Semibounded perturbation of Green function in an NTA domain

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Abstract. Let $\mathcal{L} = -\sum_{i,j} \partial_i(a_{ij}\partial_j)$ be a Hölder continuous uniformly elliptic operator on an NTA domain and let $\mathcal{L}_V = \mathcal{L} + V$ be a generalized Schrödinger operator for a nonnegative function V. Let G and G_V be the Green function for the Dirichlet problem with respect to $\mathcal{L}$ and $\mathcal{L}_V$, respectively. In case V is a function of the distance to the boundary, a necessary sufficient condition for G and G_V to have comparable decay near the boundary is given. This is based on an integral identity of $\mathcal{L}$-harmonic functions and integral inequalities of $\mathcal{L}$-superharmonic and $\mathcal{L}$-subharmonic functions, which may be of independent interest.

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1. Introduction

Let $\mathcal{L} = -\sum_{i,j} \partial_i(a_{ij}\partial_j)$ be a smooth uniformly elliptic operator on a domain $D \subset \mathbb{R}^n$, $n \geq 2$, such that $a_{ij}(x) = a_{ji}(x)$ is Hölder continuous on D and

$$\Lambda^{-1}|\xi|^2 \leq A[\xi] \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n,$$

where $A[\xi] = \sum_{i,j} a_{ij}\xi_i\xi_j$ and $\Lambda > 1$ is a constant independent of $\xi \in \mathbb{R}^n$ and $x \in D$. We also consider the Schrödinger operator $\mathcal{L}_V = \mathcal{L} + V$ for a nonnegative function V on D. Let G and G_V be the Green function for the Dirichlet problem in D with respect to $\mathcal{L}$ and $\mathcal{L}_V$, respectively. This means that for any $y \in D$, $G(\cdot, y)$ and $G_V(\cdot, y)$ vanish on ∂D and $\mathcal{L}G(\cdot, y) = \mathcal{L}_V G_V(\cdot, y) = \delta_y$, where δ_y is the Dirac measure at y. Since $\mathcal{L}$ and $\mathcal{L}_V$ are self adjoint, it follows that G and G_V are symmetric.

We are interested in the relationship between G and G_V. It is easy to see that

$$G_V(x, y) \leq G(x, y) \quad \text{for } x, y \in D.$$

Let $x_0 \in D$ be fixed. We say that V is semibounded perturbation of $\mathcal{L}_V$ if

$$G(x_0, y) \leq C G_V(x_0, y) \quad \text{for } y \in D$$

with $C > 1$ independent of $y \in D$ (see [Mu, §3]). The main purpose of this paper is to give a necessary and sufficient condition for V of a certain form to be semibounded perturbation in the case when D is an NTA domain (see [JK]). By the symbol C we denote an absolute positive constant whose value is unimportant and may change from line to line. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $C \geq 1$ such that $C^{-1} f_1 \leq f_2 \leq C f_1$. The main result of this paper is the following.

Theorem 1. *Let D be an NTA domain. Let $v(r)$ be a locally bounded nonnegative function for $0 < r \leq r_0 = \operatorname{diam}(D)$ such that $v(r) \approx v(2r)$ and $r^2 v(r)$ is nondecreasing for small $r > 0$. Set $V(x) = v(\delta(x))$ with $\delta(x) = \operatorname{dist}(x, \partial D)$. Then V is semibounded perturbation of $\mathcal{L}_V$ if and only if*

$$\int_0 r v(r) dr < \infty. \tag{1}$$

It is known that the Martin boundary of an NTA domain with respect to $\mathcal{L}$ is the Euclidean boundary and every boundary point is minimal ([JK, Theorem 5.9 and p.158]). Hence, in view of [Su, Theorem 2], we have the following corollary immediately.

Corollary 1. *Let D be an NTA domain and let v and V be as above. If (1) holds, then the Martin boundary of D with respect to $\mathcal{L}_V$ is the Euclidean boundary of D and every boundary point is minimal.*

When $\mathcal{L} = -\Delta$ and D is smooth, Suzuki [Su] showed the sufficiency part of the above theorem and we gave an alternative proof as well as the necessity part ([A3, Theorems 2 and 3]). Both proofs use the smoothness of D and do not apply to an NTA domain. Instead, we shall employ the following integral identity of $\mathcal{L}$-harmonic functions and integral inequalities of $\mathcal{L}$-superharmonic and $\mathcal{L}$-subharmonic functions, which may be of independent interest. These are generalizations of our previous results [A1] (see also [A2] and [AE, pp.171–182]), which is based on the coarea formula.

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Theorem 2. *Let D be a regular domain. Let $\varphi(t)$ be a nonnegative function for $t > 0$. Let $x_0 \in D$ and $g(x) = G(x_0, x)$. Suppose $0 \le a < b \le \infty$. Then the following statements hold:*

(i) *If h is $\mathcal{L}$-harmonic on D, then*

$$\int_{\{x \in D: a < g(x) < b\}} h(x)\varphi(g(x))A[\nabla g(x)]dx = h(x_0)\int_a^b \varphi(t)dt.$$

(ii) *If u is $\mathcal{L}$-superharmonic on D, then*

$$\int_{\{x \in D: a < g(x) < b\}} u(x)\varphi(g(x))A[\nabla g(x)]dx \le u(x_0)\int_a^b \varphi(t)dt.$$

(iii) *If s is $\mathcal{L}$-subharmonic on D, then*

$$\int_{\{x \in D: a < g(x) < b\}} s(x)\varphi(g(x))A[\nabla g(x)]dx \ge s(x_0)\int_a^b \varphi(t)dt.$$

We shall estimate $|\nabla g(x)|$ to obtain some corollaries of Theorem 2, which will be needed for the proof of Theorem 1. The upper estimate is easy. The lower estimate is more difficult. If D is an NTA domain, then $|\nabla g(x)|$ can be estimated by $g(x)/\delta(x)$ in a certain sense with the help of the boundary Harnack principle ([JK, Theorem 5.1 and p.138]). For details see Section 2.

For the proof of Theorem 1 we also need the decay estimate of $g(x)$ near the boundary. We find $0 < \beta \le 1 \le \alpha < \infty$ such that

$$C^{-1}\delta(x)^\alpha \le g(x) \le C\delta(x)^\beta \quad \text{for } x \in D \text{ close to } \partial D. \tag{2}$$

The above inequalities are rather easy. The first inequality actually holds for a John domain (see e.g. [B, p.185 and (2.6)]) and the second does for a domain satisfying the capacity density condition (see e.g. [An]). In general, $\beta < 1 < \alpha$. Only in the case when D is smooth, we have $\alpha = \beta = 1$, which was essential in [A3]. Surprisingly, the weak estimate (2) is sufficient for the proof of Theorem 1.

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2. Proof of Theorem 2 and corollaries

Proof of Theorem 2. The proof is essentially the same as in [A1]. We shall prove only (i), since the remaining can be proved in the same fashion. Let h be $\mathcal{L}$-harmonic on D. Let $D_t = \{x \in D : g(x) > t\}$ for $t > 0$. Observe that D_t is a relatively compact subset of D. We see that the Green function G^t for D_t with respect to $\mathcal{L}$ satisfies that $G^t(x_0, x) = g(x) - t$. By the Sard theorem (see e.g. [Ma, Corollary on p.35]), ∂D_t is a smooth surface for a.e. $t > 0$. For such t we see that the outward normal $\mathbf{n}$ of ∂D_t is given by $-\nabla g/|\nabla g|$. On the other hand the Poisson integral formula for D_t with respect to $\mathcal{L}$ shows that

$$h(x_0) = -\int_{\partial D_t} h(x)(\nabla g(x), A(x)\mathbf{n}(x))d\sigma(x),$$

where $A(x)\mathbf{n}(x) = (\sum_j a_{ij}n_j)$ is the conormal vector (see [Mi, (10.4) on p.21]). Hence

$$h(x_0) = \int_{\partial D_t} h(x)\frac{A[\nabla g(x)]}{|\nabla g(x)|}d\sigma(x).$$

By the coarea formula (see e.g. [Ma, pp.37–39])

$$\begin{aligned} h(x_0)\int_a^b \varphi(t)dt &= \int_a^b dt \int_{\partial D_t} h(x)\varphi(g(x))\frac{A[\nabla g(x)]}{|\nabla g(x)|}d\sigma(x) \\ &= \int_{\{x\in D: a<g(x)<b\}} h(x)\varphi(g(x))A[\nabla g(x)]dx. \end{aligned}$$

Thus the required identity holds.

Let us estimate $\nabla g(x)$ and give some corollaries of Theorem 2, which will be needed for the proof of Theorem 1. Since $g(x)$ is positive and $\mathcal{L}$-harmonic near ∂D, it follows that $|\nabla g(x)| \le Cg(x)/\delta(x)$. Since $A = (a_{ij})$ is uniformly elliptic, we have the following corollary from the above proof.

Corollary 2. *Let $0 \le a < b \le \infty$. Suppose that s is a positive $\mathcal{L}$-subharmonic function on $\{x \in D : g(x) > a\}$. Then*

$$s(x_0)\int_a^b \varphi(t)dt \le C\int_{\{x\in D: a<g(x)<b\}} s(x)\varphi(g(x))\frac{g(x)^2}{\delta(x)^2}dx,$$

where C is independent of s, a and b.

The inequality $|\nabla g(x)| \ge Cg(x)/\delta(x)$ does not hold in a pointwise sense. However, it holds in a certain weak sense for an NTA domain. In [A1] this was observed for $\mathcal{L} = -\Delta$. The key ingredient was the boundary Harnack principle. For a general uniformly elliptic operator $\mathcal{L}$ the boundary Harnack principle is available also (see [JK, Theorem 5.1 and p.138]) and hence the same proof as in [A1] works for the present case. We have the following corollary.

Corollary 3. *Let D be an NTA domain. Let $\varphi(t) \approx \varphi(2t)$ for $t > 0$ and let $0 \le a < b < \infty$. Suppose that u is a positive $\mathcal{L}$-superharmonic function on D. Then*

$$\int_{\{x\in D: a<g(x)<b\}} u(x)\varphi(g(x))\frac{g(x)^2}{\delta(x)^2}dx \le Cu(x_0)\int_a^b \varphi(t)dt,$$

where C is independent of u, a and b.

Remark. Note that the simplified proofs in [A2] and [AE, pp.171–182] use the analyticity of harmonic functions ([A2, Lemma 1] and [AE, Lemma 9.6.2 on p.180]). In general, $\mathcal{L}$-harmonic functions need not be analytic. Instead of the analyticity, we can use the unique continuation property of those functions for the present situation.

3. Proof of Theorem 1

Proof of Theorem 1. Sufficiency: Let us recall the resolvent equation

$$G(x,y) = G_V(x,y) + \int_D G_V(x,z)V(z)G(z,y)dz \quad \text{for } x,y \in D. \tag{3}$$

In view of [Su, §4] and [Mu, Theorem 1.5], it is sufficient to show that for any $\varepsilon > 0$ there is a compact subset K of D such that

$$\int_{D\setminus K} g(z)V(z)G(z,y)dz \le \varepsilon g(y) \quad \text{for } y \in D, \tag{4}$$

where we recall $g(y) = G(x_0, y)$.

Suppose (1) holds. Let $\varphi(t) = t^{2/\alpha-1}v(t^{1/\alpha})$, in other words, $v(r) = r^{\alpha-2}\varphi(r^\alpha)$ with $\alpha > 1$ in (2). By assumption $t\varphi(t)$ is nondecreasing. Let $b > 0$ be sufficiently small such that

$$V(z) = v(\delta(z)) = \delta(z)^{\alpha-2}\varphi(\delta(z)^\alpha) \le Cg(z)\varphi(g(z))\delta(z)^{-2} \quad \text{for } g(z) < b,$$

where (2) and the monotonicity of $t\varphi(t)$ are used. Hence, Corollary 3 with $u = G(\cdot, y)$ yields

$$\begin{aligned}\int_{\{z\in D: g(z)<b\}} g(z)V(z)G(z,y)dz &\le C\int_{\{z\in D: g(z)<b\}} G(z,y)\varphi(g(z))\frac{g(z)^2}{\delta(z)^2}dz \\ &\le Cg(y)\int_0^b \varphi(t)dt.\end{aligned}$$

By an elementary calculation

$$\int_0^b \varphi(t)dt = \alpha\int_0^{b^{1/\alpha}} rv(r)dr < \infty.$$

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Hence (4) holds with $K = \{z \in D : g(z) \geq b\}$ for sufficiently small $b > 0$.

Necessity: Suppose V is semibounded perturbation of $\mathcal{L}_V$. Then by the resolvent equation (3)

$$\int_D g(z)V(z)G(z,y)dz \leq Cg(y) \quad \text{for } y \in D. \tag{5}$$

Let $\psi(t) = t^{2/\beta-1}v(t^{1/\beta})$, in other words, $v(r) = r^{\beta-2}\psi(r^\beta)$ with $0 < \beta < 1$ in (2). By assumption $t\psi(t)$ is nondecreasing. We choose $b > 0$ sufficiently small such that

$$V(z) = v(\delta(z)) = \delta(z)^{\beta-2}\psi(\delta(z)^\beta) \geq Cg(z)\psi(g(z))\delta(z)^{-2} \qquad \text{for } g(z) < b,$$

where (2) and the monotonicity of $t\psi(t)$ are used. Let $0 < a < b$ and take $y \in D$ close to the boundary such that $g(y) < a$. Then $G(\cdot, y)$ is $\mathcal{L}$-harmonic on $\{z \in D : g(z) > a\}$. Hence, Corollary 2 with $G(\cdot, y)$ and $\psi(t)$ in place of s and $\varphi(t)$ yields

$$\begin{aligned}\int_{\{z\in D: a<g(z)<b\}} g(z)V(z)G(z,y)dz &\geq C\int_{\{z\in D: a<g(z)<b\}} G(z,y)\psi(g(z))\frac{g(z)^2}{\delta(z)^2}dz \\ &\geq Cg(y)\int_a^b \psi(t)dt.\end{aligned}$$

In view of (5) we have

$$\int_a^b \psi(t)dt \leq C,$$

where C is independent of a and hence

$$\infty > \int_0^b \psi(t)dt = \beta\int_0^{b^{1/\beta}} rv(r)dr.$$

Thus (1) holds. The proof is complete.

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