POSITIVE HARMONIC FUNCTIONS IN DENJOY TYPE DOMAINS

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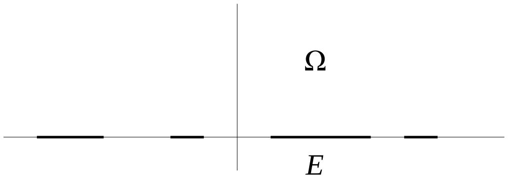
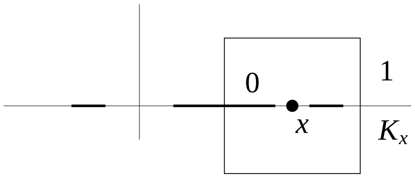
1. Introduction

Let $E \subset \{x = (x_1, \dots, x_n) : x_n = 0\}$ be closed. $\Omega = \mathbb{R}^n \setminus E$ is called a Denjoy domain.



Let $\mathscr{P}$ be the family of positive harmonic functions in Ω vanishing on $\partial\Omega$.

Benedicks [Ben80] proved the following:
$\dim \mathscr{P} = 1$ or 2.
Criterion in terms of harmonic measure $\beta_E(x) = \omega(x, \partial K_x, K_x \setminus E)$.
K_x: cube center at x, side $\alpha|x|$.



$$\dim \mathscr{P} = 1 \iff \int_{|x|\ge 1} \frac{\beta_E(x)}{|x|^{n-1}} dx = \infty.$$

$$\dim \mathscr{P} = 2 \iff \int_{|x|\ge 1} \frac{\beta_E(x)}{|x|^{n-1}} dx < \infty.$$

- Monotonicity:

If $E \subset E'$, $\dim \mathscr{P}_E = 2$, then $\dim \mathscr{P}_{E'} = 2$.

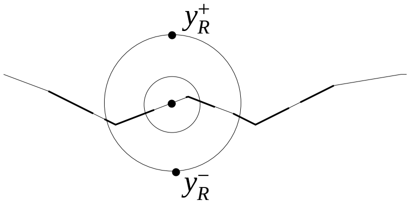
Location	Topics	Authors
C^2 surface	$\dim \mathscr{P} \le 2$	Ancona (79)
Hyperplane	Harmonic Measure	Benedicks (80)
Lipschitz surface	$\dim \mathscr{P} \le 2$ WBHP	Ancona (84)
Real line	Lebesgue Measure	Segawa (88)
Hyperplane	Lebesgue Measure	Gardiner (89)
$C^{1,1}$ surface	Harmonic Measure	Chevallier (89)
$C^{1,\alpha}$ surface	Harmonic Measure	Ancona (90)
Lipschitz surface	Non Monotonicity	Ancona (90)
Real line	Quasi-conformal	Segawa (90)
Sectorial	Harmonic Measure	Cranston-Salisbury (93)
Half space	Harmonic Major.	Eiderman-Essén (96)
Quasi-Sectorial	Schrödinger Equation	Lömker (00)

Weak boundary Harnack principle.
Ancona [Anc79].
$B(x,r)$, $S(x,r)$ the open ball and the sphere with center at x and radius r.
$B(r) = B(0,r)$, $S(r) = S(0,r)$.

$\mathscr{P}_\xi$: kernel functions h at ξ, i.e.,
$h > 0$ harmonic on Ω,
$h = 0$ q.e on $\partial\Omega$,
bounded outside ξ.
$E \subset S$: Lipschitz surface.
$h_0, h_1, h_2 \in \mathscr{P}_\xi$. Then

$$h_0(x) \le A\left(\frac{h_0(y_R^+)}{h_1(y_R^+)} h_1(x) + \frac{h_0(y_R^+)}{h_2(y_R^+)} h_2(x)\right)$$

for $x \in \Omega \cap B(\xi, R) \setminus B(\xi, R/2)$.



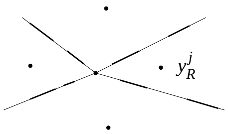
If $h_0, h_1, h_2 \in \mathscr{P}_\xi$, then $^\exists i$ s.t.

$$h_i \le A \sum_{j \ne i} h_j;$$

dim $\mathscr{P}_\xi \le 2$.

Sectorial domain.

Cranston-Salisbury [CS93].



If $h_0, \dots, h_N \in \mathscr{P}_\xi$. Then

$$h_0(x) \le A \left(\sum_{j=1}^{N} \frac{h_0(y_R^j)}{h_j(y_R^j)} h_j(x) \right)$$

for $x \in \Omega \cap B(\xi, R) \setminus B(\xi, R/2)$; $^\exists i$ s.t.

$$h_i \le A \sum_{j \ne i} h_j;$$

dim $\mathscr{P}_\xi \le N$.

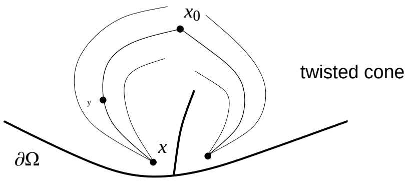
Quasi-sectorial domain (higher dimension).

2. Extension to a John domain

John domain. twisted cone condition:

$^\forall x \in \Omega$, $^\exists \gamma : x \to x_0$ s.t.

$$\delta_\Omega(y) \ge c_J \ell(\gamma(x, y)) \quad \text{for all } y \in \gamma,$$



$$\left.\begin{array}{l}\text{Denjoy domain}\\ \text{Sectorial domain}\\ \text{Quasi-Sectorial}\end{array}\right\} \Longrightarrow \text{John domain}$$

Theorem 1

Let Ω be a John domain with John constant c_J. Let $\xi \in \partial\Omega$. Then

(i) dim $\mathscr{P}_\xi \le N(c_J) < \infty$.

(ii) If $c_J > \sqrt{3}/2$, then dim $\mathscr{P}_\xi \le 2$.

Remark 1

$c_J > \sqrt{3}/2$ is sharp.

Quasihyperbolic metric:

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_\Omega(z)}.$$

where inf is taken over all curves γ connecting x to y in Ω.

$$k_\Omega(x, y) \approx \text{length of Harnack chain.}$$

If $h > 0$ is harmonic on Ω, then

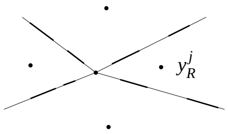
$$\exp(-A k_\Omega(x, y)) \le \frac{h(x)}{h(y)} \le \exp(A k_\Omega(x, y))$$

Local reference points:

$^\exists y_R^1, \dots, {}^\exists y_R^N \in S(\xi, R) \cap \Omega$ s.t. $\delta_\Omega(y_R^i) \approx R$ and

$$\min_{i=1,\dots,N} \{k_{\Omega_R}(x, y_R^i)\} \le A \log \frac{R}{\delta_\Omega(x)} + A'$$

for $x \in B(\xi, \eta R) \cap \Omega$, where $\Omega_R = \Omega \cap B(\xi, AR)$.



- If $h \in \mathscr{P}_\xi$, then 0-extension to Ω^c is subharmonic in $\mathbb{R}^n \setminus \{\xi\}$.

Lemma 1 (Domar [Dom57]**)**

Let $u \geq 0$ be subharmonic in D s.t.

$$I = \int_D (\overset{+}{\log} u)^{n-1+\varepsilon} dx < \infty$$

for $^\exists\varepsilon > 0$. Then

$$u(x) \leq \exp(AI^{1/\varepsilon}\, \mathrm{dist}(x, \partial D)^{-n/\varepsilon}).$$

Lemma 2

$^\exists\tau > 0$ s.t.

$$\int_{\Omega\cap B(\xi,R)} \left(\frac{R}{\delta_\Omega(x)}\right)^\tau dx \leq AR^n.$$

Lemma 3

Let $h \in \mathscr{P}_\xi$ for $\xi \in \partial\Omega$. Then

$$h(x) \leq A|x-\xi|^{-\lambda}.$$

Proof. By local reference points

$$h(x) \leq A\left(\frac{R}{\delta_\Omega(x)}\right)^\lambda \sum_{i=1}^N h(y_R^i).$$

Apply Lemma 1 to $D = B(\xi, AR) \setminus \overline{B(\xi, A^{-1}R)}$. with the help of Lemma 2. Then

$$(1) \qquad h(x) \leq A \sum_{i=1}^N h(y_R^i)$$

on $S(\xi, R)$, and hence on $\Omega \setminus B(\xi, R)$ by the maximum principle. Since $\delta_\Omega(y_R^i) \approx R$, we have $h(y_R^i) \leq AR^{-\lambda}$. Hence

$$h(x) \leq AR^{-\lambda} \quad \text{on } \Omega \setminus B(\xi, R),$$

i.e. $h(x) \leq A|x-\xi|^{-\lambda}$ on Ω. □

Tract argument (Friedland-Hayman [FH76]) implies

$$\dim \mathscr{P}_\xi \leq N.$$

By box argument initiated by Bass-Burdzy [BB91] (see [Aik01, Lemma 2]) we have

$$\omega(x, \Omega \cap S(\xi, AR), \Omega \cap B(\xi, AR)) \leq AR^{2-n} \sum_{i=1}^N G_R(x, y_R^i)$$

for $x \in \Omega \cap B(\xi, R)$, where G_R is the Green function for $\Omega \cap B(\xi, A'R)$. Combine with (1). Then

$$h(x) \leq AAR^{2-n} \sum_{i=1}^N G_R(x, y_R^i) \sum_{j=1}^N h(y_R^j).$$

Apply this inequality to $h(x) = G_R(x, y)$. Then

$$G_R(x, y) \leq AR^{2-n} \sum_{i=1}^N G_R(x, y_i) \sum_{j=1}^N G_R(y_j, y).$$

Now let $c_J > \sqrt{3}/2$. Then $N \leq 2$. Ancona's ingenious trick [Anc84, Théorème 7.3] gives

$$G_R(x, y) \leq AR^{2-n} \sum_{i=1}^2 G_R(x, y_i) G_R(y_i, y).$$

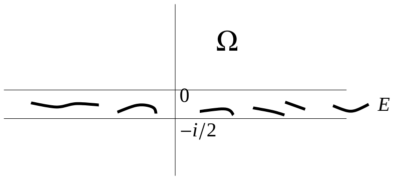
This yields the WBHP: Let $h_0, h_1, h_2 \in \mathscr{P}_\xi$. Then

$$h_0(x) \leq A \sum_{i=1}^2 \frac{h_0(y_i)}{h_i(y_i)} h_i(x) \quad \text{for } x \in \Omega.$$

This immediately means $\dim \mathscr{P}_\xi \leq 2$.

3. Extension to Denjoy type domain

Poggi-Corradini [PC02] gave an extension:
- E is included in a strip,
- harmonic functions of finite order.



Let Ω be an unbounded domain.
$u > 0$ is of order λ if

$$\lambda = \limsup_{r\to\infty} \frac{\log \sup_{B(r)\cap\Omega} u}{\log r}.$$

Hayman-Kennedy [HK76, Definition 4.1].

Let $\mathscr{P}$ be the family of locally bounded positive harmonic functions on Ω vanishing q.e. on $\partial\Omega$, $\mathscr{F}$ the subfamily of functions in $\mathscr{P}$ of finite order.

Poggi-Corradini proved

Theorem A

Suppose $\Omega = \mathbb{C}\setminus E$ is a planar domain with closed $E \subset \{-1/2 < \operatorname{Im} z < 0\}$ of positive capacity s.t. $\partial\Omega$ is regular.

(i) Suppose $u \in \mathscr{P}$. Then $u \in \mathscr{F}$ if and only if

$$\limsup_{r\to\infty} \omega(i, S(r), \Omega\cap B(r)) \max_{S(r)} u < \infty.$$

Moreover, in this case $\max_{S(r)} u \le Ar$.

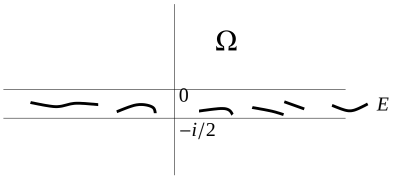
(ii) $\dim \mathscr{F} = 1$ or 2.

(iii) If

$$\sup_{x\in\mathbb{R}} \max\{\omega(x, \mathbb{R}+i, \Omega), \omega(x, \mathbb{R}-i, \Omega)\} < 1,$$

then $\dim \mathscr{F} = 2$.

(iv) Suppose that $\Omega + 1 = \Omega$. Then $\mathscr{P} = \mathscr{F}$.



Extend Theorem A to domain $\Omega \subset \mathbb{R}^n$ whose complement is included in a set wider than a strip.

Theorem 2

Let $n \ge 2$. Let $\Omega \subset \mathbb{R}^n$ satisfy

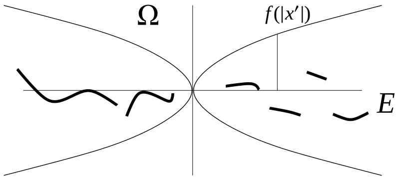
$$\Omega^c \subset \{x \in \mathbb{R}^n : |x_n| \le f(|x'|)\},$$

where $f(t) \ge 0$ for $t \ge 0$ s.t.

$$\lim_{t\to\infty} \frac{f(t)}{t} = 0.$$

(i) Suppose $u \in \mathscr{P}$. Then $u \in \mathscr{F} \iff u$ is of order at most 1.

(ii) $\dim \mathscr{F} = 1$ or 2.

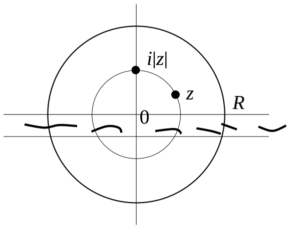


4. Positive harmonic functions on a cone

Poggi-Corradini [PC02] used a lemma after Ancona [Anc79, Lemme 1]: a Carleson estimate. Depends on symmetry.

Lemma 4.10. *Let* $v = \omega(\cdot, \Omega \cap S(R), \Omega \cap B(R))$, $n = 2$. *Then*

$$v(z) \le Av(i|z|).$$



Rather easy estimates of harmonic measures on a cone.
Write the Laplacian as

$$\Delta = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\Lambda_n,$$

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where Λ_n is the Laplace-Beltrami operator. Let E be a (relatively) open set on $S(1)$. Laplace-Beltrami equation:

$$\begin{aligned} \Lambda_n F + \lambda F &= 0 \quad \text{on } E, \\ F &= 0 \quad \text{on } \partial E, \end{aligned}$$

where $\lambda = \lambda(E)$ is the first positive eigenvalue.
F_E: the positive eigenfunction corresponding to λ.
The characteristic constant $\alpha = \alpha(E)$ is the positive root of

$$\alpha(\alpha + n - 2) = \lambda$$

Let $\Gamma(E) = \{x \in \mathbb{R}^n : x/|x| \in E\}$ be the cone subtended by E with vertex at the origin.

(2) $$h_E(x) = |x|^\alpha F_E\left(\frac{x}{|x|}\right)$$

is a positive harmonic function on $\Gamma(E)$ vanishing on $\partial\Gamma(E)$.

In fact, h_E corresponds to the Martin kernel at infinity.

If E is $\Sigma(\theta) = \{x \in S(1) : x_n > \cos\theta\}$, then write $\Gamma(\theta)$ and $\alpha(\theta)$ for $\Gamma(\Sigma(\theta))$ and $\alpha(\Sigma(\theta))$.

$\Sigma(\theta)$ has the least characteristic constant among open sets on $S(1)$ with the same surface measure (Sperner [Spe73]). See Friedland and Hayman [FH76].

Note $\alpha(\theta)$ is a strictly decreasing function of θ s.t. $\alpha(\pi/2) = 1$ and $\alpha(\theta) \uparrow \infty$ as $\theta \downarrow 0$.

Lemma 4

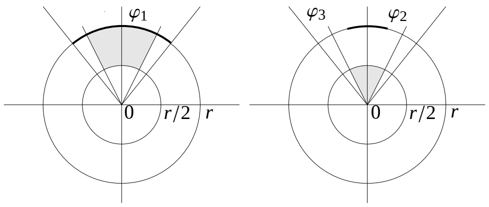
Let $0 < \varphi_1 < \varphi_2 < \varphi_3 < \pi/2$. Then

$$\omega(\cdot, \Gamma(\varphi_2) \cap S(r), \Gamma(\varphi_2) \cap B(r)) \ge A$$

on $\overline{\Gamma(\varphi_1) \cap B(r)} \setminus B(r/2)$;

$$\omega(x, \Gamma(\varphi_1) \cap S(r), \Gamma(\varphi_3) \cap B(r)) \approx \left(\frac{|x|}{r}\right)^{\alpha(\varphi_3)}$$

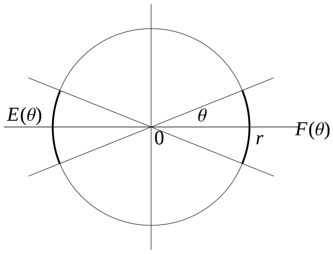
for $x \in \overline{\Gamma(\varphi_2) \cap B(r/2)}$.



Lemma 5

Let $E(\theta) = \{x \in S(1) : |x_n| < \sin\theta\}$, $F(\theta) = \{x : |x_n| < |x|\sin\theta\}$ for $0 < \theta < \pi/2$. Let $0 < \beta < \alpha(E(\theta))$. Then

$$\begin{aligned}&\omega(x, \partial F(\theta) \cap B(2r) \setminus B(r), F(\theta))\\ &\le \omega(x, F(\theta) \cap S(r), F(\theta) \cap B(r))\\ &\le A\left(\frac{|x|}{r}\right)^{\beta} \quad \text{for } x \in F(\theta) \cap B(r).\end{aligned}$$



Proof. Let $\theta' > \theta$ be close to θ so that $\beta \le \alpha' = \alpha(E(\theta')) < \alpha(E(\theta))$. Consider $h_{E(\theta')}$ given by (2).

Then

$$h_{E(\theta')}(x) \approx |x|^{\alpha'} \quad \text{for } x \in F(\theta).$$

The maximum principle gives

$$\omega(\cdot, F(\theta) \cap S(r), F(\theta) \cap B(r)) \le Ar^{-\alpha'} h_{E(\theta')}$$

on $F(\theta)$, so that

$$\omega(x, F(\theta) \cap S(r), F(\theta) \cap B(r)) \le A\left(\frac{|x|}{r}\right)^{\beta}$$

for $x \in F(\theta) \cap B(r)$ by $\beta \le \alpha'$. □

Dilation yields

Lemma 6

For $^{\forall}\eta > 0$, $^{\exists}\varepsilon > 0$ s.t. if $0 < \theta < \varepsilon$, then

$$\omega(\cdot, F(\theta) \cap S(r), B(r)) < \eta \quad \text{on } B(0, r/2).$$

Repeated application of the Harnack inequality along a Harnack chain gives

Lemma 7

For $0 < \theta < \pi/2$, $^{\exists}\gamma = \gamma(\theta)$ s.t.

$$\frac{h(0,\ldots,0,r)}{h(0,\ldots,0,1)} \le Ar^{\gamma},$$

if $1 \le r \le R/2$ and $h > 0$ is harmonic on $\Gamma(\theta) \cap B(R)$. Moreover, $\gamma(\theta) \downarrow 1$ as $\theta \uparrow \pi/2$.

5. Proof of Theorem 2

Estimate of the harmonic measure for Ω.
We may assume that f is nondecreasing, $f(t) = 0$ for $0 \le t \le 1$ and $B(2) \subset \Omega$.

Lemma 8

Let $0 < \theta \le \pi$. Then

$$\liminf_{r\to\infty} r^{1+\eta}\omega(0, \Gamma(\theta) \cap S(r), \Omega \cap B(r)) = \infty$$

for $^{\forall}\eta > 0$.

For $0 < \theta < \pi/2$ let $I^+(\theta) = \{x : x_n > |x|\sin\theta\}$, $I^-(\theta) = \{x : x_n < -|x|\sin\theta\}$, $I(\theta) = I^+(\theta) \cup I^-(\theta)$.

Proof of Theorem 2 (i). Let $u \in \mathscr{F}$ and let $M_j = \sup_{B(2^j)} u$. Then $^{\exists}S > 1$ s.t.

(3) $$M_j \le S^j \quad \text{for } j \ge 0.$$

Let $\eta > 0$. We shall show that

(4) $$M_j \le A2^{(1+\eta)j} \quad \text{for sufficiently large } j,$$

where $A > 0$ may depend u and η but not on j.
Then u is of order at most 1.
By Lemma 6 $^{\exists}\theta > 0$ s.t.

$$\omega(\cdot, S(2^{j+1}) \setminus I(\theta), B(2^{j+1})) \le \frac{1}{2^{1+\eta}S}$$

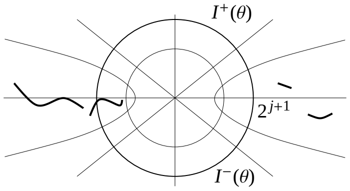
on $B(2^j)$. By assumption we may assume $S(r) \cap I(\theta') \subset \Omega$ for large r with $0 < \theta' < \theta$. The maximum principle over $\Omega \cap B(2^{j+1})$ gives

$$\begin{aligned}u &\le M_{j+1}\omega(\cdot, \Omega \cap S(2^{j+1}) \setminus I(\theta), \Omega \cap B(2^{j+1}))\\ &\quad + \sup_{S(2^{j+1}) \cap I(\theta)} u,\end{aligned}$$

so that

(5) $$u \le \frac{1}{2^{1+\eta}S} M_{j+1} + \sup_{S(2^{j+1})\cap I(\theta)} u$$

on $\Omega \cap B(2^j)$.



The Harnack inequality and Lemma 8 yield

$$u(0) \ge \omega(0, I^+(\theta)\cap S(2^j), \Omega\cap B(2^j)) \inf_{I^+(\theta)\cap S(2^j)} u$$
$$\ge A2^{-(1+\eta)j} \sup_{I^+(\theta)\cap S(2^{j+1})} u$$

for sufficiently large j. Similarly, estimate $\sup_{I^-(\theta)\cap S(2}$
Then

$$u(0) \ge A2^{-(1+\eta)j} \sup_{I(\theta)\cap S(2^{j+1})} u.$$

Substitute this to (5). Take sup over $B(2^j)$.

$$M_j \le \frac{1}{2^{1+\eta}S} M_{j+1} + A2^{(1+\eta)j} u(0),$$

so that

$$\begin{aligned} 2^{-(1+\eta)j} M_j &\le \frac{1}{S} 2^{-(1+\eta)(j+1)} M_{j+1} + Au(0) \\ &\le \frac{1}{S}\left(\frac{1}{S} 2^{-(1+\eta)(j+2)} M_{j+2} + Au(0)\right) \\ &\quad + Au(0) \\ &= \frac{1}{S^2} 2^{-(1+\eta)(j+2)} M_{j+2} + A(1+\frac{1}{S})u(0). \end{aligned}$$

Repeating this, we obtain

$$2^{-(1+\eta)j} M_j \le \frac{1}{S^k} 2^{-(1+\eta)(j+k)} M_{j+k} + Au(0) \sum_{i=0}^{k-1} \frac{1}{S^i}$$

for $k \ge 1$. Let $k \to \infty$. Then (3) yields

$$2^{-(1+\eta)j} M_j \le Au(0) \sum_{i=0}^{\infty} \frac{1}{S^i} = \frac{A}{1-1/S} u(0).$$

This implies (4). □

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