Positive harmonic functions in nonsmooth domains

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Contents

1. Harmonic functions

Let $D \subset \mathbb{R}^n$ ($n \geq 2$). A C^2 function u is harmonic on D if it enjoys the Laplace equation

$$\Delta u = \Big(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\Big) u = 0.$$

- No differentiability is needed. Distribution sense.
- C^∞, C^ω.
- Beautiful properties.
- Characterization without differentiation (mean value property).
- Extension to networks.

- $B(x, r)$: open ball with center at x, radius r.
- $S(x, r)$: sphere with center at x, radius r.
- $d\sigma$ stands for the surface element.

Theorem 1

u is harmonic in D if and only if the following two conditions hold:

(i) u is continuous in D.

(ii) The mean value principle holds, i.e., for every $x \in D$ and $0 < r <$ **dist**$(x, \partial D)$

$$u(x) = \frac{1}{\sigma(S(x, r))} \int_{S(x,r)} u(y) d\sigma(y).$$

Definition 1 (Super(sub)harmonic functions)

- Replace the mean value equality by the inequality:

$$u(x) \geq \frac{1}{\sigma(S(x,r))} \int_{S(x,r)} u(y) d\sigma(y).$$

  Replace the continuity by the lower semi-continuity: superharmonic functions.
- The inverse inequality and the upper semi-continuity: subharmonic functions.

Remark 1

- If $\Delta u \leq 0$ in the distribution sense, then the l.s.c. version of u is superharmonic.
- If $\Delta u \geq 0$in the distribution sense, then the u.s.c. version of u is subharmonic.
- Harmonic functions are rather rigid by the real analyticity, while super(sub)harmonic functions are flexible and widely applicable.
- The necessity of the Perron solution to the Dirichlet problem.

Various extensions and generalizations:

- General equations (non-linear).
- General space (metric measure spaces).
- De Giorgi-Nash-Moser theory (variational methods).

We concentrate on classical harmonic functions.

- harmonic functions in domains of the Euclidean space.
- Elementary and simple.
- Yet, complicated boundary behavior.
- The more general domains, the more open problems.
- “Approach to the boundary” gives new frontiers.

Positive harmonic functions $\Longrightarrow$

- rule out pathological harmonic functions.
- fruitful arguments.

2. The Martin boundary

The Martin boundary describes the family of all positive harmonic functions in completely general domains.

First, define the Green function: $G(x,y)$. Let ϕ_y be the fundamental harmonic function with pole at y:

$$\phi_y(x) = \begin{cases} \log \dfrac{1}{|x-y|} & (n = 2), \\ |x-y|^{2-n} & (n \geq 3). \end{cases}$$

Definition 2 (Green 1828 [Gre28])

$G(x, y)$ is the Green function for D if it is a function of $x \in D$ and $y \in D$ such that for each fixed $y \in D$,

(i) $G(\cdot, y)$ is harmonic in $D \setminus \{y\}$

(ii) $G(\cdot, y) - \phi_y(x)$ has a harmonic extension to D.

(iii) $G(\cdot, y) = 0$ on ∂D.

Suppose D is sufficiently smooth. Then every harmonic function on D continuous up to the boundary has a Poisson integral representation.

Definition 3 (Poisson 1823 [Poi27])

Let D be smooth and let G be the Green function for D. For $x \in D$ and $y \in \partial D$ put $P(x,y) = -\frac{1}{e_n}\frac{\partial}{\partial n_y}G(x,y)$. This is the Poisson kernel for D. Here, $e_2 = 2\pi$ and if $n \geq 3$, then $e_n = (n-2)\sigma_n$ and σ_n is the area of a unit sphere. Moreover, for a boundary function f

$$P[f](x) = \int_{\partial D} P(x,y) f(y) d\sigma(y)$$

is called the Poisson integral.

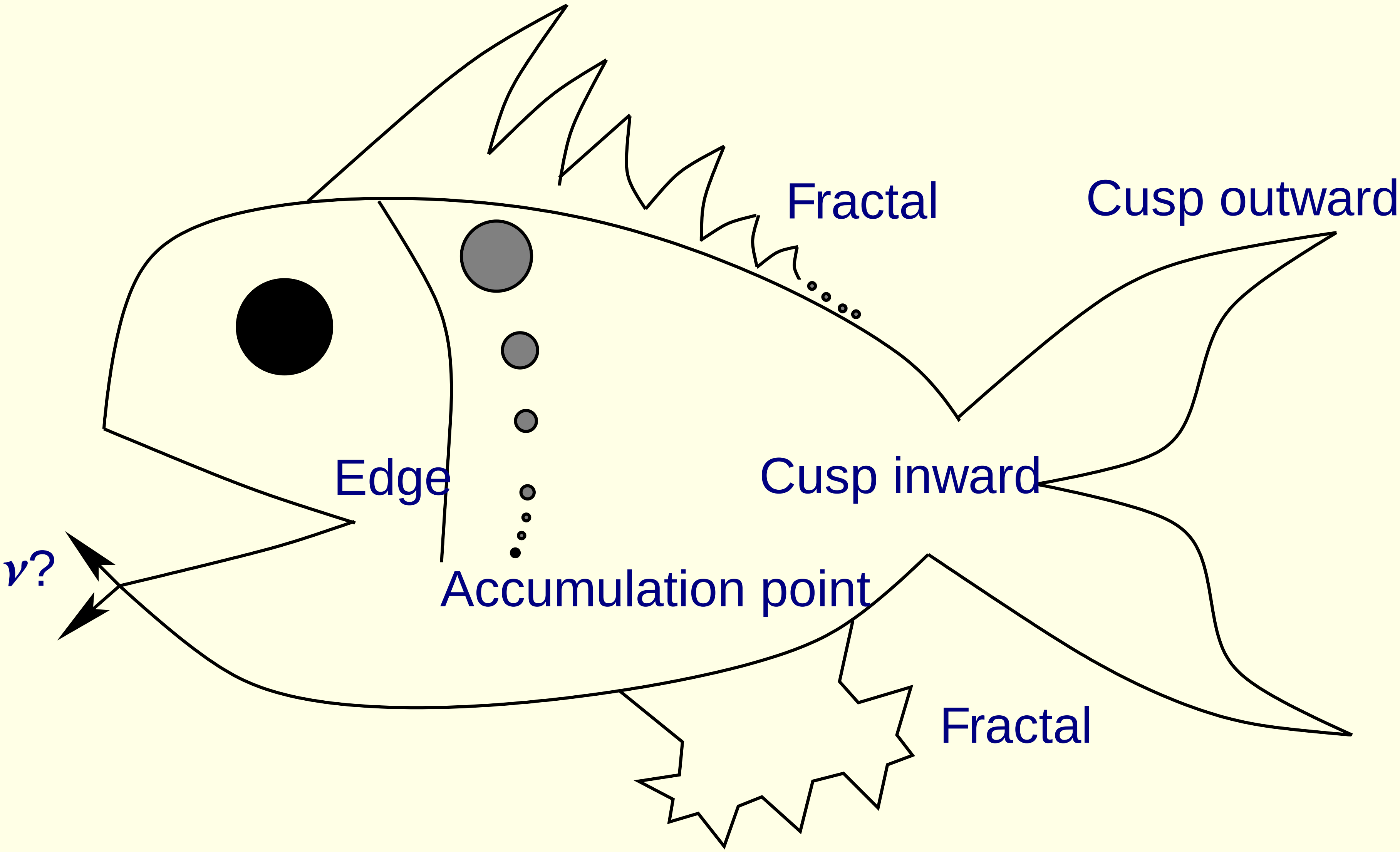
Theorem 2

Let D be smooth and let h be a harmonic function on D continuous up to ∂D. Then

$$h(x) = P[h](x) = \int_{\partial D} P(x,y)h(y)d\sigma(y) \quad (x \in D).$$

- If h is a positive harmonic on D, then there is a unique measure μ_h on ∂D such that $h = P[\mu_h]$.
- No explicit formula for the Green function. The Poisson integral representation and its estimates available.
- Every positive harmonic function has nontangential boundary values for a.e. boundary point (Fatou 1906 [Fat06]).

Nonetheless, the situation is complicated for nonsmooth domains.



Non-existent Poisson integral.

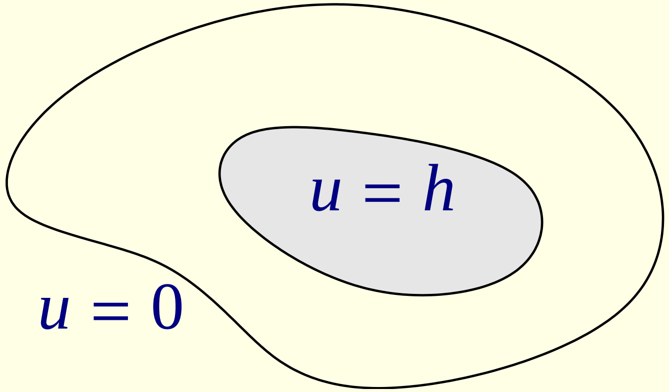
- Lipschitz domains have corners at which no normals are defined.
- Fractal domains have the boundaries of dimension greater than $n - 1$. Surface measure cannot be defined.

Martin (1941) [Mar41] introduced the Martin boundary for arbitrary domains with Green function. Every positive harmonic function is represented as the integral over the Martin boundary.

- Take an exhaustion D_j of D. $D_j \uparrow D$.
- Consider the balayage $u = \hat{R}_h^{D_j}$ (regularized reduced function).

Namely, $h = h$ on $\overline{D_j}$ and, on $D \setminus \overline{D_j}$, u is the solution to the Dirichlet problem:

$$\begin{aligned} \Delta u(x) &= 0 && (x \in D \setminus \overline{D_j}), \\ u(x) &= h(x) && (x \in \partial D_j), \\ u(x) &= 0 && (x \in \partial D) \end{aligned}$$



- u is superharmonic in the whole D.
- u is harmonic in $D \setminus \partial D_j$
- $u = 0$ on ∂D.
- There is a measure ν_j on ∂D_j such that $u = G\nu_j = \int_D G(\cdot, y) d\nu_j(y)$, a Green potential on D.

Fix $x_0 \in D$. Since $\hat{R}_h^{D_j} = h$ on D_j, we have the integral representation:

$$h(x) = \int G(x,y) d\nu_j(y) = \int \frac{G(x,y)}{G(x_0,y)} G(x_0,y) d\nu_j(y) \quad x \in D_j.$$

- Let μ_j be the measure defined by $d\mu_j(y) = G(x_0, y)d\nu_j(y)$.
- Letting $x = x_0$, we observe that $\|\mu_j\| = h(x_0)$ (constant).
- A subsequence of $d\mu_j$ w^*-converges. The limit μ_h satisfies

the integral representation: $h(x) = \int \frac{G(x,y)}{G(x_0,y)} d\mu_h(y) \quad (x \in D)$

- Unfortunately, this is not sufficient.
- Does the ratio $G(x,\cdot)/G(x_0,\cdot)$ extends continuously to the boundary?
- Consider the opposite direction: the smallest ideal boundary Δ and the extension$K(x,\cdot)$ to which $G(x,\cdot)/G(x_0,\cdot)$ continuously extends.

Integral representation: $h(x) = \int_\Delta K(x,y) d\mu_h(y).$

- Δ is called the Martin boundary, $K(x, y)$ the Martin kernel.
- $K(\cdot, y)$ is a positive harmonic function in D with $K(x_0, y) = 1$.
- A Martin kernel is regarded as an ideal boundary point.
- Unfortunately, the measure μ_h is not unique.
- Consider minimal Martin boundary points.
- A positive harmonic function u is minimal if every positive harmonic function less than u is a constant multiple of u.
- If the Martin kernel $K(\cdot, y)$ is minimal, then y is a minimal Martin boundary point. The set of all minimal Martin boundary points is referred to as the minimal Martin boundary (Δ_1).
- The rest $\Delta \setminus \Delta_1$ is called the non-minimal Martin boundary (Δ_0).
- Poisson integral representation is now generalized as follows:

Martin integral representation (1941) [Mar41]

For every positive harmonic function h in D, there exists a unique measure μ_h on Δ_1 such that

$$h(x) = \int_{\Delta_1} K(x,y)d\mu_h(y).$$

- What is the Martin boundary of a specific domain?
- How do we identify it?

3. Boundary Harnack principle

Suppose D is smooth. Then

- $G(x_0, y) \approx \delta_D(y) = \textbf{dist}(y, \partial D)$.
- Martin kernel $K(x, y) = P(x, y) \times$ positive function.
- Martin representation = Poisson integral representation.
- $\Delta = \Delta_1 = \partial D$.

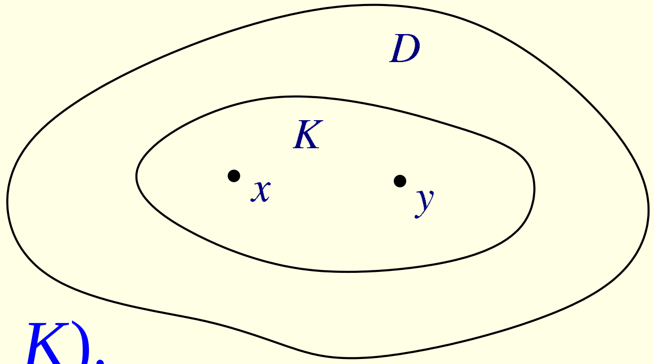
Suppose D is general.

- $\Delta = \Delta_1 = \partial D$? ${}^{\exists}\Delta_0$?
- Several families of domains between "smooth" and "general".
- How do you identify the Martin boundary?
- The more general D, the coarser information.

Recall the Harnack principle.

Theorem 3 (Harnack principle (1886) [Har86])

Let K be a compact subset of D. Then $^{\exists}C > 1$ depending only on K and D such that $^{\forall}u, v >$ harmonic in D

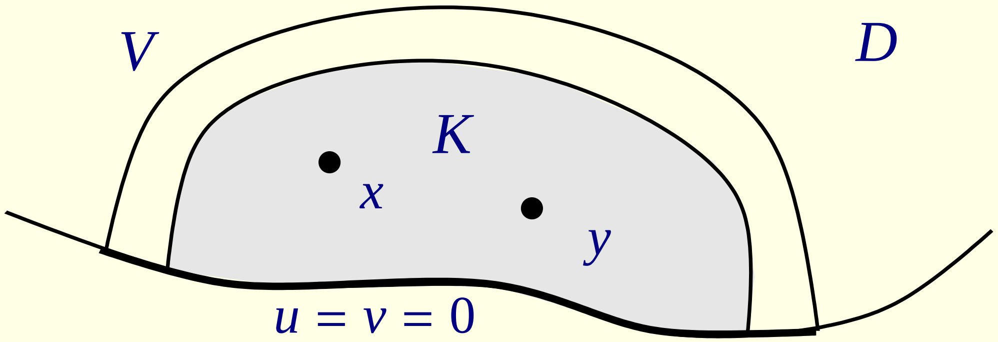


$$\frac{u(x)/v(x)}{u(y)/v(y)} \le C \quad (x, y \in K).$$

Remove the compactness of K and suppose K touches ∂D. What happens? The hypothesis $u, v >$ harmonic in D is insufficient, the boundary values of u, v affect the comparison. So, we assume that $u = v = 0$ on $K \cap \partial D$.

Definition 4 (Boundary Harnack principle)

Let $K \cap \partial D \neq \emptyset$ and let $K \subset V$ open. Then $^{\exists}C > 1$ depending only on K, V and D such that $^{\forall}u, v >$ harmonic in D with $u = v = 0$ on $\partial D \cap V$



$$\frac{u(x)/v(x)}{u(y)/v(y)} \leq C \quad (x, y \in D \cap K).$$

Remark 2

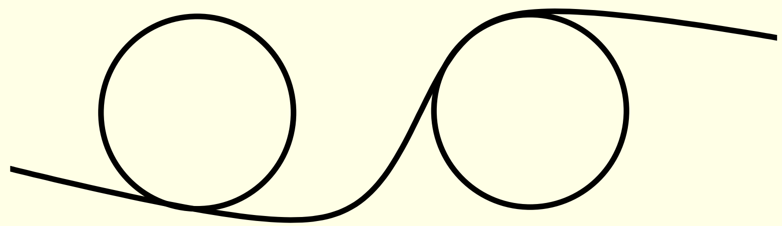
The geometry of D is crucial for the BHP.

- What does $u = 0$ on ∂D mean?
- D may be irregular for the Dirichlet problem.
- We cannot assume $u = 0$ on ∂D continuously.
- Milder assumption: u is bounded and $u = 0$ q.e. (quasi everywhere), i.e. the equality holds outside a polar set.
- A polar set is a small set on which $u = \infty$, ${}^{\exists}u$ superharmonic.
- The Hausdorff dimension of a polar set is $n - 2$ ([AG01, Theorem 5.9.6]). The $n-1$ dimensional Hausdorff and the Lebesgue measure are equal to 0.

4. From smooth domain to Lipschitz and NTA domains

A smooth domain usually means $C^{2,\alpha}$-domain ([GT01]).

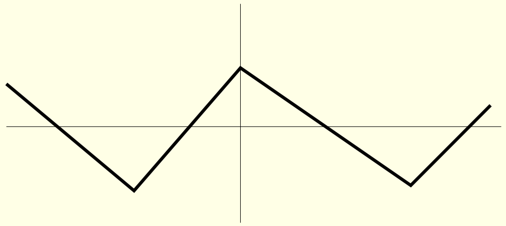
- Interior ball condition. $^\exists r_0$ such that every boundary point ξ has an interior ball touching ∂D at ξ.
- Exterior ball condition.
- Interior and Exterior ball conditions $\implies$ Ball condition.



It turns out $C^{1,1}$-domain [AKSZ07]

- Ball condition $\implies$ If $u > 0$ in D and $u = 0$ on a portion E of ∂D, then $u(x) \approx \delta_D(x)$ near E.
- The BHP holds (p-)harmonic functions.
- $C^{1,\alpha}$-domain (Widman (1967) [Wid67])

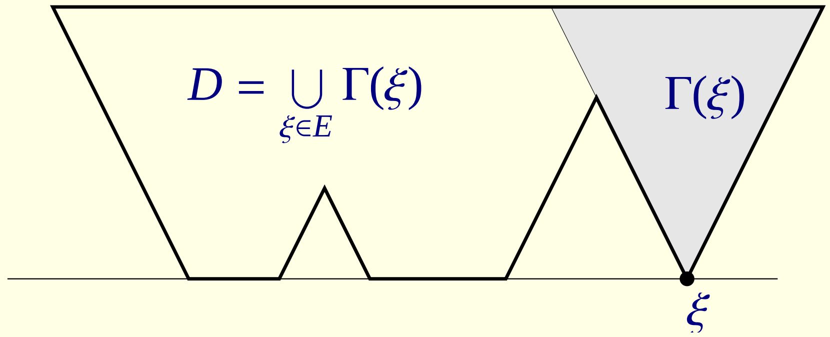
4.1. Lipschitz domain. ∂D is given locally as the graph of a Lipschitz continuous function.



Lipschitz domains are necessary even if you study smooth domains. Illustrate by the local Fatou theorem.

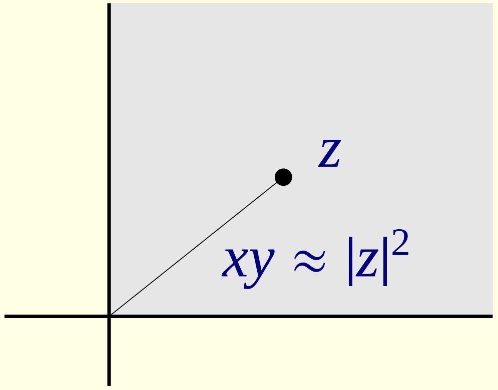
Theorem 4 (Local Fatou theorem (Carleson (1962) [Car62]))

Let h be harmonic in $\mathbb{R}^n_+$ and suppose h is nontangentially bounded on $E \subset \partial\mathbb{R}^n_+$. Then h has nontangential boundary values for a.e. $\xi \in E$.



- h is nontangentially bounded if ${}^{\forall}\xi \in E\ {}^{\exists}\Gamma(\xi)$ of vertex at ξ on which h is bounded.
- Nontangential cones $\Gamma(\xi)$ have the same height and aperture.
- $h > 0$ on $\Gamma(\xi)$ by adding a constant.
- The union of nontangential cones a Lipschitz domain.
- Fatou's theorem reduces to the existence of nontangential boundary values of positive harmonic functions in a Lipschitz domain.

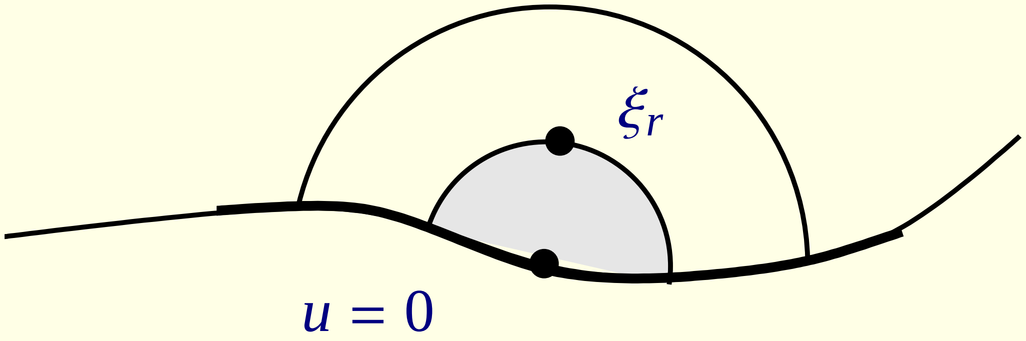
Note $u(x) \not\approx \delta_D(x)$ for a Lipschitz domain.
Identify $\mathbb{R}^2$ and $\mathbb{C}$. Then the positive harmonic function $h(z) = xy$ in the first quadrant decays at the speed of $|z|^2$.



Carleson proved the local Fatou theorem by the Carleson estimate inextricably related to the BHP.

Definition 5 (Carleson estimate [Car62])

Let D be a Lipschitz domain. Suppose $\xi \in \partial D$ and $r > 0$ is small. Let $\xi_r \in S(\xi, r) \cap D$ be a nontangential point, i.e. $\delta_D(\xi_r) \approx r$. If $u > 0$ is harmonic in D and $u = 0$ on $\partial D \cap B(\xi, Cr)$, then

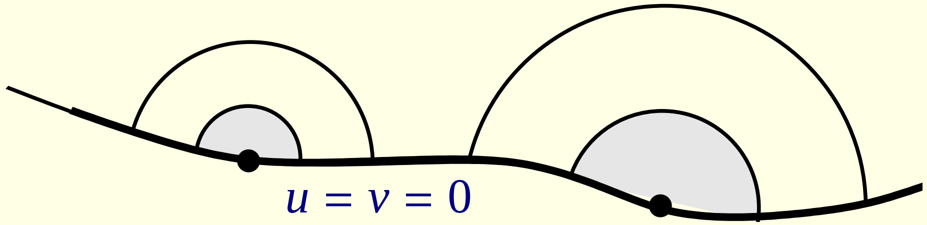


$$u(x) \leq Cu(\xi_r) \quad (x \in D \cap B(\xi, r)).$$

- Interior and exterior cone condition. At each $\xi \in \partial D$ there are interior and exterior cones of vertex at ξ with fixed height and aperture.
- Hunt-Wheeden [HW68], [HW70] applied Carleson's method to a Lipschitz domain and proved $\partial D = \Delta = \Delta_1$.
- Kemper [Kem72] formulated the BHP for a Lipschitz domain, though his proof had a gap.
- The BHP for a Lipschitz domain was established by Ancona [Anc78], Dahlberg [Dah77], Wu [Wu78], independently.
- There are two types of BHP. Local BHP or uniform BHP is important.

Definition 6 (Uniform BHP)

Let $\xi \in \partial D$ and let $r > 0$ be small. If u, v are positive and harmonic in $D \cap B(\xi, Cr)$ and $u = v = 0$ on $\partial D \cap B(\xi, Cr)$, then



$$\frac{u(x)/v(x)}{u(y)/v(y)} \le C \quad (x, y \in D \cap B(\xi, r)).$$

Here $C > 1$ is independent of ξ, r, u, v.

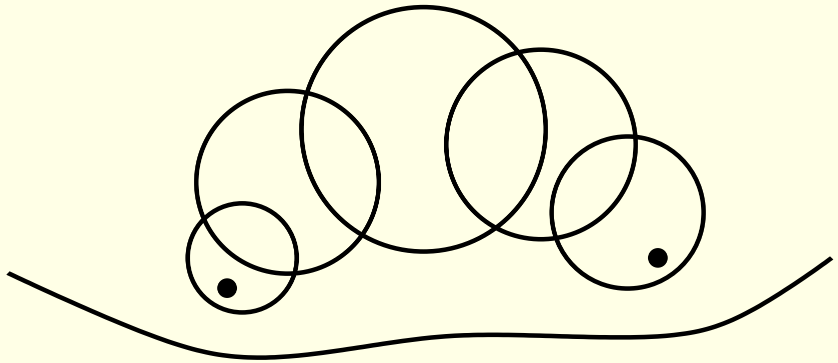
Theorem 5

The uniform BHP holds for a Lipschitz domain.

4.2. NTA domain. Harnack chain of length N connecting x and y is a sequence $\{B(x_j, \frac{1}{2}\delta_D(x_j))\}_{j=1}^N$ of consecutive balls in D such that

- $B(x_j, \frac{1}{2}\delta_D(x_j)) \cap B(x_{j+1}, \frac{1}{2}\delta_D(x_{j+1})) \neq \emptyset$,
- $x \in B(x_1, \frac{1}{2}\delta_D(x_1))$, $y \in B(x_N, \frac{1}{2}\delta_D(x_N))$.

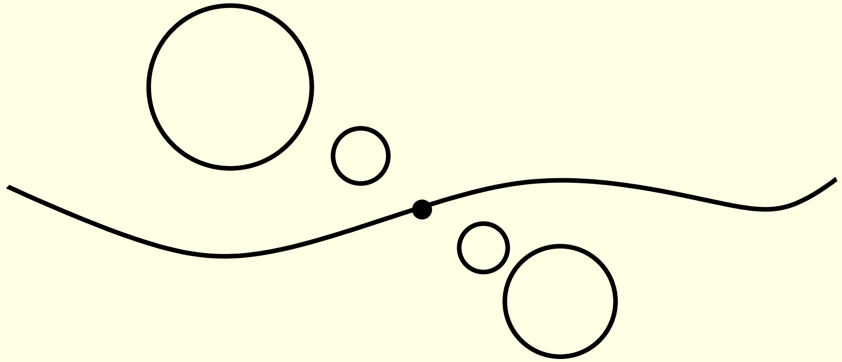
$^\exists C > 1$ depending only on n such that if $h > 0$ is harmonic in D and if x and y is connected by a Harnack chain of length N, then $h(x)/h(y) \le C^N$.

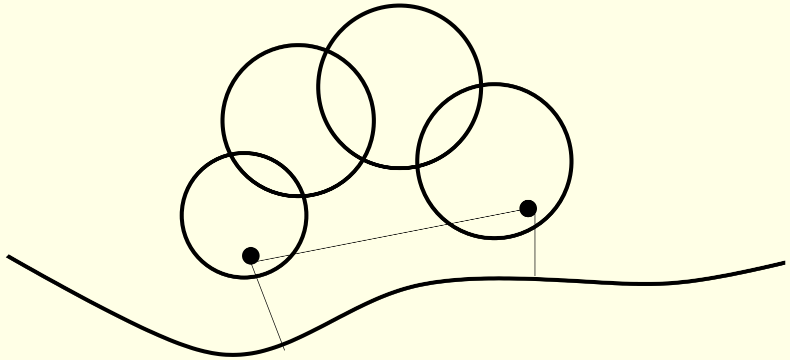


Jerison-Kenig [JK82] generalized Lipschitz domain to NTA domain (Non-Tangentially Accessible domain).

$^\exists C > 1$ and $^\exists r_0 > 0$ with the following conditions:

- Corkscrew condition.
  $^\forall \xi \in \partial D$, $0 < {}^\forall r < r_0$, $D \cap B(\xi, r)$ contains a ball of radius r/C.
- Exterior corkscrew condition. $^\forall \xi \in \partial D$, $0 < {}^\forall r < r_0$, $B(\xi, r) \setminus D$ contains a ball of radius r/C.
- Harnack chain condition.
  If $x, y \in D$ and $|x - y| \approx \delta_D(x) \approx \delta_D(y)$, then x and y is connected by a Harnack chain of bounded length.





- Carleson's method is applicable to NTA domains.
- Uniform BHP holds.
- $\partial D = \Delta = \Delta_1$.
- Harmonic measure is doubling.
- Local Fatou theorem holds.
- Yet, NTA domains much nastier than Lipschitz domains.
- ∂D has Hausdorff dimension $> n - 1$.
- Harmonic measure need not absolutely continuous to the surface measure.
- Harmonic analysis on NTA domains are developed, based on harmonic measure.

5. Various nonsmooth domains

5.1. Hölder domain. α-Hölder domain: ∂D is given locally as the graph of an α-Hölder function $(0 < \alpha \le 1)$. If α is not important, we call it a Hölder domain.

If $\alpha = 1$, then a 1-Hölder domain is a Lipschitz domain. If $0 < \alpha < 1$, then an α-Hölder domain need not be regular for the Dirichlet problem in the higher dimensional case.

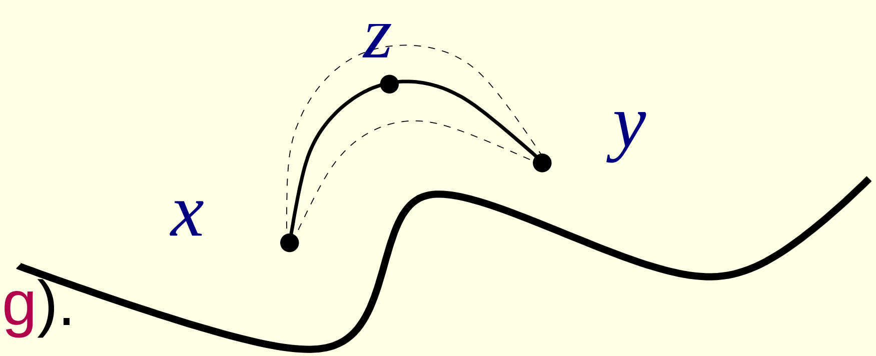
Surprisingly, Bass-Burdzy [BB91] proved a BHP for a Hölder domain.

- Probabilistic.
- No uniform barrier. Interior conditions are crucial.
- Their BHP is not strong enough to identify the Martin boundary.

5.2. Uniform domain. BMO-extension domain, Sobolev-extension domain, quasiconformal mappings. ([GO79], [Jon80], [Jon81], [GM85], [Väi88]).

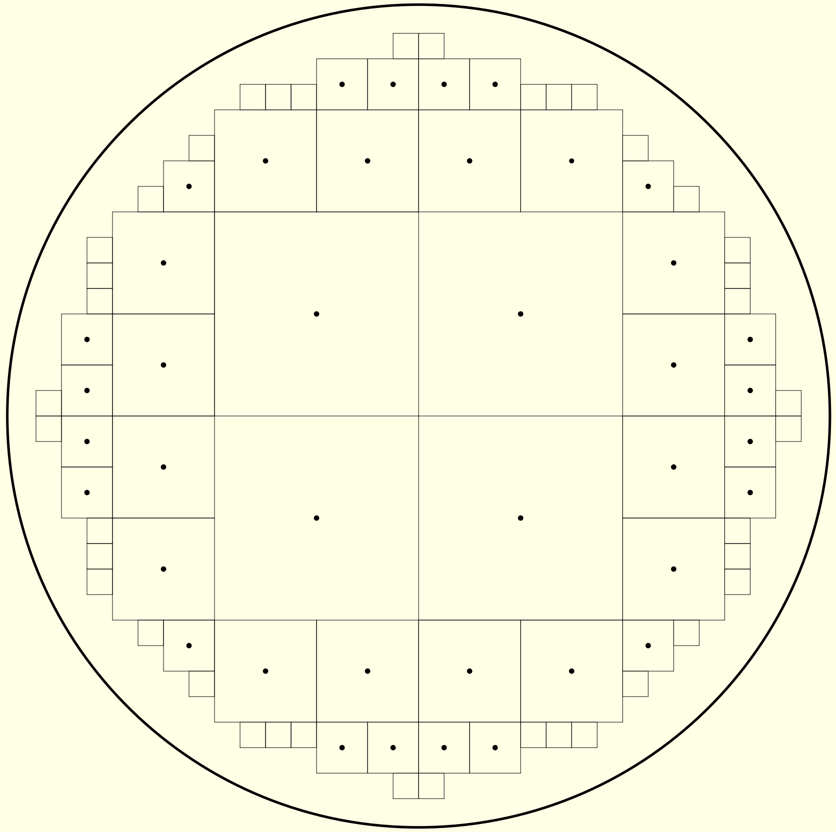
Definition 7 (Uniform domain)

D is a uniform domain if $^\forall x, {}^\forall y \in D$ can be connected by a curve $\gamma \subset D$ such that



- $\ell(\gamma) \le C|x-y|$ (Bounded Turning).
- $\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \le C\delta_D(z) \quad (z \in \gamma)$ (Cigar).

An example of uniform domains is given by an open ball minus the centers of Whitney cubes.

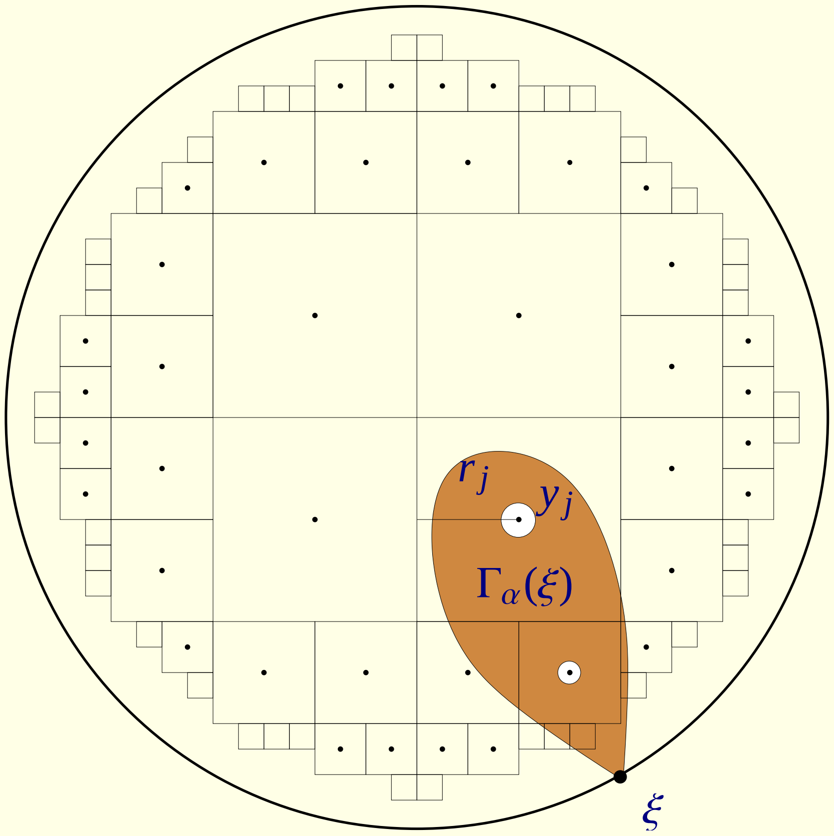


Uniform domains are nice domains:

- (i) Cork screw + (iii) Harnack chain of NTA $\Longleftrightarrow$ Uniform.
- The unifrom BHP holds.
- $\Delta = \Delta_1 = \partial D$.
- The ratio u/v of two positive harmonic functions u, v vanishing on $E \subset \partial D$ is Hölder continuous up to E, although u and v need not continuously vanish on E. Unifrom domains may be irregular.
- Under the CDC given below, the uniform BHP characterizes uniform domains.

Theorem 6 (No Local Fatou Theorem)

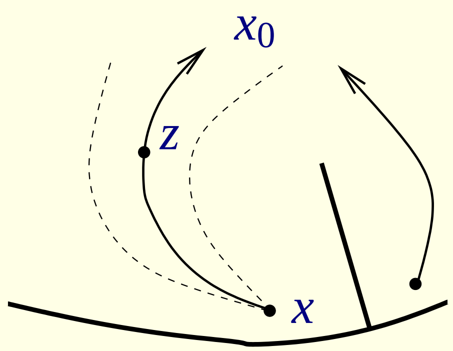
$^{\exists}D$ uniform, $^{\exists}E \subset \partial D$, $^{\exists}u$ harmonic on D nt-bdd on $\partial D \setminus E$, and yet no nt-limits on $\partial D \setminus E$.



5.3. John domain. The carrot condition is close, but no cigar. Fix $y = x_0$ and move x only. Then we have the carrot condition, which characterizes John domains.

Definition 8 (John domain)

D is a John domain with John constant c_J if $\forall x \in D$ can be connected to x_0 by a curve $\gamma \subset D$ such that $\delta_D(z) \geq c_J \ell(\gamma(x,z)) \quad (z \in \gamma)$. If x_0 is replaced by a fixed compact set, then D is called a generalized John domain.

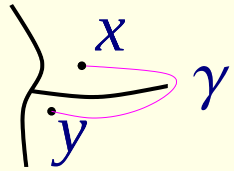


John domains are nice domains:

- $0 < c_J \leq 1$. The closer c_J to 1, the smoother D.
- Remove the bounded turning condition from the definition of a uniform domain. Then we have a John domain.
- John domains satisfy the weak boundary Harnack principle. The number of minimal Martin boundary points over a topological boundary point is estimated.
- Under the CDC, John domains are characterized by estimates of harmonic measure.

5.4. Inner uniform domain. Define the inner length distance $\lambda_D(x,y)$ and internal metric $\rho_D(x,y)$ by

$$\lambda_D(x,y) = \mathbf{inf}\{\ell(\gamma)\},$$
$$\rho_D(x,y) = \mathbf{inf}\{\mathbf{diam}(\gamma)\}.$$



- $|x-y| \le \rho_D(x,y) \le \lambda_D(x,y)$.
- $\rho_D(x,y) \approx \lambda_D(x,y)$ for John domains (Väisälä (1998) [Väi98]).

Definition 9 (Inner Uniform)

$^\forall x, {}^\forall y \in D$ can be connected by $\gamma \subset D$ with $\ell(\gamma) \le C\rho_D(x,y)$ (Weak BT) and Cigar. Balogh-Volberg [BV96a], [BV96b], Bonk-Heinonen-Koskela [BHK01].

Proposition 1

- The completion D^* with respect to ρ_D ([ALM03, Proposition 2.1]): $\partial^* D$ the ideal boundary.
- For $\xi^* \in \partial^* D$, $^\exists \xi \in \partial D$ and $^\exists\{x_j\} \subset D$ converging to ξ with respect to the Euclidean metric as well as converging to ξ^* with respect to ρ_D.
- ξ^* lies over ξ. Define π from D^* to $\overline{D}$ by $\pi(\xi^*) = \xi$ for $\xi^* \in \partial^* D$ and $\pi|_D = \mathbf{id}\,|_D$.
- Let $B_\rho(\xi, R)$ be the connected component of $B(\xi, R) \cap D$ from which ξ^* is accessible.
- $B_\rho(\xi, R)$ plays a role of a ball with center at ξ^* in the completion D^* ([ALM03, Lemma 2.2]).

Inner uniform domains are nice domains:

- Inner uniform domains are John domains.
- The unifrom BHP wrt ρ_D holds.
- The Martin boundary is identified: $\Delta = \Delta_1 = \partial^* D$.
- $B_\rho(\xi^*, r)$: the component of $B(\xi, r) \cap D$ accessible to ξ^* is a neighborhood of ξ^*.

5.5. Semi-uniform domain. In the definition of uniform domains, restrict one point in ∂D. Then we obtain a semi-uniform domain. A Denjoy domain is a typical exapmple.

Definition 10 (Semi-uniform domain)

D is a semi-uniform domain if $^{\forall}x \in D$ and $^{\forall}y \in \partial D$ can be connected by a cigar curve γ such that $\gamma \setminus \{y\} \subset D$, $\ell(\gamma) \leq C|x - y|$.

Definition 11 (Inner semi-uniform domain)

D is a semi-uniform domain if $^{\forall}x \in D$ and $^{\forall}y \in \partial D$ can be connected by a cigar curve γ such that $\gamma \setminus \{y\} \subset D$, $\ell(\gamma) \leq C\rho_D(x, y)$.

Semi-uniform domains are nice domains:

- Under the CDC, D is semi-uniform if and only if the harmonic measure is doubling in a certain sense.
- Inner semi-uniform domains correspond to inner-doubling property of harmonic measure.

5.6. Quasihyperbolic boundary condition (QHB). Let D be a general domain s.t. $\partial D \neq \emptyset$.

Definition 12

The quasihyperbolic metric $k_D(x,y)$ is defined by

$$k_D(x,y) = \inf_{\widetilde{xy}} \int_{\widetilde{xy}} \frac{ds}{\delta_D(z(s))},$$

where the infimum is taken over all curves $\widetilde{xy}$ connecting x and y in D.

$k_D(x,y)$ is comparable to the shortest length of Harnack chain connecting x and y in D.

Theorem 7

Let D be a general domain s.t. $\partial D \neq \emptyset$. Then ${}^{\forall}x, {}^{\forall}y \in D$

$$k_D(x,y) \geq \mathbf{log}\left(\frac{|x-y|}{\mathbf{min}\{\delta_D(x), \delta_D(y)\}} + 1\right).$$

Proof. Let γ be a curve connecting x and y in D of length L. Then $|x-y| \leq L$. Moreover, $\delta_D(z(s)) \leq \delta_D(x) + s$ for $0 \leq s \leq L$ with arc length parameter s. Hence

$$\int_\gamma \frac{ds}{\delta_D(z(s))} \geq \int_0^L \frac{ds}{\delta_D(x)+s} = \Big[\mathbf{log}(\delta_D(x)+s)\Big]_0^L \geq \mathbf{log}\left(1 + \frac{|x-y|}{\delta_D(x)}\right).$$

Replacing x and y proves the theorem. □

The reverse inequality (up to a multiplicative constant) characterizes uniform domains.

Theorem 8 ([GO79])

D is a uniform domain if and only if

$$k_D(x,y) \le C\,\mathbf{log}\left(\frac{|x-y|}{\mathbf{min}\{\delta_D(x), \delta_D(y)\}} + 1\right) + C'$$

for ${}^\forall x, {}^\forall y \in D$

Theorem 9

John domains D satisfy quasihyperbolic boundary condition:

$$k_D(x, x_0) \le C\,\mathbf{log}\,\frac{\delta_D(x_0)}{\delta_D(x)} + C' \quad (x \in D).$$

Proof. Let γ be a John curve connecting x and x_0 in D of length L. Then

$$\int_\gamma \frac{ds}{\delta_D(z)} = \int_0^{\delta_D(x)/4} + \int_{\delta_D(x)/4}^{L} \frac{ds}{\delta_D(z)}$$

$$\le \frac{\delta_D(x)/4}{3\delta_D(x)/4} + \int_{\delta_D(x)/4}^{L} \frac{ds}{c_J s} \le \frac{1}{c_J}\,\mathbf{log}\,\frac{\delta_D(x_0)}{\delta_D(x)} + C.$$

□

Remark 3

The converse of the above theorem does not hold, i.e. $^{\exists}D$ satisfying the QHB condition and yet D is not John. Smith-Stegenga [SS91] called a domain satisfying the QHB condition a Hölder domain. This is different from Hölder domain treated by Bass-Burdzy.

Domains with the QHB are nice domains:

- $k_D(x, y)$ is exponentially integrable.
- The global (non-uniform) BHP holds.

5.7. Summary of interior conditions. Interior conditions are summarized as follows:

$$C^{2,\alpha} \subsetneqq C^{1,1} = \text{Ball condition} \subsetneqq \text{Lipschitz} \subsetneqq \text{NTA} \subsetneqq \text{Uniform} \subsetneqq \text{John} \subsetneqq \text{QHB}.$$

From Uniform to John we have finer classifications:

Inner uniform

$\subsetneqq$ $\subsetneqq$

Uniform Inner semi-uniform $\subsetneqq$ John.

$\subsetneqq$ $\subsetneqq$

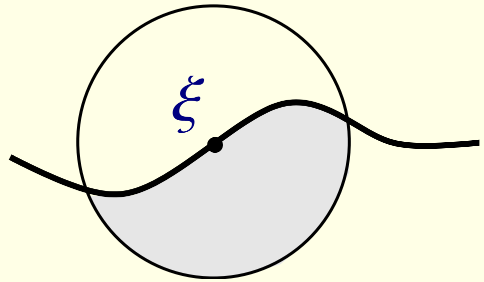
Semi-uniform

5.8. Capacity density condition. By $\textbf{Cap}_U(E)$ we denote the Green capacity of E relative to an open set U with Green function.

Definition 13 (Capacity density condition)

We say D satisfies the capacity density condition, if $\exists\lambda > 0$ and $\exists r_0 > 0$ such that for $\xi \in \partial D$ and $0 < r < r_0$

$$\frac{\textbf{Cap}_{B(\xi,2r)}(B(\xi,r)\setminus D)}{\textbf{Cap}_{B(\xi,2r)}(B(\xi,r))} \geq \lambda.$$



Remark 4

- Replacing **Cap** by the logarithmic capacity ($n = 2$) and the Newtonian capacity ($n \geq 3$) gives
$$\mathbf{Cap}(B(\xi, r) \setminus D) \geq \begin{cases} Cr & \text{if } n = 2, \\ Cr^{n-2} & \text{if } n \geq 3. \end{cases}$$
- Volume density condition: $\dfrac{|B(\xi, r) \setminus D|}{|B(\xi, r)|} \geq C$ is sufficient for the CDC.
- The exterior cone condition and the exterior corkscrew condition are sufficient as well.
- Smooth domains and Lipschitz domains satisfy the CDC.

Remark 5

- A positive superharmonic function s in D is called strong barrier if $\exists\varepsilon > 0$ such that $\Delta s(x) + \dfrac{\varepsilon}{\delta_D(x)^2} s(x) \le 0$.
- $\exists$ strong barrier $\iff$ the Hardy inequality:

$$\int_D \left| \frac{\psi(x)}{\delta_D(x)} \right|^2 dx \le C \int_D |\nabla \psi(x)|^2 dx \quad \text{for } \psi \in W_0^{1,2}(D).$$

- The CDC yield a strong barrier (Ancona [Anc86]).
- Ancona [Anc87] applied strong barrier extensively. Particularly, he identified the Martin boundary of a Cartan-Hadmard manifold of negative curvature.
- The Hardy inequality $\implies$ the quasiadditivity of capacity (HA [Aik91], [Aik93]).

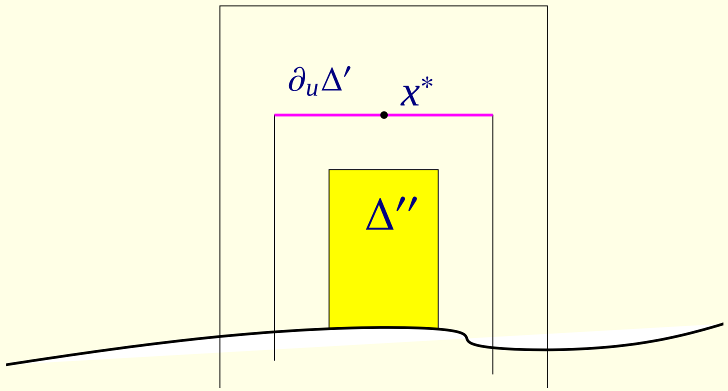
5.9. Summary of exterior conditions. Exterior conditions are summarized as follows:

$$\text{Exterior Ball} \subsetneqq \text{Exterior cone} \subsetneqq \text{Exterior corkscrew} \subsetneqq \text{Volume density} \subsetneqq \text{CDC} \subsetneqq \text{Strong barrier}$$

6. Uniform BHP and the box argument — heart of the proof —

Carleson's proof and successors

Lipschitz domain $\Longrightarrow$ exterior cone $\Longrightarrow$ uniform barrier.



- Carleson estimate: If $u \in H^+(\Delta)$ and $u = 0$ on $\Delta \cap \partial D$, then

$$u \le Cu(x^*) \quad \text{on } \Delta'.$$

- Box estimate: $\partial_u \Delta'$ is the main part:

$$\omega(\partial\Delta' \cap D, \Delta') \le C\omega(\partial_u\Delta', \Delta') \quad \text{on } \Delta''.$$

Suppose $u, v \in H^+(\Delta)$ and $u = v = 0$ on $\Delta \cap \partial D$. Then

- $u \le Cu(x^*)$ on Δ (Carleson estimate).
- $u \le Cu(x^*)\omega(\partial\Delta' \cap D, \Delta')$ on Δ' (maximum principle).
- $u \le Cu(x^*)\omega(\partial_u\Delta', \Delta')$ on Δ'' (box estimate).
- $v \approx v(x^*)$ on $\partial_u\Delta'$ (Harnack inequality).
- $v \ge Cv(x^*)\omega(\partial_u\Delta', \Delta')$ on Δ' (maximum principle).
- Hence $\dfrac{u}{u(x^*)} \le C\omega(\partial_u\Delta', \Delta') \le C\dfrac{v}{v(x^*)} \quad$ on Δ''.

- Uniform domains and John domains satisfy only interior conditions. No (uniform) barrier nor regularity wrt the Dirichlet problem.
- Bass-Burdzy [BB91] found the box argument:
  - No exterior conditions are needed.
  - Interior conditions are sufficient.
  - Big difference from NTA domains.
- Capacitary width explains why no exterior condition is needed.

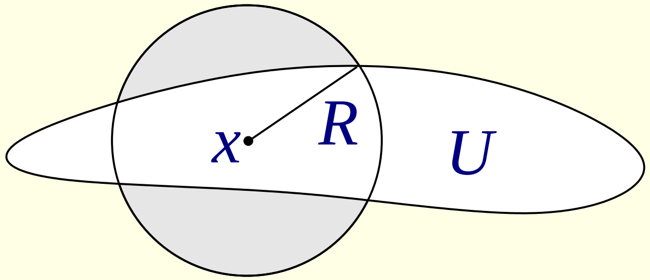
Define the Green capacity by

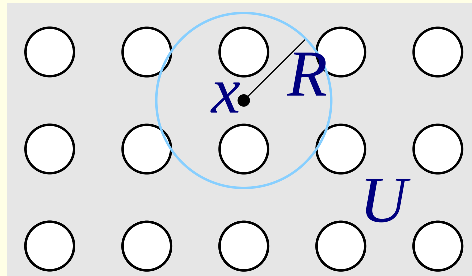
$$\mathbf{Cap}_U(E) = \mathbf{sup}\{\mu(E) : G_U\mu \le 1 \text{ on } U,\ \mu \text{ is on } E\}.$$

Definition 14 (Capacitary width)

Let $0 < \eta < 1$. Define the capacitary width for an open set U by

$$w_\eta(U) = \inf\left\{R > 0 : \frac{\mathbf{Cap}_{B(x,2R)}(B(x,R) \setminus U)}{\mathbf{Cap}_{B(x,2R)}(B(x,R))} \geq \eta \quad (x \in U)\right\}.$$





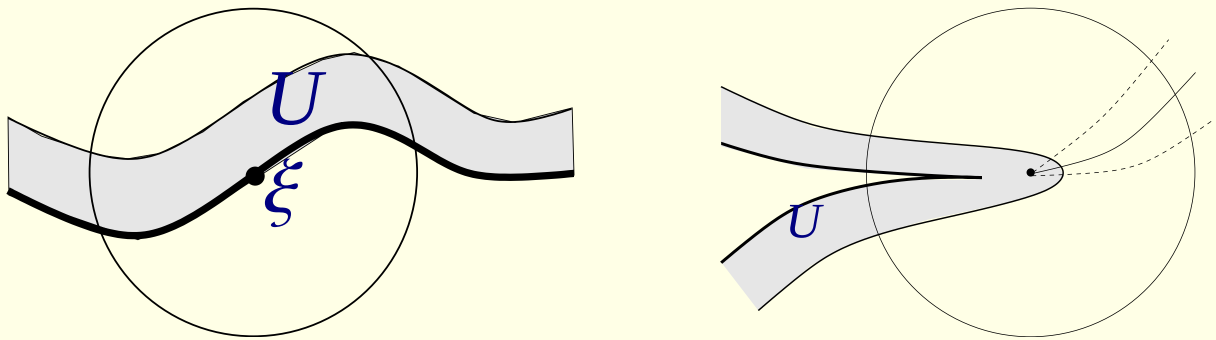
Remark 6

$0 < \eta < 1$ has no significance. Different η' yields comparable capacitary width.

Lemma 1

Let D be a John domain or let D satisfy the CDC. Then for small $r > 0$

$$w_\eta(\{x \in D : \delta_D(x) < r\}) \leq Cr.$$



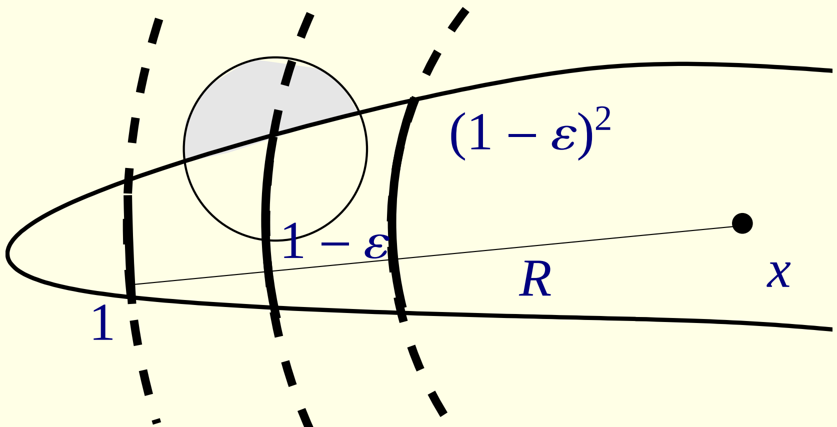
Proof. Let $U = \{x \in D : \delta_D(x) < r\}$. Suppose D is a John domain. Then $^\forall x \in U$ can be connected to x_0 by a John curve γ. Observe if $z \in \gamma$ and $|x - z| \geq 3r/c_J$, then $\delta_D(z) \geq c_J \ell(\gamma(x, z)) \geq c_J|x - z| \geq 3r$. This means $B(z, r) \cap U = \emptyset$. Hence (1) holds with $C = 3/c_J + 1$.

If D satisfies the CDC, then (1) holds with $C = 2$ by definition. □

Lemma 2

$\exists C_1 > 1$ depending only on n and η such that for an open set $U \neq \emptyset$, $x \in U$ and $R > 0$

$$\omega(x; U \cap S(x,R), U \cap B(x,R)) \le \mathbf{exp}\left(2 - C_1 \frac{R}{w_\eta(U)}\right).$$



- Estimate of the harmonic measure by capacitary width.
- Compare the Green function and the harmonic measure.

  Box argument.

- $\xi \in \partial D$.
- $\xi_R \in D \cap S(\xi, 4R)$, i.e., $\delta_D(\xi_R) \approx R$.
- Let G_R be the Green function for $D \cap B(\xi, CR)$.

Lemma 3

If $C > 10$ is sufficiently large, then

$$\omega(y; D \cap S(\xi, 2R), D \cap B(\xi, 2R)) \le CR^{n-2} G_R(y, \xi_R) \quad (y \in D \cap B(\xi, R)).$$

Proof. Let

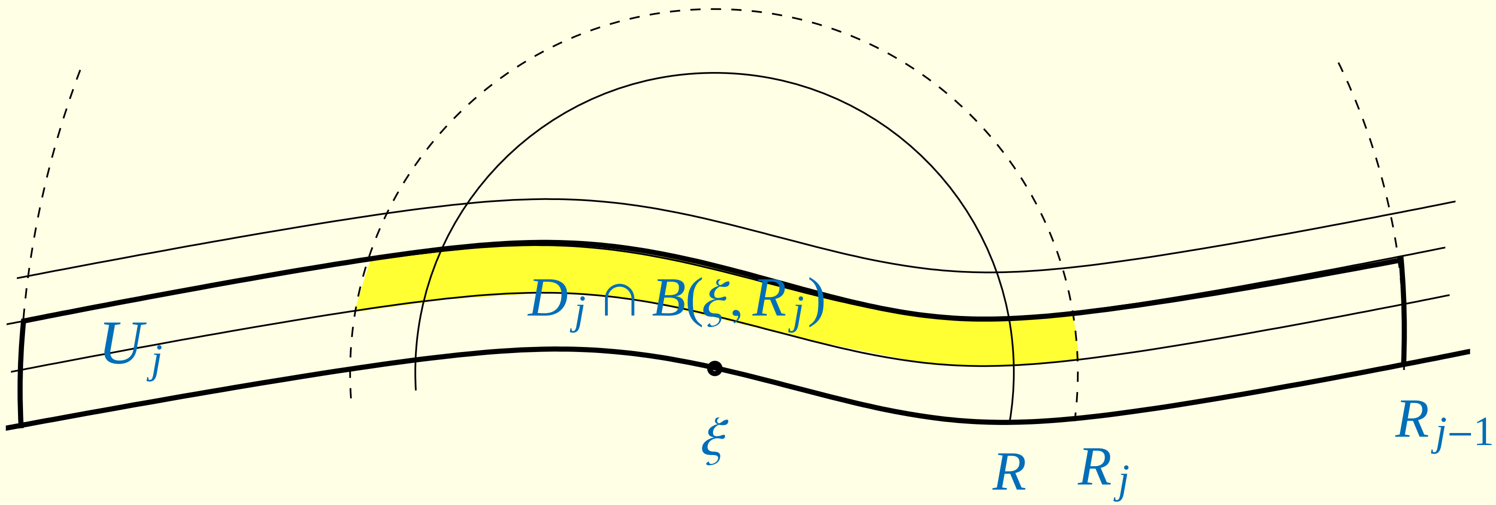
$$D_j = \{x \in D : \mathbf{exp}(-2^{j+1}) \le R^{n-2} G_R(x, \xi_R) < \mathbf{exp}(-2^j)\},$$

$$U_j = \{x \in D : R^{n-2} G_R(x, \xi_R) < \mathbf{exp}(-2^j)\}.$$

Observe U_j is very slim, i.e. $w_\eta(U_j) \le CR\,\mathbf{exp}\left(-\frac{2^j}{\lambda}\right)$. Let $R_j \downarrow R$ slowly; $R_{j-1} - R_j \approx j^{-2}R$. Put $\omega_0 = \omega(D \cap S(\xi, 2R), D \cap B(\xi, 2R))$ and

$$d_j = \sup_{x \in D_j \cap B(\xi, R_j)} \frac{\omega_0(x)}{R^{n-2} G_R(x, \xi_R)}.$$

We want to show $\sup_{j \ge 0} d_j \le C < \infty$.



The maximum principle over $U_j \cap B(\xi, R_{j-1})$ says

$$\omega_0 \leq \omega(\cdot, U_j \cap S(\xi, R_{j-1}), U_j \cap B(\xi, R_{j-1})) + d_{j-1} R^{n-2} G_R(\cdot, \xi_R)$$

on $U_j \cap B(\xi, R_{j-1})$. Hence

$$d_j \leq C \,\mathbf{exp}\left(2^{j+1} - C j^{-2} \,\mathbf{exp}\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1}$$

with $\sum_{j=1}^{\infty} \mathbf{exp}\left(2^{j+1} - C j^{-2} \,\mathbf{exp}\left(\frac{2^j}{\lambda}\right)\right) < \infty.$ □

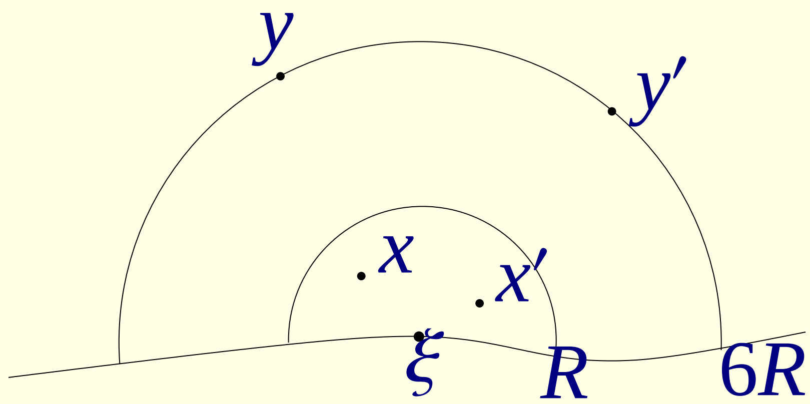
The Green function obviously satisfies

$$G_R(x, y) \leq C R^{2-n} \text{ for } x \in D \cap B(\xi, 2R),\ y \in D \cap B(\xi, 9R) \setminus B(\xi, 3R).$$

This estimate, together with the previous lemma, gives

Lemma 4 (4G inequality)

$$\frac{G_R(x,y)}{G_R(x',y)} \approx \frac{G_R(x,y')}{G_R(x',y')} \qquad (x, x' \in D \cap B(\xi, R),\ y, y' \in D \cap S(\xi, 6R))$$



▶ Contents

Theorem 10

Uniform domains satisfy the uniform BHP, i.e., $^{\exists}C_2$, $^{\exists}C_3 > 1$, $r_0 > 0$ depending only on D:

- $\xi \in \partial D$, $0 < r < r_0$,
- $u, v > 0$ are harmonic on $D \cap B(\xi, C_2 r)$,
- $u = v = 0$ on $\partial D \cap B(\xi, C_2 r)$,

then

$$\frac{u(x)/u(y)}{v(x)/v(y)} \le C_3 \text{ for } x, y \in D \cap B(\xi, r).$$

Moreover, $\Delta = \Delta_1 = \partial D$.

Proof. Let us prove the uniform BHP. We may assume

$$u(x) = \widehat{R}_u^{D\cap S(\xi,6R)}(x) = \int_{D\cap S(\xi,6R)} G_R(x,y)d\mu(y) \quad \text{for } x \in D \cap B(\xi,6R).$$

Let $x, x' \in D \cap B(\xi, R)$ and $y, y' \in D \cap S(\xi, 6R)$. Then the 4G lemma gives

$$G_R(x,y) \approx \frac{G_R(x,y')}{G_R(x',y')} G_R(x',y).$$

Hence

$$u(x) \approx \frac{G_R(x,y')}{G_R(x',y')} \int_{D\cap S(\xi,6R)} G_R(x',y)d\mu(y) = \frac{G_R(x,y')}{G_R(x',y')} u(x').$$

Therefore,

$$\frac{u(x)}{u(x')} \approx \frac{G_R(x,y')}{G_R(x',y')} \quad \text{for } y' \in D \cap S(\xi,6R).$$

Similarly,

$$\frac{v(x)}{v(x')} \approx \frac{G_R(x,y')}{G_R(x',y')}.$$

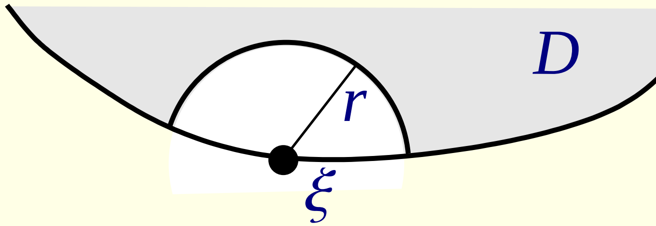
Hence the uniform BHP follows.

Let us prove the second assertion (Kemper [Kem72]).

Let $\xi \in \partial D$.

$\mathcal{H}_\xi$, Kernel function at ξ

- $h > 0$ harmonic in D and $h = 0$ q.e. on ∂D.
- $\forall r > 0$ h is bounded on $D \setminus B(\xi, r)$.
- $h(x_0) = 1$



- The uniform BHP yields $C^{-1} \le \dfrac{u}{v} \le C$ $(u, v \in \mathcal{H}_\xi)$.
- Let $c = \displaystyle\sup_{\substack{u,v\in\mathcal{H}_\xi \\ x\in D}} \frac{u(x)}{v(x)}$. Then $1 \le c < \infty$.
- Let us prove $c = 1$ by contradiction. Suppose $c > 1$.
- $\forall u,\ v \in \mathcal{H}_\xi\ v_1 = (cv - u)/(c - 1) \in \mathcal{H}_\xi$.
- $u \le cv_1 = c(cv - u)/(c - 1)$.
- Hence $(2c - 1)u \le c^2 v$, so that

$$c = \sup_{\substack{u,v\in\mathcal{H}_\xi \\ x\in D}} \frac{u(x)}{v(x)} \le \frac{c^2}{2c-1} < c,$$ Contradiction.

- $c = 1$ and $\mathcal{H}_\xi$ is singleton. $u \in \mathcal{H}_\xi$ is minimal. □

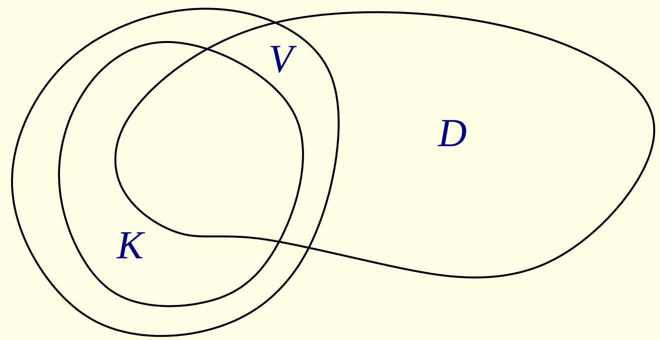
Remark 7

Under the CDC, John domains and uniform domains are characterized by an estimate of harmonic measure and the uniform BHP. ([Aik04]).

7. Further extensions

- HA [Aik08a] gives an exposition for the foundation of the Martin boundary and the results stated above.
- The BHP and the Carleson estimate are inextricably related.
- Precise formulations show they are equivalent.

Let (V, K) be a pair of a bounded open set V and a compact set K such that



(1) $$K \subset V,\ K \cap D \neq \emptyset,\ K \cap \partial D \neq \emptyset$$

Definition 15

We say D satisfies the global BHP if the following property hold: ${}^{\forall}(V, K)$ with (1) we find ${}^{\exists}C_4$ depending only on D and (V, K) such that if u and v are positive superharmonic functions in D with

(i) u and v are bounded and harmonic in $V \cap D$.

(ii) $u = v = 0$ q.e. on $V \cap \partial D$,

then ${}^{\forall}x, {}^{\forall}y \in K \cap D$ $\dfrac{u(x)/u(y)}{v(x)/v(y)} \leq C_4.$

Definition 16

We say D satisfies the global Carleson estimate if the following property hold: $^{\forall}(V, K)$ with (1) we find $^{\exists}C_5$ depending only on D and (V, K) such that if u is a positive superharmonic function in D with

(i) u is bounded and harmonic in $V \cap D$.

(ii) $u = 0$ q.e. on $V \cap \partial D$,

then $^{\forall}x \in K \cap D \; u(x) \leq C_5 u(x_0)$

Theorem 11 ([Aik08b])

D satisfies the GBHP if and only if it satisfies the global Carleson estimate.

- Domar's method is available for Carleson estimate. It is based on the mean value inequality of subharmonic functions, applicable to p-harmonic functions in a metric measure space.
- Carleson estimate hods for a domain with QHB condition. ([AHL06]).

Theorem 12

If D satisfies the QHB conditions, then it satisfies the GBHP.

Remark 8

- Carleson estimate extends to p-harmonic functions in a metric measure space. ([AS05]).
- The equivalence between the Carleson estimate and the BHP is based Brelot's remark (the relationship between the harmonic measure and the Green function), valid only for the linear case.
- The BHP for p-harmonic functions does not follow.
- Lewis-Nyström [LN07] proved the BHP for p-harmonic functions in a Lipschitz domain. Their argument is deep.

Remark 9

- Harmonicity is generalized to α-harmonicity ($0 < \alpha \leq 2$).
- $\alpha = 2$ corresponds to the classical harmonicity.
- If $0 < \alpha < 2$, then α-harmonicity is not a local property.
- α-stable process is the counterpart in probability theory.
- In case $0 < \alpha < 2$ Song-Wu [SW99] and Bogdan et.al. [BKK08] proved probabilistically the BHP for α-harmonic functions in a completely general domain.

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