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MINIMAL MARTIN BOUNDARY POINTS OF A JOHN DOMAIN

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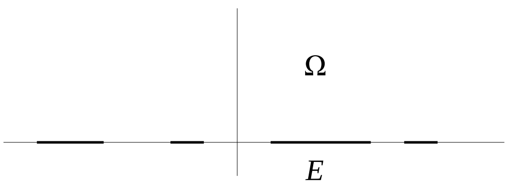
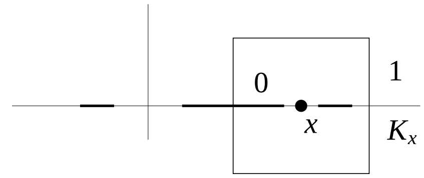
1. Introduction

Let $E \subset \{x = (x_1, \dots, x_n) : x_n = 0\}$ be closed. $\Omega = \mathbb{R}^n \setminus E$ is called a Denjoy domain.



Let $\mathscr{P}$ be the family of positive harmonic functions in Ω vanishing on $\partial\Omega$.

Benedicks [Ben80] proved the following:
$\dim \mathscr{P} = 1$ or 2.
Criterion in terms of harmonic measure $\beta_E(x) = \omega(x, \partial K_x, K_x \setminus E)$.
K_x: cube center at x, side $\alpha|x|$.



$$\dim \mathscr{P} = 1 \iff \int_{|x|\geq 1} \frac{\beta_E(x)}{|x|^{n-1}} dx = \infty.$$

$$\dim \mathscr{P} = 2 \iff \int_{|x|\geq 1} \frac{\beta_E(x)}{|x|^{n-1}} dx < \infty.$$

• Monotonicity:
If $E \subset E'$, $\dim \mathscr{P}_E = 2$, then $\dim \mathscr{P}_{E'} = 2$.

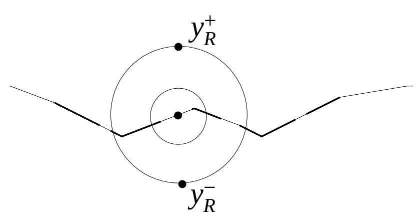
Location	Topics	Authors
C^2 surface	$\dim \mathscr{P} \leq 2$	Ancona (79)
Hyperplane	Harmonic Measure	Benedicks (80)
Lipschitz surface	$\dim \mathscr{P} \leq 2$ WBHP	Ancona (84)
Real line	Lebesgue Measure	Segawa (88)
Hyperplane	Lebesgue Measure	Gardiner (89)
$C^{1,1}$ surface	Harmonic Measure	Chevallier (89)
$C^{1,\alpha}$ surface	Harmonic Measure	Ancona (90)
Lipschitz surface	Non Monotonicity	Ancona (90)
Real line	Quasi-conformal	Segawa (90)
Sectorial	Harmonic Measure	Cranston-Salisbury (93)
Half space	Harmonic Major.	Eiderman-Essén (96)
Quasi-Sectorial	Schrödinger Equation	Lömker (00)

Weak boundary Harnack principle.
Ancona [Anc84].
$B(x,r)$, $S(x,r)$ the open ball and the sphere with center at x and radius r.
$B(r) = B(0,r)$, $S(r) = S(0,r)$.

$\mathscr{P}_\xi$: kernel functions h at ξ, i.e.,
$h > 0$ harmonic on Ω,
$h = 0$ q.e on $\partial\Omega$,
bounded outside ξ.
$E \subset S$: Lipschitz surface.
$h_0, h_1, h_2 \in \mathscr{P}_\xi$. Then

$$h_0(x) \leq A\left(\frac{h_0(y_R^+)}{h_1(y_R^+)} h_1(x) + \frac{h_0(y_R^-)}{h_2(y_R^-)} h_2(x)\right)$$

for $x \in \Omega \cap B(\xi, R) \setminus B(\xi, R/2)$.



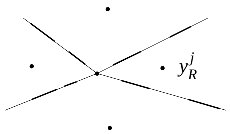
If $h_0, h_1, h_2 \in \mathscr{P}_\xi$, then $\exists i$ s.t.

$$h_i \le A \sum_{j \ne i} h_j;$$

$\dim \mathscr{P}_\xi \le 2$.

Sectorial domain.

Cranston-Salisbury [CS93].



If $h_0, \dots, h_N \in \mathscr{P}_\xi$. Then

$$h_0(x) \le A \left(\sum_{j=1}^{N} \frac{h_0(y_R^j)}{h_j(y_R^j)} h_j(x) \right)$$

for $x \in \Omega \cap B(\xi, R) \setminus B(\xi, R/2)$; $\exists i$ s.t.

$$h_i \le A \sum_{j \ne i} h_j;$$

$\dim \mathscr{P}_\xi \le N$.

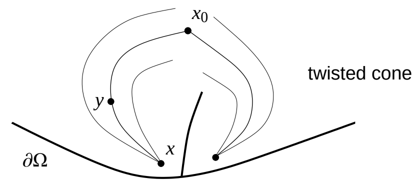
Quasi-sectorial domain (higher dimension).

2. Extension to a John domain

John domain. twisted cone condition:

$\forall x \in \Omega$, $\exists \gamma : x \to x_0$ s.t.

$$\delta_\Omega(y) \ge c_J \ell(\gamma(x, y)) \quad \text{for all } y \in \gamma,$$



Denjoy domain
Sectorial domain
Quasi-Sectorial

$\Longrightarrow$ John domain

Theorem 1

Let Ω be a John domain with John constant c_J. Let $\xi \in \partial\Omega$. Then

(i) $\dim \mathscr{P}_\xi \le N(c_J) < \infty$.

(ii) If $c_J > \sqrt{3}/2$, then $\dim \mathscr{P}_\xi \le 2$.

Remark 1

$c_J > \sqrt{3}/2$ is sharp.

Quasihyperbolic metric:

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_\Omega(z)}.$$

where inf is taken over all curves γ connecting x to y in Ω.

$$k_\Omega(x, y) \approx \text{length of Harnack chain.}$$

If $h > 0$ is harmonic on Ω, then

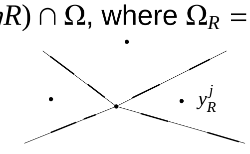
$$\exp(-Ak_\Omega(x, y)) \le \frac{h(x)}{h(y)} \le \exp(Ak_\Omega(x, y))$$

Local reference points:

$\exists y_R^1, \dots, \exists y_R^N \in S(\xi, R) \cap \Omega$ s.t. $\delta_\Omega(y_R^i) \approx R$ and

$$\min_{i=1,\dots,N} \{k_{\Omega_R}(x, y_R^i)\} \le A \log \frac{R}{\delta_\Omega(x)} + A'$$

for $x \in B(\xi, \eta R) \cap \Omega$, where $\Omega_R = \Omega \cap B(\xi, AR)$.



• If $h \in \mathscr{P}_\xi$, then 0-extension to Ω^c is subharmonic in $\mathbb{R}^n \setminus \{\xi\}$.

Lemma 1 (Domar [Dom57]**)**

Let $u \geq 0$ be subharmonic in D s.t.

$$I = \int_D (\overset{+}{\log} u)^{n-1+\varepsilon} dx < \infty$$

for $^\exists\varepsilon > 0$. Then

$$u(x) \leq \exp(AI^{1/\varepsilon}\, \mathrm{dist}(x, \partial D)^{-n/\varepsilon}).$$

Lemma 2

$^\exists\tau > 0$ s.t.

$$\int_{\Omega \cap B(\xi,R)} \left(\frac{R}{\delta_\Omega(x)}\right)^\tau dx \leq AR^n.$$

Lemma 3

Let $h \in \mathscr{P}_\xi$ for $\xi \in \partial\Omega$. Then

$$h(x) \leq A|x - \xi|^{-\lambda}.$$

Proof. By local reference points

$$h(x) \leq A\left(\frac{R}{\delta_\Omega(x)}\right)^\lambda \sum_{i=1}^N h(y_R^i).$$

Apply Lemma 1 to $D = B(\xi, AR) \setminus \overline{B(\xi, A^{-1}R)}$. with the help of Lemma 2. Then

$$(1) \qquad h(x) \leq A \sum_{i=1}^N h(y_R^i)$$

on $S(\xi, R)$, and hence on $\Omega \setminus B(\xi, R)$ by the maximum principle. Since $\delta_\Omega(y_R^i) \approx R$, we have $h(y_R^i) \leq AR^{-\lambda}$. Hence

$$h(x) \leq AR^{-\lambda} \quad \text{on } \Omega \setminus B(\xi, R),$$

i.e. $h(x) \leq A|x - \xi|^{-\lambda}$ on Ω. □

Tract argument (Friedland-Hayman [FH76]) implies

$$\dim \mathscr{P}_\xi \leq N.$$

By box argument initiated by Bass-Burdzy [BB91] (see [Aik01, Lemma 2]) we have

$$\omega(x, \Omega \cap S(\xi, AR), \Omega \cap B(\xi, AR)) \leq AR^{2-n} \sum_{i=1}^N G_R(x, y_R^i)$$

for $x \in \Omega \cap B(\xi, R)$, where G_R is the Green function for $\Omega \cap B(\xi, A'R)$. Combine with (1). Then

$$h(x) \leq AR^{2-n} \sum_{i=1}^N G_R(x, y_R^i) \sum_{j=1}^N h(y_R^j).$$

Apply this inequality to $h(x) = G_R(x, y)$. Then

$$G_R(x, y) \leq AR^{2-n} \sum_{i=1}^N G_R(x, y_R^i) \sum_{j=1}^N G_R(y_R^j, y).$$

Now let $c_J > \sqrt{3}/2$. Then $N \leq 2$. Ancona's ingenious trick [Anc84, Théorème 7.3] gives

$$G_R(x, y) \leq AR^{2-n} \sum_{i=1}^2 G_R(x, y_R^j) G_R(y_R^j, y).$$

This yields the WBHP: Let $h_0, h_1, h_2 \in \mathscr{P}_\xi$. Then

$$h_0(x) \leq A \sum_{i=1}^2 \frac{h_0(y_R^i)}{h_i(y_R^i)} h_i(x) \quad \text{for } x \in \Omega.$$

This immediately means $\dim \mathscr{P}_\xi \leq 2$.

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