BOUNDARY HARNACK PRINCIPLE WITH APPLICATIONS

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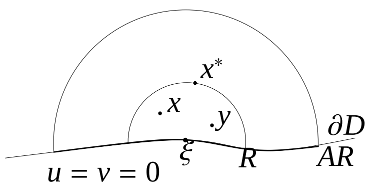
1. Boundary Harnack Principle (BHP)

Talk based on HA [Aik01].
$\xi \in \partial D$, $R > 0$ small,
u, v positive harmonic on D.
If $u = v = 0$ on $\partial D \cap B(\xi, AR)$, then

$$\frac{u(x)}{v(x)} \approx \frac{u(y)}{v(y)} \quad \text{for } x, y \in D \cap B(\xi, R)$$

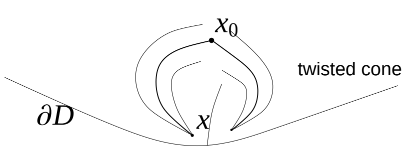


Domain	Topics	Authors
Half space	Carleson Estimates	Carleson (61)
Smooth	$u(x) \approx \delta_D(x)$	Widman (67)
Lipschitz	Martin boundary	Hunt-Wheeden (70)
Lipschitz	Harmonic analysis, $d\omega/d\sigma \in A_\infty$	Dahlberg (77)
Lipschitz	Uniform elliptic equation	Ancona (78)
Lipschitz	Harmonic measure of box	Wu (78)
Lipschitz	Reflection, Uniform elliptic	Caffarelli et al (81)
NTA	Harmonic analysis	Jerison-Kenig (82)
Manifold	special superharmonic function	Anderson-Schoen (85)
Hölder, John	Probability, No exterior	Bass-Burdzy (91)
Uniformly John	Uniform wrt internal metric	Balogh-Volberg (96)
Inner Uniform	Gromov hyperbolic	Bonk-Heinonen-Koskela (01)

Theorem (Analytic proof of [BB91])
Non uniform BHP holds for a John domain.

John domain. twisted cone condition:
$^\forall x \in D$, $^\exists \gamma : x \to x_0$ s.t.

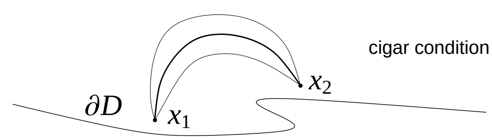
$$\delta_D(y) \geq c_J \ell(\gamma(x, y)) \quad \text{for all } y \in \gamma,$$



Uniform domain. cigar condition:
$^\forall x_1, x_2 \in D$, $^\exists \gamma : x_1 \to x_2$ s.t.

$$\delta_D(y) \geq A \min\{\ell(\gamma(x_1, y)), \ell(\gamma(x_2, y))\},$$

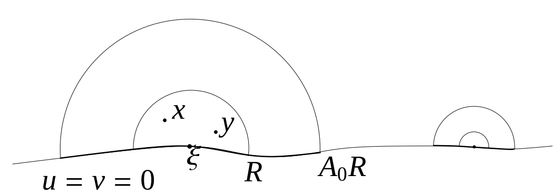
$$\ell(\gamma) \leq A|x_1 - x_2|.$$



Theorem
Uniform BHP holds for a uniform domain.
$\xi \in \partial D$, $R > 0$ small,
u, v bounded positive harmonic on $D \cap B(\xi, A_0 R)$.
If $u = v = 0$ q.e. on $\partial D \cap B(\xi, A_0 R)$, then

$$\frac{u(x)}{v(x)} \leq A_1 \frac{u(y)}{v(y)} \quad \text{for } x, y \in D \cap B(\xi, R).$$

A_0 and A_1 are independent of ξ and R.



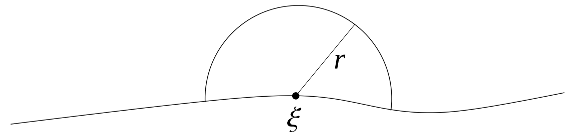
Remark

- $\Delta = \Delta_1 = \partial D$.
- Martin kernel is Hölder continuous.
- u/v is Hölder continuous at the boundary where they vanish.
- u and v are not continuous at irregular boundary points!

Difficulties and new aspects.

- No Exterior condition. Carleson argument does not work.
- Surface ball $\partial D \cap B(\xi, r)$ may be polar.
- Harmonic measure needs not doubling.
- Dahlberg's estimate does not hold.
- Marin kernel cannot be retrieved by the ratio of harmonic measures.
- Estimate the ratio of Green functions.
- Carleson estimates trivially holds for the Green function.
- Jones' geometric localization can be avoided.
- Further extensions to (uniformly) John domains.

2. Application: Hölder continuity of u/v



Let

$$M(r) = \sup_{D\cap B(\xi,r)} \frac{u}{v}, \quad m(r) = \inf_{D\cap B(\xi,r)} \frac{u}{v}.$$

Then $\mathrm{osc}_{D\cap B(\xi,r)}\, u/v = M(r) - m(r)$. BHP says

$$1 \le \frac{M(r)}{m(r)} \le A \text{ independent of } r.$$

Moser technique:

$$M(r)v - u > 0 \text{ on } D \cap B(\xi, r),$$
$$u - m(r)v > 0 \text{ on } D \cap B(\xi, r).$$

$\Longrightarrow$

$$\operatorname*{osc}_{D\cap B(\xi,r/A_0)} \frac{u}{v} \le (1-\delta) \operatorname*{osc}_{D\cap B(\xi,r)} \frac{u}{v}$$

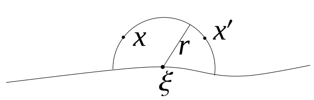
$\Longrightarrow$

$$\operatorname*{osc}_{D\cap B(\xi,r)} \frac{u}{v} \le A\left(\frac{r}{R}\right)^{\varepsilon} \operatorname*{osc}_{D\cap B(\xi,R)} \frac{u}{v}.$$

3. Application: Martin boundary

$$\mathscr{H}_\xi = \{h > 0 : h(x_0) = 1, h = 0 \text{ q.e. on } \partial D,$$
$$\text{bounded on } D \setminus B(\xi, r)\ {}^{\forall} r > 0\}$$

$\cdot\, x_0$



BHP says

$$\frac{u(x)}{v(x)} \approx \frac{u(x')}{v(x')},$$

$x' \to x_0$ by the maximum principle

$$A^{-1} \le \frac{u}{v} \le A \quad \text{for } u, v \in \mathscr{H}_\xi.$$

$\Longrightarrow$

$$\mathscr{H}_\xi \text{ is singleton. } \Delta = \Delta_1 = \partial D.$$

Ancona's series argument or

Short proof:

$$1 \le c = \sup_{\substack{u,v\in\mathscr{H}_\xi \\ x\in D}} \frac{u(x)}{v(x)} < \infty.$$

We show $c > 1 \implies$ contradiction.
If $u, v \in \mathscr{H}_\xi$, then $v_1 = \dfrac{cv - u}{c-1} \in \mathscr{H}_\xi$. Hence

$$u \le cv_1 = c\frac{cv - u}{c - 1}$$

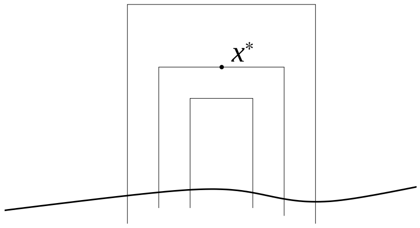
by definition. This means

$$\frac{u}{v} \le \frac{c^2}{2c - 1} < c,$$

since $c > 1$. This yields a contradiction

$$c = \sup_{\substack{u,v\in\mathscr{H}_\xi \\ x\in D}} \frac{u(x)}{v(x)} \le \frac{c^2}{2c-1} < c.$$

4. Classical Proof for a Lipschitz domain



- Carleson estimate: If $u > 0$ is harmonic in Δ and vanishes on $\Delta \cap \partial D$, then

$$u \le Au(x^*) \quad \text{on } \Delta'.$$

- Box estimate:

$$\omega(\partial_s \Delta', \Delta') \le A\omega(\partial_u \Delta', \Delta') \quad \text{on } \Delta''.$$

Hence

$$\omega(\partial \Delta' \cap D, \Delta') \le A\omega(\partial_u \Delta', \Delta') \quad \text{on } \Delta''.$$

$\partial_u \Delta'$ is the main part.

5. Proof of BHP

Suppose u and v are positive harmonic functions on Δ which vanish on $\Delta \cap \partial D$.
By the Carleson estimate and the box estimate

$$u \le Au(x^*) \quad \text{on } \Delta'$$
$$u \le Au(x^*)\omega(\partial \Delta' \cap D, \Delta') \quad \text{on } \Delta'$$
$$u \le Au(x^*)\omega(\partial_u \Delta', \Delta') \quad \text{on } \Delta''.$$

On the other hand, the Harnack inequality yields

$$v \approx v(x^*) \quad \text{on } \partial_u \Delta',$$
$$v \ge Av(x^*)\omega(\partial_u \Delta', \Delta') \quad \text{on } \Delta'.$$

Hence

$$\frac{u}{u(x^*)} \le A\omega(\partial_u \Delta', \Delta') \le A\frac{v}{v(x^*)} \quad \text{on } \Delta''.$$

6. Box Estimate: An Outline

$\xi \in \partial D$ and R small, ξ_R: reference point.
G_R: Green function for $D \cap B(\xi, AR)$.

Lemma 1

On $D \cap B(\xi, R)$

$$\omega(\cdot, D \cap S(\xi, 2R), D \cap B(\xi, 2R)) \le AR^{n-2} G_R(\cdot, \xi_R).$$

Proof.

$$D_j = \{\exp(-2^{j+1}) \le R^{n-2} G_R(x, \xi_R) < \exp(-2^j)\},$$
$$U_j = \{R^{n-2} G_R(x, \xi_R) < \exp(-2^j)\}.$$

Observe U_j is very slim, i.e.

$$w_\eta(U_j) \le AR \exp\left(-\frac{2^j}{\lambda}\right).$$

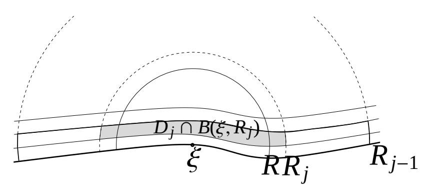
Let $R_j \downarrow R$ slowly; $R_{j-1} - R_j \approx j^{-2}$.
Put $\omega_0 = \omega(\cdot, D \cap S(\xi, 2R), D \cap B(\xi, 2R))$ and

$$d_j = \sup_{x \in D_j \cap B(\xi, R_j)} \frac{\omega_0(x)}{R^{n-2} G_R(x, \xi_R)}.$$

We want to show $\sup_{j \ge 0} d_j \le A < \infty$.
The maximum principle over $U_j \cap B(\xi, R_{j-1})$ says

$$\omega_0 \le \omega(\cdot, U_j \cap S(\xi, R_{j-1}), U_j \cap B(\xi, R_{j-1})) + d_{j-1} R^{n-2} G_R(\cdot, \xi_R)$$

on $U_j \cap B(\xi, R_{j-1})$.



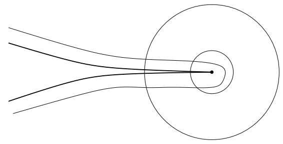
Then

$$d_j \le A \exp\left(2^{j+1} - Aj^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1}$$

with

$$\exp\left(2^{j+1} - Aj^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) < \infty.$$

The width argument works for an irregular domain:



□

Obvious Carleson estimate for the Green function:

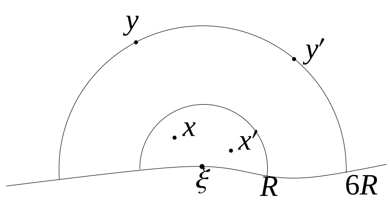
$$G_R(x,y) \le AR^{2-n}$$

for $x \in D \cap B(\xi, 2R)$, $y \in D \cap B(\xi, 9R) \setminus B(\xi, 3R)$.

Lemma 2

For $x, x' \in D \cap B(\xi, R)$ and $y, y' \in D \cap S(\xi, 6R)$

$$\frac{G_R(x,y)}{G_R(x',y)} \approx \frac{G_R(x,y')}{G_R(x',y')}.$$



Proof. We may assume

$$u(x) = \widehat{R}_u^{D\cap S(\xi,6R)}(x) = \int_{D\cap S(\xi,6R)} G_R(x,y)d\mu(y)$$

for $x \in D \cap B(\xi, 6R)$. Let $x, x' \in D \cap B(\xi, R)$ and $y, y' \in D \cap S(\xi, 6R)$. Then

$$G_R(x,y) \approx \frac{G_R(x,y')}{G_R(x',y')} G_R(x',y).$$

Hence

$$u(x) \approx \frac{G_R(x,y')}{G_R(x',y')} \int_{D\cap S(\xi,6R)} G_R(x',y)d\mu(y) = \frac{G_R(x,y')}{G_R(x',y')} u(x').$$

Therefore,

$$\frac{u(x)}{u(x')} \approx \frac{G_R(x,y')}{G_R(x',y')} \quad \text{for } y' \in D \cap S(\xi, 6R).$$

Similarly,

$$\frac{v(x)}{v(x')} \approx \frac{G_R(x,y')}{G_R(x',y')}.$$

Hence the theorem follows. □

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