John domains and the doubling property of the harmonic measure

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Contents

1. Nonsmooth domains

[AHL06]: H. Aikawa, K. Hirata, and T. Lundh, Martin boundary points of a John domain and unions of convex sets, J. Math. Soc. Japan **58** (2006), no. 1, 247–274.

Notation: $D \subset \mathbb{R}^n$ with $n \geq 2$, $\delta_D(x) = \textbf{dist}(x, \partial D)$.

$B(x, r)$: open ball center at x and radius r;

$S(x, r)$: sphere center at x and radius r.

A: general positive constant.

John domain

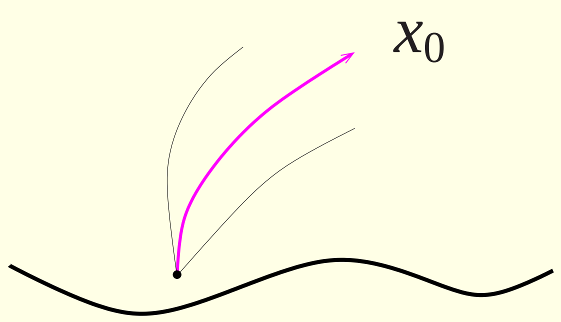
$\exists_{c_J} > 0$: John constant and
$\exists_{x_0} \in D$: John center s.t.
$\forall_{x} \in D$ can be joined to x_0 by γ with

$$\delta_D(y) \geq c_J \ell(\gamma(x,y)) \quad \text{for all } y \in \gamma;$$

- $\gamma(x,y)$ is the subarc of γ connecting x and y.
- $\ell(\gamma(x,y))$ is its length.

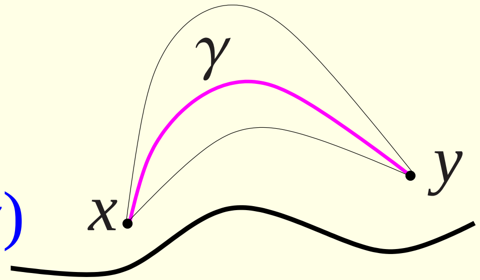
In general, $0 < c_J < 1$.

Visualized as a twisted cone condition.



Uniform domain

$\forall x, y \in D$ can be joined by $\gamma \subset D$ s.t.
$\ell(\gamma) \le A|x-y|$ (Bounded Turning) and



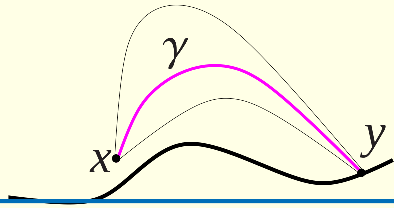
$$\textbf{min}\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \le A\delta_D(z) \quad (1)$$

for $\forall z \in \gamma$ (Cigar condition).

The connectivity of a uniform domain can be extended from $x, y \in D \implies x, y \in \overline{D}$.

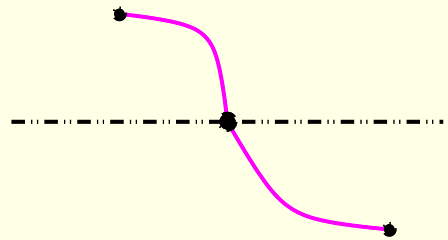
Semi-uniform domain

$\forall x \in D$ and $\forall y \in \partial D$ can be joined by γ s.t. $\gamma \setminus \{y\} \subset D$, $\ell(\gamma) \le A|x-y|$ and (1) holds.



Remark 1

A Denjoy domain is a typical semi-uniform domain which is not necessarily uniform.



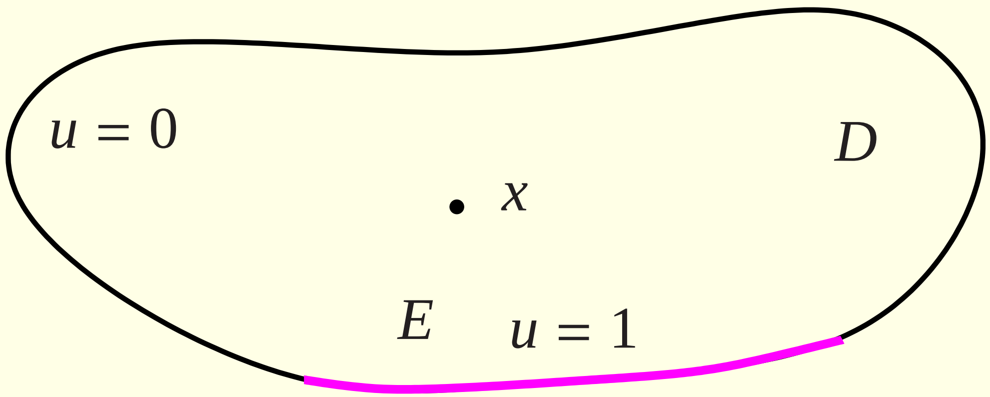
2. Harmonic measure

Consider the Dirichlet problem:

$$\Delta u = 0 \text{ in } D$$

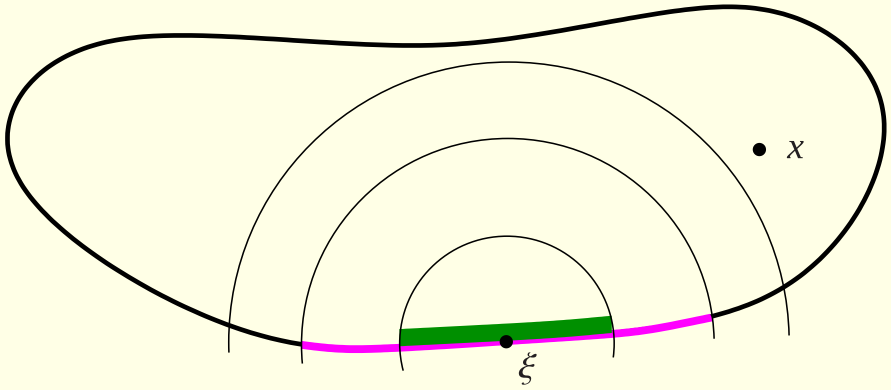
$$u = f \text{ on } \partial D$$

If $f = \chi_E$, then the solution is called the harmonic measure of E with respect to D and is denoted by $\omega(x, E, D)$.



Strong doubling condition

Jerison-Kenig [JK82] proved that harmonic measure of an NTA domain D satisfies the strong doubling condition: $\exists A_0 > 2$ such that



(2)

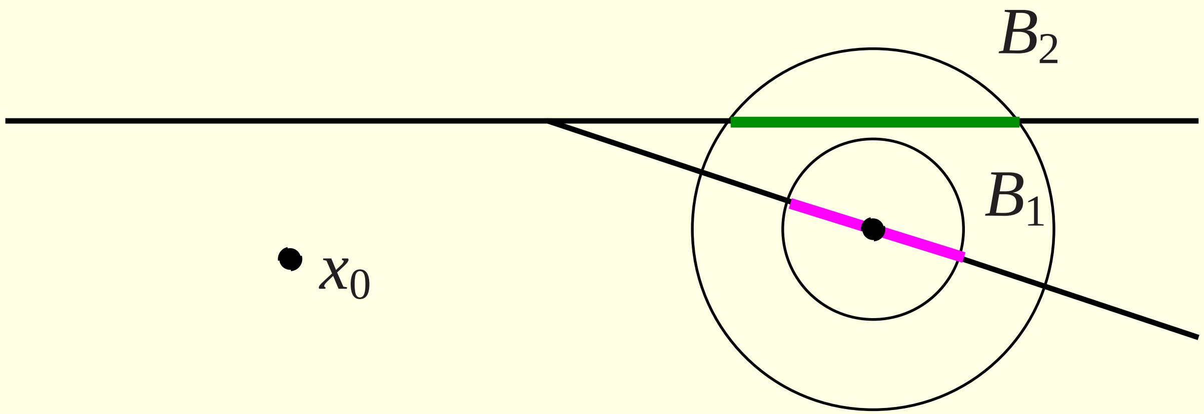
$$\omega(x, B(\xi, 2R)\cap\partial D, D) \le A\omega(x, B(\xi, R)\cap\partial D, D) \quad \text{for } x \in D\setminus B(\xi, A_0 R),$$

where $\xi \in \partial D$ and $R > 0$ small.

If (2) holds only for some fixed point $x = x_0$, we say that the harmonic measure of D satisfies the doubling condition.

- Strong doubling condition $\Longrightarrow$ the doubling condition.
- In $\mathbb{R}^2$.
  - a simply connected domain D is an NTA domain $\Longleftrightarrow$ the harmonic measures both for D and $\overline{D}^c$ satisfy the doubling condition (Jerison-Kenig [JK82, Theorem 2.7]).
  - Kim and Langmeyer [KL98] gave the one-sided analogue; a bounded planar Jordan domain is a John domain $\Longleftrightarrow$ the harmonic measure only for D satisfies the doubling condition.
  - Balogh-Volberg [BV96a], [BV96b] showed a doubling condition similar to (2) in a planar uniformly John domain, or inner uniform domain.
  - All arguments are based on complex analysis.

Balogh-Volberg's counter example [BV96a]:



The complement of $[-1,1]$ and $L_\theta = \{te^{-i\theta} : 0 \le t \le 1\}$ with $0 < \theta < \pi/2$.
Let $B_1 = B(te^{-i\theta}, ct)$ and $B_2 = B(te^{-i\theta}, 2ct)$, where $\frac{1}{2}\sin\theta < c < \sin\theta$.
Since $B_1 \cap [-1,1] = \emptyset$ and $B_2 \cap [-1,1] \neq \emptyset$, we have

$$\omega(x_0, B_1 \cap \partial D, D) \approx t^{\pi/(\pi-\theta)}, \quad \omega(x_0, B_2 \cap \partial D, D) \approx t$$

as $t \to 0$. Hence $\dfrac{\omega(x_0, B_2 \cap \partial D, D)}{\omega(x_0, B_1 \cap \partial D, D)} \to \infty.$

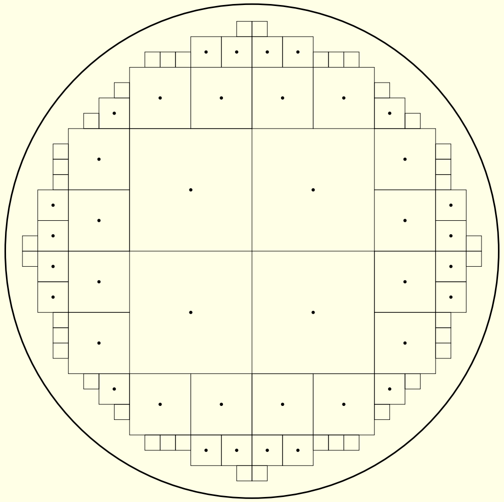
3. Harmonic measure and semi-uniform domain

Characterize

John domains whose harmonic measures satisfy the strong doubling condition.

Remark 2

There is a John domain with polar boundary whose harmonic measure vanishes. For such domains any doubling conditions for harmonic measure is hopeless.



To avoid such pathological domains, we assume the capacity density condition (abbreviated to CDC).

Definition 1

By **Cap** we denote the logarithmic capacity if $n = 2$, and the Newtonian capacity if $n \geq 3$. We say that the CDC holds if $^{\exists}A > 0$ s.t.

$$\mathbf{Cap}(B(\xi, R) \setminus D) \geq \begin{cases} AR & \text{if } n = 2, \\ AR^{n-2} & \text{if } n \geq 3, \end{cases}$$

whenever $\xi \in \partial D$ and $R > 0$ is small.

Theorem 1

Let D be a John domain and suppose the CDC holds. Then

semi-uniform domain $\Longleftrightarrow$ strong doubling harmonic measure.

4. Green function and harmonic measure

Let $G(x,y)$ be the Green function for D:

- $\Delta_x G(\cdot,y) = -e_n \delta_y$.
- $G(\cdot,y) = 0$ on ∂D.

Lemma 1 (Green function and connecting curve)

Let D satisfy the CDC. Suppose $\delta_D(y) = R > 0$ is small and $G(x,y) > A_1 R^{2-n}$. Then $\exists \gamma$ connecting x and y in D s.t.

- $\ell(\gamma) \le AR$,
- $\delta_D(z) \ge R/A$ for all $z \in \gamma$.

We define the quasihyperbolic metric $k_D(x, y)$ by

$$k_D(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable curves γ connecting x to y in D. We observe that the shortest length of the Harnack chain connecting x and y is comparable to $k_D(x, y) + 1$. Therefore, the Harnack inequality yields that there is a constant $A > 1$ depending only on n such that

$$\exp(-A(k_D(x, y) + 1)) \le \frac{h(x)}{h(y)} \le \exp(A(k_D(x, y) + 1)) \tag{3}$$

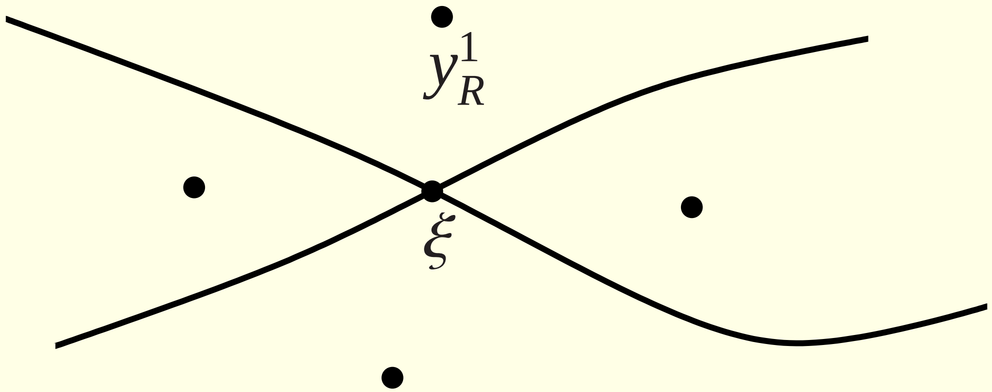
for every positive harmonic function h on D. We say that D satisfies a quasihyperbolic boundary condition if

$$k_D(x, x_0) \le A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A \quad \text{for all } x \in D. \tag{4}$$

It is easy to see that a John domain satisfies the quasihyperbolic boundary condition (see [GM85, Lemma 3.11]). We have more precise estimate ([AHL06, Proposition 2.1]).

Lemma A (Satellite points)

Let D be a John domain with John constant c_J. Then $^\exists N$ s.t. for $^\forall \xi \in \partial D$ and R small, N points $y_1^R, \dots, y_N^R \in D \cap S(\xi, R)$ s.t.



- $A^{-1}R \le \delta_D(y_i^R) \le R$,
- $\min_{i=1,\dots,N} \{k_{D_R}(x, y_i^R)\} \le A \log \dfrac{R}{\delta_D(x)} + A$ for $x \in D \cap B(\xi, R/2)$, where $D_R = D \cap B(\xi, 8R)$,
- $^\forall x \in D \cap B(\xi, R/2)$ can be connected to $^\exists y_i^R$ by $^\exists \gamma \subset D_R$ with $\ell(\gamma(x,z)) \le A\delta_D(z)$ for all $z \in \gamma$.

If the conclusion of the above lemma holds, then we say that ξ has a system of local reference points $y_1^R, \ldots, y_N^R$ of order N.

Lemma 2 (Lower estimate)

Let D be a John domain with the CDC. Let $\xi \in \partial D$ have a system of local reference points $y_1^R, \ldots, y_N^R \in D \cap S(\xi, R)$ of order N for $0 < R < R_D$. Then

$$R^{n-2} \sum_{i=1}^{N} G(x, y_i^R) \le A\omega(x, \partial D \cap B(\xi, 2AR), D) \quad \text{for } x \in D \setminus B(\xi, 2R),$$

Lemma 3 (Upper estimate)

Let D be a John domain. Let $\xi \in \partial D$ have a system of local reference points $y_1^R, \dots, y_N^R \in D \cap S(\xi, R)$ of order N for small $R > 0$. Then

$$\omega(x, \partial D \cap B(\xi, R/8), D) \le AR^{n-2} \sum_{i=1}^{N} G(x, y_i^R) \quad \text{for } x \in D \setminus B(\xi, R/4)$$

5. Proof of Theorem 1

Theorem 1

Let D be a John domain and suppose the CDC holds. Then

semi-uniform domain $\Longleftrightarrow$ strong doubling harmonic measure.

Proof of Theorem 1. $\Longrightarrow$ is easy.

Let us prove $\Longleftarrow$. Let $x \in D$ and $\xi \in \partial D$. We may assume that $|x-\xi| = R$ is small. Then by Lemma A and scaling we find a system of local reference points $y_1^R, \dots, y_N^R \in D \cap S(\xi, R)$ and $y_1^{2R}, \dots, y_N^{2R} \in D \cap S(\xi, 2R)$.

Claim: ${}^\forall y_i^{2R}$ can be connected to ${}^\exists y_j^R$ by a curve γ with $\ell(\gamma) \le AR$ and $\delta_D(z) \ge R/A$ for all $z \in \gamma$.

▶ Contents

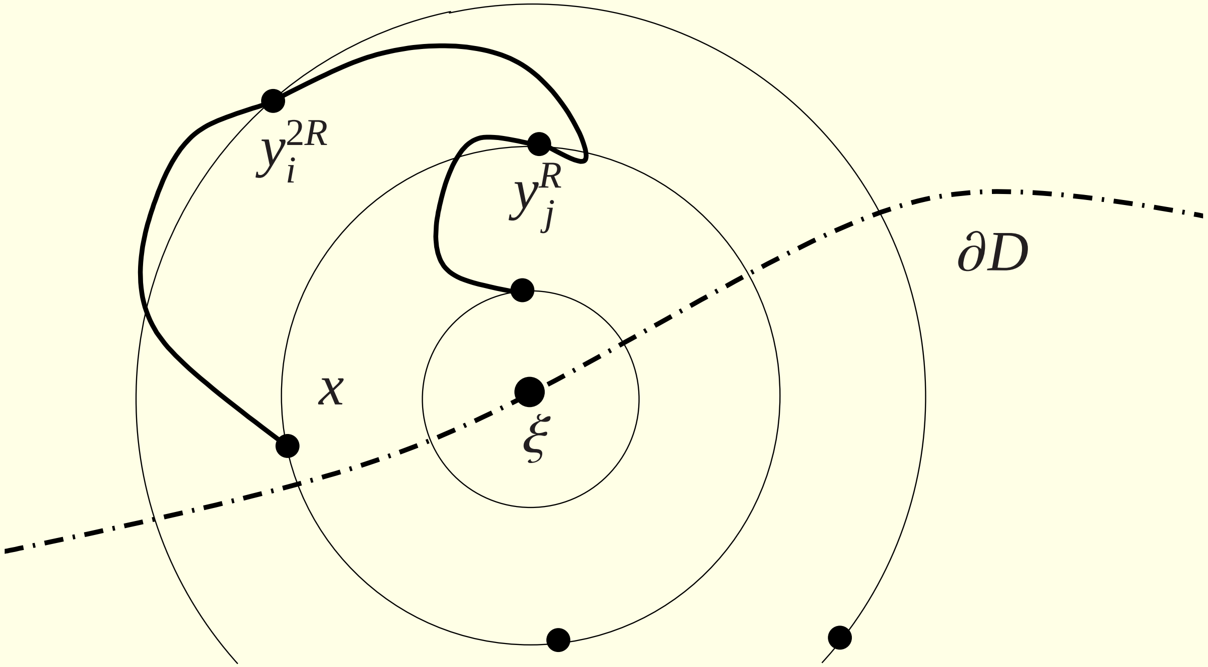
By Lemmas 2 and 3,

$$\frac{1}{A} \le \omega(y_i^{2R}, \partial D \cap B(\xi, R/8), D) \le AR^{n-2} \sum_{j=1}^{N} G(y_i^{2R}, y_j^R).$$

Hence $\exists y_j^R$ s.t. $G(y_i^{2R}, y_j^R) \ge AR^{2-n}$. Lemma 1 gives a curve γ connecting y_i^{2R} to y_j^R in D such that $\ell(\gamma) \le AR$ and $\delta_D(z) \ge R/A$ for all $z \in \gamma$.

Now the proof is easy. By Lemma A, $\exists y_i^{2R}$ which can be connected to x by a cigar curve with length bounded by AR. The claim gives $\exists y_j^R$ which can be connected to y_i^{2R} by a cigar curve with length bounded by AR.

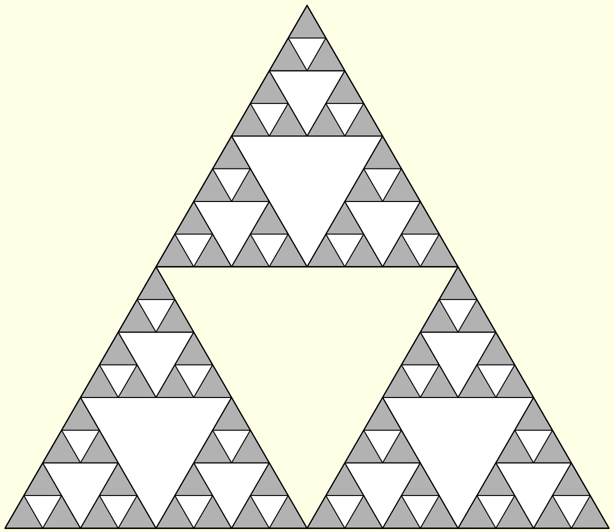
Repeat the claim again. We find a point $\exists y_k^{R/2}$ which can be connected to y_j^R by a cigar curve with length bounded by $AR/2$. Thus D is a semi-uniform domain.



□

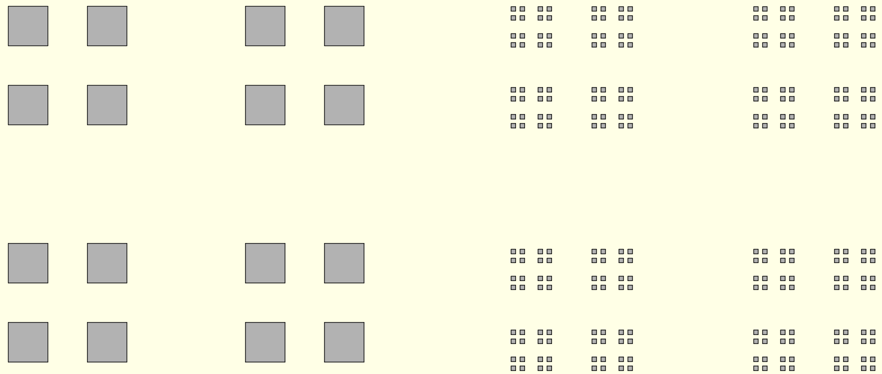
6. Nonsmooth domains as the complements of fractals

Construct nonsmooth domains as the complements of self-similar fractals F. $D = B \setminus F$. Assume D is connected. Rule out the 2-dimensional Sierpiński gasket.

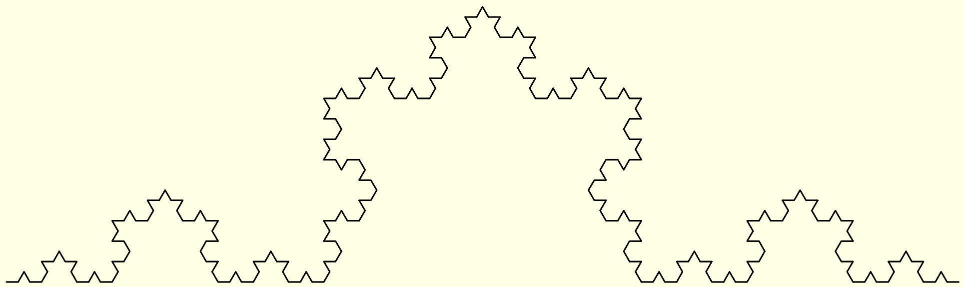


Sierpiński gasket. The complement has infinitely many component.

The complement of the usual Cantor set is a uniform domain.



Cantor set. Totally disconnected.

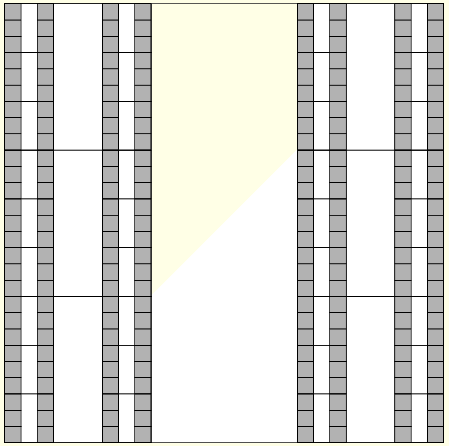


Snow flake (NTA).

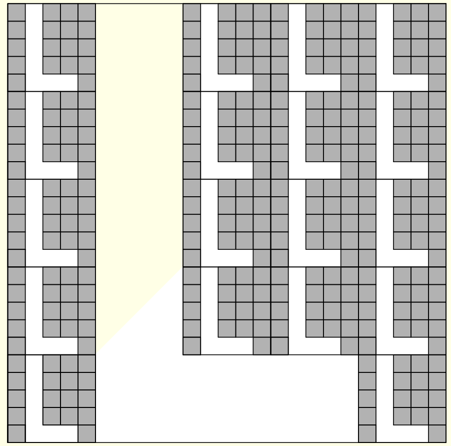
Question

Is the complement of a fractal a John domain?

$B \setminus F$ needs not be a John domain.



Filled Cantor.



L-Cantor.

Question

What condition?

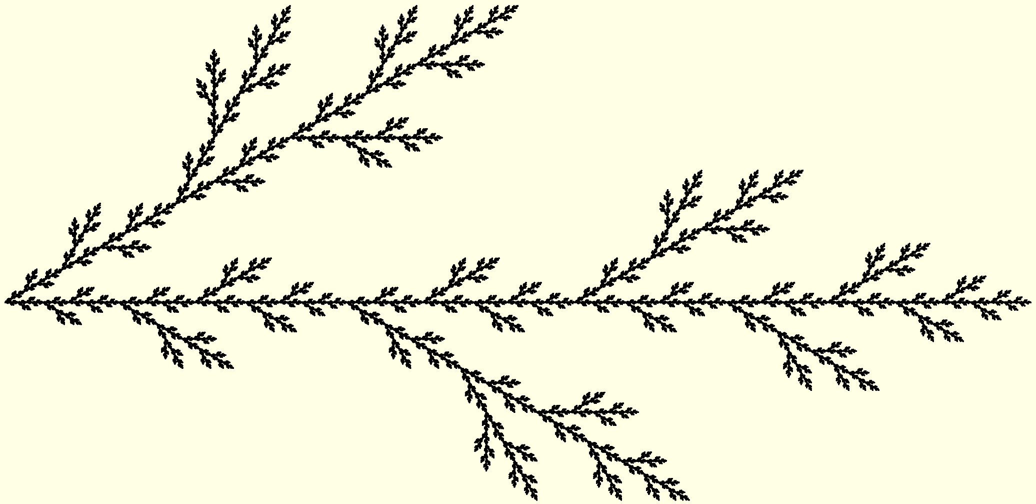
- Starting with a polygon.
- Certain Nesting Axiom.
- Pockets Axiom.

$\Psi = \{\psi_1, \ldots, \psi_m\}$: self-similar mappings.

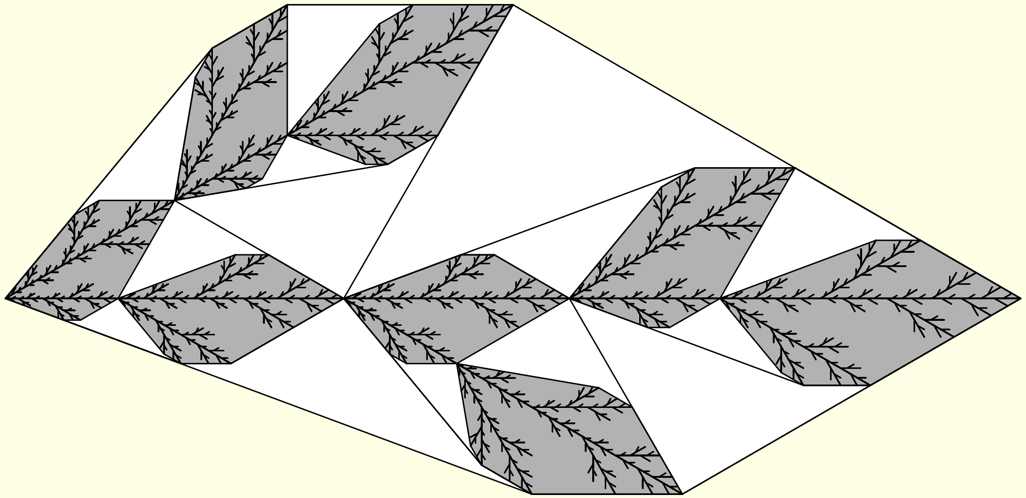
- Starting with a polygon H.

$$H \supset \Psi(H) \supset \Psi^2(H) \supset \cdots \to F.$$

Find H. Easy or Difficult? What is H for Hata's tree: $\Psi = \{\psi_1, \psi_2\}$ with $\psi_1(z) = \omega\bar{z}$, $\psi_2(z) = (2\bar{z} + 1)/3$.



H is a heptangular. $\Psi^3(H)$ is as follows.



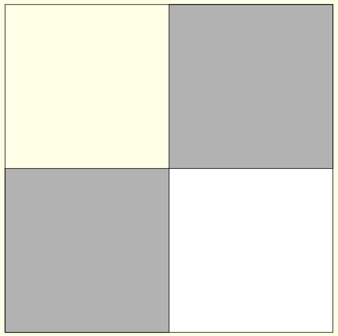
- Nesting Axiom

If $i \neq j$, then $\psi_i(H) \cap \psi_j(H) = \psi_i(F) \cap \psi_j(F)$.

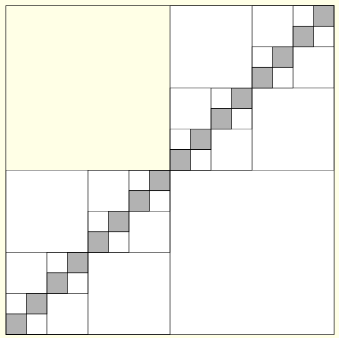
Remark 3 (Indefinite nesting)

$I = (i_1, \ldots, i_n)$, $|I| = n$. If $|I| = |J|$ and $I \neq J$, then $\psi_I(H) \cap \psi_J(H) = \psi_I(F) \cap \psi_J(F)$.

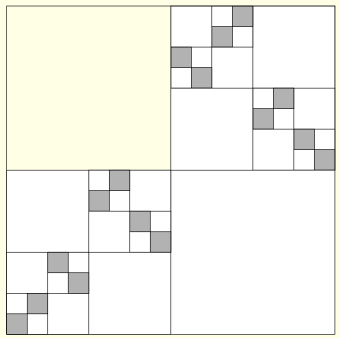
Need to verify only the first step.



The same 0 step.



Straight 3rd step.



Twisted 3rd step.

Remark 4

Similar to Lindström [Lin90] nesting axiom.

F_0: essential fixed points of Ψ.

If $|I| = |J|$ and $I \neq J$, then $\psi_I(F) \cap \psi_J(F) = \psi_I(F_0) \cap \psi_J(F_0)$.

- Lindström $\implies$ $\psi_I(F) \cap \psi_J(F)$ is finite.
- Our assumption allows infinite $\psi_I(F) \cap \psi_J(F)$.

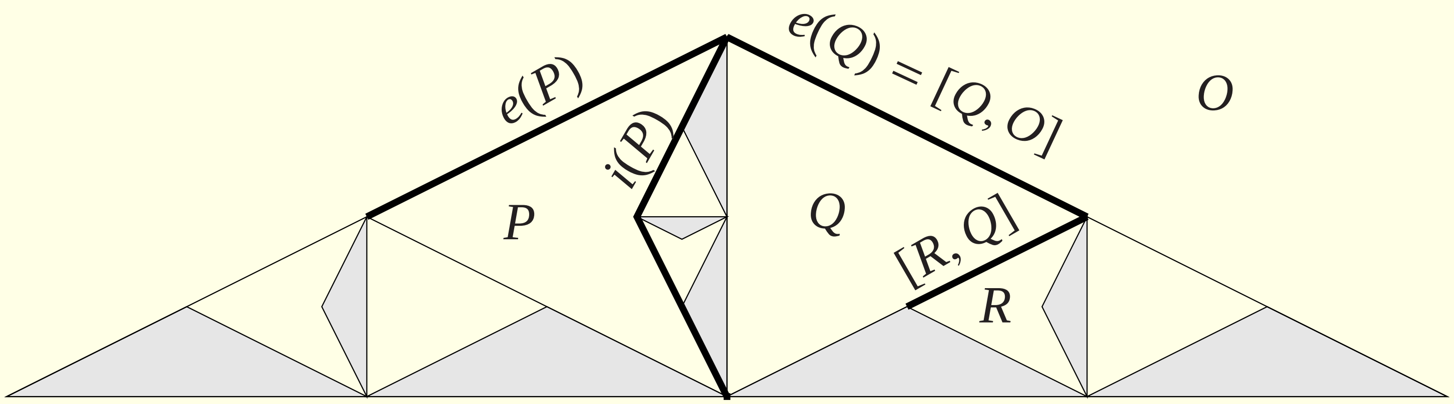
▶ Pockets Axiom.

$H \setminus \Psi(H)$ consists of finitely many polygons (pockets) with

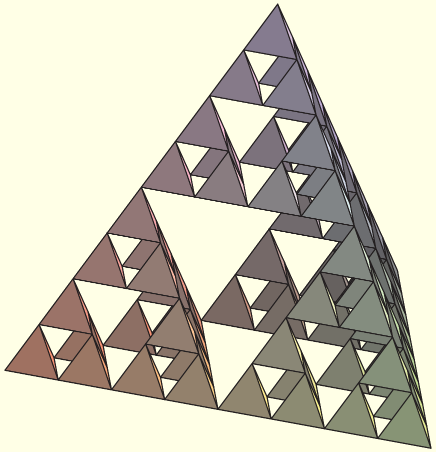
(i) $e(P^i) \neq \emptyset$ and consists of finitely many open subfaces of H.

(ii) $i(P^i)$ is the union of finitely many polygons appearing in $\Psi(H)$.

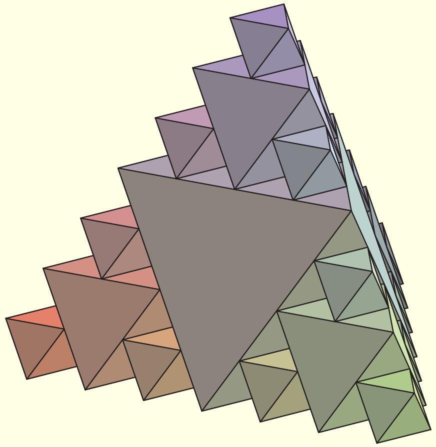
(iii) $i(P^i) \cap \partial H \subset F$.



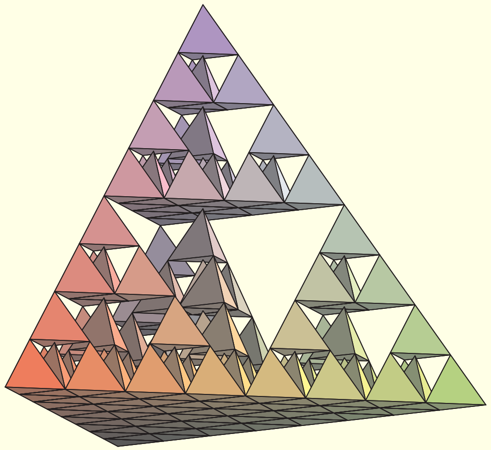
Examples. 3-dimensional Sierpiński gasket,



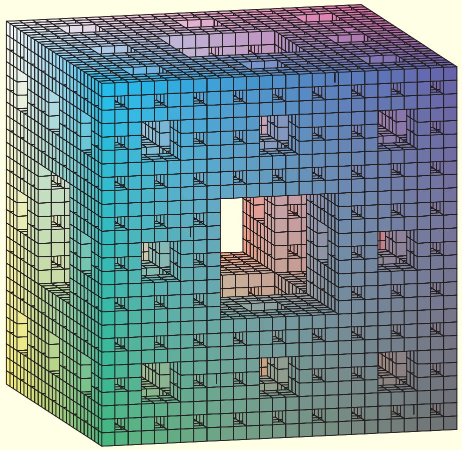
3rd step.



$H \setminus F$ consists of octahedra.



Base covered Sierpiński Gasket.



Menger sponge. No nesting axiom.

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