(\mathbb{R}^n)$, $\Phi \in L^p(\mathbb{R}^n)$ for $1 < p \le \infty$.
$\Phi_t * \chi_\Omega(x) \sim \log 1/t$ as $t \to 0$ for $x \in \Omega$.
A sharp contrast between Ψ and Φ.
No need of the explicit form $(1+|x|^2)^{-n/2}$.
Suppose

- $\Phi(x) > 0$ is a doubling function of $|x|$.
- $\Phi \notin L^1(\mathbb{R}^n)$, $\Phi \in L^p(\mathbb{R}^n)$ for $1 < p \le \infty$.

Let

$$\varphi(r) = \int_{|x|<r} \Phi(x)\,dx.$$

Then $\varphi(r) \uparrow \infty$ is doubling. Assume

$$(3) \qquad \lim_{r\to\infty} \frac{\varphi(2r)}{\varphi(r)} = 1.$$

Looks technical; but it turns out to be crucial.

Fix a bounded open set $\Omega \subset \mathbb{R}^n$. Study the boundary behavior of the normalization

$$(\mathscr{P}_0 f)(x,t) = \frac{\Phi_t * f(x)}{\Phi_t * \chi_\Omega(x)}.$$

Proposition 1

Condition (3) holds if and only if

$$\lim_{t\to 0} (\mathscr{P}_0 f)(x,t) = f(x) \quad \text{for } x \in \Omega$$

for ${}^\forall f \in C_0(\mathbb{R}^n)$.

Theorem 1

Let $1 \le p \le \infty$. Suppose, for small $t > 0$,

(4) $h(t) \lesssim t\varphi(1/t)^{p/n}$ if $1 \le p < \infty$,

(5) $\displaystyle \lim_{t\to 0} \frac{\varphi(h(t)/t)}{\varphi(1/t)} = 0$ if $p = \infty$.

If $f \in L^p(\mathbb{R}^n)$, then

$$\lim_{\substack{t\to 0\\ (x,t)\in\mathcal{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) = f(\xi) \quad \text{for a.e. } \xi \in \Omega.$$

- (4) $\Longrightarrow$ (5).
- If $\Phi(x) = (1+|x|^2)^{-n/2}$, then
  - $\varphi(r) \sim \log r$ for large $r > 0$
  - (1) $\Longleftrightarrow$ (4), (2) $\Longleftrightarrow$ (5).

What is a Littlewood type theorem?

Theorem 2

Let $1 \le p < \infty$. If (4) does not hold, i.e.,

$$(6) \qquad \limsup_{t\to 0} \frac{h(t)}{t\varphi(1/t)^{p/n}} = \infty.$$

then ${}^\exists f \in L^p(\Omega)$ such that for ${}^\forall\xi \in \Omega$,

$$-\infty = \liminf_{\substack{t\to 0\\ (x,t)\in\mathcal{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) < \limsup_{\substack{t\to 0\\ (x,t)\in\mathcal{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) = \infty.$$

Theorem 3

If (5) does not hold, i.e.,

$$\limsup_{t\to 0} \frac{\varphi(h(t)/t)}{\varphi(1/t)} > 0. \tag{7}$$

then $^{\exists}f \in L^{\infty}(\Omega)$ such that for $^{\forall}\xi \in \Omega$,

$$\liminf_{\substack{t\to 0\\ (x,t)\in\mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) < \limsup_{\substack{t\to 0\\ (x,t)\in\mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t).$$

- $B(x,r)$: open ball with center at x, radius r.
- $\delta_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$, $\mathrm{diam}\,\Omega$: diameter of Ω.

Proof of Proposition 1. For $^{\forall}x \in \Omega$, $^{\exists}\Gamma(x) \subset \Omega$ with vertex at x, fixed aperture α, radius r_0. Change of variable gives

$$A\varphi\left(\frac{r_0}{t}\right) \le \Phi_t * \chi_\Omega(x) \le \varphi\left(\frac{\mathrm{diam}\,\Omega}{t}\right).$$

Since φ is doubling, it follows that

$$\Phi_t * \chi_\Omega(x) \sim \varphi\left(\frac{1}{t}\right) \quad \text{for } x \in \Omega. \tag{8}$$

Let $x \in \Omega$ and let $0 < \varepsilon < \delta_\Omega(x)$. Then (8) and the doubling of φ gives

$$\begin{aligned}\frac{\varphi(\delta_\Omega(x)/t) - \varphi(\varepsilon/t)}{\varphi(\varepsilon/t)} &\lesssim (\mathscr{P}_0\chi_{\Omega\setminus B(x,\varepsilon)})(x,t)\\ &\lesssim \frac{\varphi(\mathrm{diam}\,\Omega/t) - \varphi(\varepsilon/t)}{\varphi(\varepsilon/t)}.\end{aligned}$$

Hence $\lim_{t\to 0}(\mathscr{P}_0\chi_{\Omega\setminus B(x,\varepsilon)})(x,t) = 0$ if and only if (3) holds.

□

4. Ingredients of Proof of Theorem 1

Local part for $p = \infty$.

Lemma 1

Suppose h satisfies (5). Then

$$\lim_{(x,t)\to(\xi,0)} (\mathscr{P}_0\chi_{B(x,4h(t))})(x,t) = 0 \quad \text{for } \xi \in \Omega.$$

Local part for $1 \le p < \infty$. The Lebesgue point argument gives a.e. result.

Lemma 2

Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Suppose h satisfies (4). Then for a.e. $\xi \in \Omega$,

$$\lim_{\substack{t\to 0\\ (x,t)\in\mathscr{A}_h(\xi)}} (\mathscr{P}_0[\chi_{B(x,4h(t))} f])(x,t) = 0.$$

Global part is controlled by maximal functions. Define the truncated maximal function by

$$M_t f(x) = \sup_{r>t} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

with $t \ge 0$. $Mf(x) = M_0 f(x)$ is the classical Hardy-Littlewood maximal function.
Define another maximal function $\mathscr{M}_h f(\xi)$ by

$$\sup_{(x,t)\in\mathscr{A}_h(\xi)} \left| \frac{1}{\Phi_t * \chi_\Omega(x)} \int_{|x-y|\ge 4h(t)} \Phi_t(x-y) f(y)dy \right|$$

associated with the approach region $\mathscr{A}_h(\xi)$.

Lemma 3

There is A such that

$$\mathscr{M}_h f(\xi) \le AMf(\xi) \quad \text{for } \xi \in \Omega$$

for arbitrary $h(t) > 0$.

Lemma 4

Let $f \in L^p(\Omega)$ with $1 \le p < \infty$. Then

$$\lim_{t\to 0} \|(\mathscr{P}_0 f)(\cdot, t) - f\|_p = 0.$$

As a result, for a.e. $x \in \Omega$, some subsequence $\{(\mathscr{P}_0 f)(x, t_j)\}_j$ converges to $f(x)$.

5. Ingredients of Proof of Theorem 2

Lower Estimate

We find $0 < {}^{\exists}A_0 < 1$ s.t.

$$(\mathscr{P}_0 \chi_{B(x,r)})(x,t) \ge A_0 \frac{\varphi(r/t)}{\varphi(1/t)}$$

for $x \in \Omega$, $t > 0$, $r > 0$ small.

Upper Estimate

If $f \in L^1(\Omega)$, then

$$|(\mathscr{P}_0 f)(x,t)| \lesssim M_t f(x) \quad \text{for } x \in \Omega.$$

Proof of Theorem 2. By (6) we find ${}^{\exists}t_j \downarrow 0$ s. t.

$$\frac{t_j \varphi(1/t_j)^{p/n}}{h(t_j)} \to 0.$$

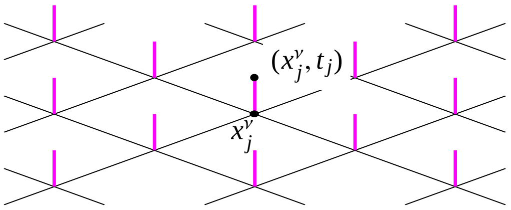
Let $\{x_j^\nu\}_\nu$ be lattice points $(h(t_j)/\sqrt{n})\mathbb{Z}^n$. Observe x_j^ν are vertices of cubes of side length $h(t_j)/\sqrt{n}$. Hence ${}^{\exists}x_j^\nu \in B(\xi, h(t_j))$.

If $\xi \in \Omega$, then

$$(x_j^\nu, t_j) \in \mathscr{A}_h(\xi) \ \text{ with } x_j^\nu \in \Omega,$$

provided j is sufficiently large.

Put vertical line segments connecting $(x_j^\nu, 0)$ and (x_j^ν, t_j). We obtain a bed of thorns.



$\mathscr{A}_h(\xi)$ cannot touch Ω without being pierced by some thorn.

Construct f_j s.t. $(\mathscr{P}_0 f_j)(x,t)$ is large on each "thorn". Put

$$f_j = \varphi\left(\frac{1}{t_j}\right)\chi_{D_j} \quad \text{with } D_j = \bigcup_\nu B(x_j^\nu, t_j) \cap \Omega.$$

Extract subsequence, find $c_j \uparrow \infty$ and let

$$f = \sum_{j=1}^{\infty} (-1)^j c_j f_j \in L^p(\mathbb{R}^n).$$

If j is even and $j \to \infty$, then

$$(\mathscr{P}_0 f)(x_j^\nu, t_j) \to \infty;$$

if j is odd and $j \to \infty$, then

$$(\mathscr{P}_0 f)(x_j^\nu, t_j) \to -\infty.$$

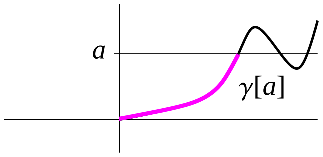
Since $\mathscr{A}_h(\xi)$ cannot touch Ω without being pierced by some thorn. we have

$$-\infty = \liminf_{\substack{t\to 0\\ (x,t)\in\mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) < \limsup_{\substack{t\to 0\\ (x,t)\in\mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) = \infty.$$

□

6. Oscillating limits along curves

Stronger form of Theorem 3. Let γ be a curve in $\mathbb{R}^{n+1}_+$ ending at the boundary. Let $\gamma[a]$ be the connected component of $\gamma \cap \{(x,t) : 0 \le t \le a\}$ containing the end point of γ.



Theorem 4

Assume $\varphi(2r)/\varphi(r)$ is nonincreasing of r. Suppose γ is more tangential than (5), i.e.,

(9) $$\limsup_{t\to 0} \frac{\varphi(\mathrm{diam}(\gamma[t])/t)}{\varphi(1/t)} > 0.$$

Then ${}^{\exists}f \in L^\infty(\Omega)$ s. t. for every $\xi \in \Omega$,

$$\liminf_{\substack{t\to 0\\ (x,t)\in\gamma+\xi}} (\mathscr{P}_0 f)(x,t) < \limsup_{\substack{t\to 0\\ (x,t)\in\gamma+\xi}} (\mathscr{P}_0 f)(x,t).$$

Proof of Theorem 4. Let π be the projection from $\mathbb{R}^{n+1}_+$ to $\mathbb{R}^n$. By assumption ${}^\exists A_1$, ${}^\exists\{a_j\}$ and ${}^\exists\{b_j\}$ such that $b_0 > a_0 > b_1 > a_1 > \cdots \to 0$ and that one of the connected component of $\gamma \cap \{(x,t) : a_j \le t \le b_j\}$, say γ_j, satisfies

$$\frac{\varphi(\mathrm{diam}(\pi(\gamma_j))/b_j)}{\varphi(1/b_j)} \ge A_1,$$
$$\mathrm{diam}(\pi(\gamma_j)) \le a_{j-1} \quad \text{for } j \ge 1.$$

For simplicity let $d_j = \mathrm{diam}(\pi(\gamma_j))$ and let

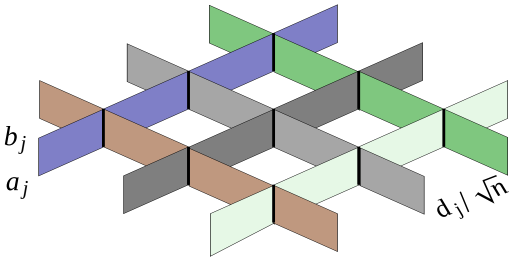
$$M_j = \bigcup_{k=1}^{n} \bigcup_{\ell=-\infty}^{\infty} \{(x_1, \dots, x_n) : x_k = \frac{\ell d_j}{\sqrt{n}}\}.$$

Observe that $\mathbb{R}^n \setminus M_j$ consists of congruent open cubes of diameter d_j and $M_j \times [a_j, b_j]$ is a set of grid shape.

By construction $(\gamma + \xi) \cap (M_j \times [a_j, b_j]) \ne \emptyset$ for ${}^\forall\xi \in \mathbb{R}^n$. Let $G_j = (M_j \cap \Omega) \times [a_j, b_j]$. Then

$$(\gamma + \xi) \cap G_j \ne \emptyset \quad \text{if } \xi \in \Omega \text{ and } j \text{ is large.}$$

In other words,



${}^\forall\gamma + \xi$ is captured by G_j before Ω.

Let $R_j > 0$ be such that $\varphi(R_j/b_j) = \frac{1}{2}\varphi(d_j/b_j)$. Then (3) yields $R_j/d_j \to 0$. Let

$$F_j = \{x \in \mathbb{R}^n : \mathrm{dist}(x, M_j \cap \Omega) < R_j\}.$$

Then ${}^\exists A_2 > 0$ such that

$$(\mathscr{P}_0\chi_{F_j})(x,t) \ge A_2 \quad \text{for } (x,t) \in G_j;$$

but

$$(\mathscr{P}_0\chi_{F_j})(x,T) \lesssim M_T(\chi_{F_j})(x) \lesssim \left[\frac{R_j}{d_j}\right]^n \to 0$$

for $x \in \Omega$ and $T \ge d_j$. Taking a subsequence, we may assume that $|F_j| < 2^{-j}$ and

$$(\mathscr{P}_0\chi_{F_j})(x,t) < 2^{-j} \quad \text{for } x \in \Omega,\ t \ge a_{j-1}.$$

Observe that $|\limsup F_j| = 0$ and

$$f_k(x) = \begin{cases} (-1)^{J_k(x)} & x \in \bigcup_{j=1}^k F_j, \\ 0 & x \notin \bigcup_{j=1}^k F_j \end{cases}$$

a.e. converges to

$$f(x) = \begin{cases} (-1)^{J(x)} & x \in \bigcup_{j=1}^\infty F_j \setminus \limsup F_j, \\ 0 & x \notin \bigcup_{j=1}^\infty F_j, \end{cases}$$

where $J_k(x)$ (resp. $J(x)$) is the last index j such that $x \in F_j$ (resp. $x \in \bigcup_{j=1}^\infty F_j \setminus \limsup F_j$). Observe if $(x,t) \in G_j$ with even j, then

$$\begin{aligned} &(\mathscr{P}_0 f)(x,t) \\ &= (\mathscr{P}_0 f_j)(x,t) + (\mathscr{P}_0(f_{j+1} - f_j))(x,t) + \cdots \\ &\ge A_2 - 2((\mathscr{P}_0\chi_{F_{j+1}})(x,t) + \cdots) \\ &\ge A_2 - 2(2^{-j-1} + 2^{-j-2} + \cdots) \ge A_2 - 2^{1-j}. \end{aligned}$$

Similarly, if $(x,t) \in G_j$ with odd j, then

$$(\mathscr{P}_0 f)(x,t) \le -A_2 + 2^{1-j}.$$

Hence

$$\liminf_{\substack{t\to 0\\(x,t)\in\gamma+\xi}} (\mathscr{P}_0 f)(x,t) < \limsup_{\substack{t\to 0\\(x,t)\in\gamma+\xi}} (\mathscr{P}_0 f)(x,t)$$

for every $\xi \in \Omega$. □

References

[1] H. Aikawa, Proc. Amer. Math. Soc. **108** no. 2 (1990), 457–464, MR **90h:31003**.

[2] ______, J. London Math. Soc. (2) **43** no. 1 (1991), 125–136, MR **92b:31005**.

[3] M. Brundin, *Boundary behaviour of eigenfunctions for the hyperbolic Laplacian*, Ph.D. thesis, Department of Mathematics, Chalmers University, 2002.

[4] F. Di Biase, J. London Math. Soc. (2) **58** no. 2 (1998), 331–341, MR **99m:31016**.

[5] P. Fatou, Acta. Math. **30** (1906), 335–400.

[6] M. Hakim and N. Sibony, Duke Math. J. **50** no. 1 (1983), 133–141, MR **84m:32011**.

[7] K. Hirata, preprint (2003).

[8] A. Korányi, Trans. Amer. Math. Soc. **135** (1969), 507–516, MR **43:3480**.

[9] J. E. Littlewood, J. London Math. Soc. **2** (1927), 172–176.

[10] ———, Proc. London Math. Soc. (2) **28** (1928), 383–394.

[11] A. J. Lohwater and G. Piranian, Ann. Acad. Sci. Fenn. Ser. A. I. **1957** no. 239 (1957), 17, MR **19,950c**.

[12] Y. Mizuta and T. Shimomura, preprint.

[13] A. Nagel and E. M. Stein, Adv. in Math. **54** no. 1 (1984), 83–106, MR **86a:42026**.

[14] J.-O. Rönning, Math. Scand. **81** no. 2 (1997), 219–235, MR **99e:31004**.

[15] ———, Studia Math. **125** no. 3 (1997), 219–229, MR **98k:42026**.

[16] ———, Math. Scand. **84** no. 1 (1999), 81–92, MR **2000b:32008**.

[17] M. Salvatori and M. Vignati, Potential Anal. **6** no. 3 (1997), 269–287, MR **98m:31012**.

[18] P. Sjögren, Théorie du potentiel (Orsay, 1983), Lecture Notes in Math., vol. 1096, Springer, Berlin, 1984, pp. 544–548, MR **88f:31001**.

[19] ———, Pacific J. Math. **131** no. 2 (1988), 361–391, MR **89d:43012**.

[20] ———, Bull. Austral. Math. Soc. **55** no. 3 (1997), 521–527, MR **98h:31002**.

[21] A. Zygmund, Summa Brasil. Math. **2** no. 5 (1949), 51–57, MR **12,88c**.