QUASIADDITIVITY OF RIESZ CAPACITY

Hiroaki AIKAWA

1. Introduction

Let $0 < \alpha < n$ and $k_\alpha(x) = |x|^{\alpha-n}$ the Riesz kernel on $\mathbb{R}^n$. Define the Riesz capacity by

$$R_{\alpha,p}(E) = \begin{cases} \inf\{\|f\|_p^p : k_\alpha * f(x) \ge 1 \text{ on } E, f \ge 0\} & \text{if } 1 < p < \infty, \\ \inf\{\|\mu\| : k_\alpha * \mu(x) \ge 1 \text{ on } E, \mu \ge 0\} & \text{if } p = 1. \end{cases}$$

In view of [7] we see that $R_{\alpha,1}(E)$ is equal to the usual (outer) α-capacity $C_\alpha(E)$. It is obvious that $R_{\alpha,p}$ is countably subadditive, i.e.

$$R_{\alpha,p}(E) \le \sum_k R_{\alpha,p}(E_k)$$

with $E = \bigcup_k E_k$. The main purpose of this paper is to investigate for what decompositions the inequality

$$R_{\alpha,p}(E) \ge N \sum_k R_{\alpha,p}(E_k)$$

holds with some positive constant N. We refer to this inequality as "quasiadditivity". Quasiadditivity for decompositions into spherical shells has been considered by Landkof [9, Lemma 5.5 in p.304] and Adams [1, Theorem 7.5]. In the case of Green energy (for the definition see Section 5), quasiadditivity for the Whitney decomposition (cf. [14, p.16]) of a half space is discussed in Essén [5].

We shall show that the Whitney decomposition associated with a certain closed set has quasiadditivity.

Definition. *Let F be a closed set having no interior points. Put $\delta(x) = \operatorname{dist}(x, F)$ and let m_β be the measure defined by*

$$m_\beta(E) = \int_E \delta(x)^{-\beta} dx.$$

We associate the least number $d = d(F)$ for which

$$m_\beta(C(x,r)) \le N_\beta r^{n-\beta} \tag{1.1}$$

holds for all $x \in F$ and $r > 0$ with a positive constant N_β, whenever $0 < \beta < n - d$.

The constant $d(F)$ is related to the dimension of F. In fact, if L is an m-dimensional affine subspace in $\mathbb{R}^n$, then $d(L) = m$. We can easily see that if F is an m-dimensional compact Lipschitz manifold, then $d(F) = m$. By definition if $F_1 \subset F_2$, then $d(F_1) \le d(F_2)$. The Hausdorff dimension of F is not greater than $d(F)$. They are, in general, different; if $F = \{0\} \cup \bigcup_{j=1}^{\infty} \{(j^{-1}, 0, \dots, 0)\}$, then $d(F) > 0$ and yet the Hausdorff dimension of F is equal to 0.

Our main result is

Theorem 1. *Let $1 \leq p < \infty$ and suppose $\alpha p + d(F) < n$. Let $\{Q_k\}$ be the Whitney decomposition of $\mathbb{R}^n \setminus F$. Then, for any set $E \subset \mathbb{R}^n$,*

$$R_{\alpha,p}(E) \geq N \sum_k R_{\alpha,p}(E_k)$$

holds with $E_k = E \cap Q_k$ for some positive constant N.

Let us note that $R_{\alpha,p}(F) = 0$ since the Hausdorff dimension of F is not greater than $d(F) < n - \alpha p$ (see [10, Theorem 21]). Since $d(\{0\}) = 0$, we see that Theorem 1 is a generalization of the aforementioned results of Landkof and Adams. Our proof is completely different; it relies on the following comparison between the Riesz capacity $R_{\alpha,p}$ and the measure $m_{\alpha p}$.

Theorem 2. *Let $1 \leq p < \infty$. Suppose $\alpha p + d(F) < n$. If E is measurable, then*

$$m_{\alpha p}(E) \leq N R_{\alpha,p}(E)$$

for some positive constant N.

The plan of this paper is as follows. In Section 2 we shall prove Theorem 1 assuming Theorem 2. Theorem 2 will, in turn, be proved in Section 3 as a corollary to a certain weighted norm inequality. Section 4 will be devoted to applications of Theorems 1 and 2. We shall deal with sets E for which $m_{\alpha p}(E)$ and $R_{\alpha,p}(E)$ are comparable. We shall observe that α-thin sets are characterized by Wiener type conditions associated with the Whitney decomposition. In Section 5, we shall study quasiadditivity of Green energy in connection with the notion of minimal thinness. We shall characterize minimally thin sets in terms of ordinary capacity (cf. [5] and [6, Section 1]). Also we shall observe that [4, Theorems 1 and 2] follows from our method.

The author would like to thank Professor Essén for valuable comments and showing the preprint of [6]. He also would like to thank the referee for several helpful suggestions.

2. Proof of Theorem 1

By the symbol N we denote an absolute positive constant whose value is unimportant and may change from line to line. We shall say that two positive functions f and g are comparable, written $f \approx g$, if and only if there exists a constant N such that $N^{-1}g \le f \le Ng$. By $C(x,r)$ we denote the closed ball with center at x and radius r. For the Whitney decomposition $\{Q_k\}$ of $\mathbb{R}^n \setminus F$, we write r_k for the side-length of Q_k. Note that $\operatorname{dist}(Q_k, F) \approx r_k$. By $\widetilde{Q}_k$ we denote the double of Q_k.

Lemma 1. *Let $\beta + d(F) < n$. Then*

$$0 < r < \delta(x)/2 \Rightarrow m_\beta(C(x,r)) \approx r^n \delta(x)^{-\beta}, \tag{2.1}$$

$$r \ge \delta(x)/2 \Rightarrow m_\beta(C(x,r)) \approx r^{n-\beta}. \tag{2.2}$$

In particular, (1.1) holds for all $x \in \mathbb{R}^n$ and $r > 0$; the measure m_β is a doubling measure. Let $\alpha p + d(F) < n$. Then for a Whitney cube Q_k

$$R_{\alpha,p}(Q_k) \approx m_{\alpha p}(Q_k) \approx r_k^{n-\alpha p}. \tag{2.3}$$

Proof. By definition (2.1) is obvious. Let $r \ge \delta(x)/2$. Then we find $x_0 \in F$ such that $C(x,r) \subset C(x_0, 3r)$. Hence by (1.1) we have $m_\beta(C(x,r)) \le Nr^{n-\beta}$. Let us prove the opposite inequality. Since $\delta(y) \le r + \delta(x) \le 3r$ for all $y \in C(x,r)$, it follows that $m_\beta(C(x,r)) \ge (3r)^{-\beta} \int_{C(x,r)} dx \ge r^{n-\beta}$. Thus (2.2) is proved. It is well known that $R_{\alpha,p}(C(x,r)) = Nr^{n-\alpha p}$ for $\alpha p < n$. Hence (2.3) follows.

Let us put

$$\widetilde{R}_{\alpha,p}(E) = \sum_k R_{\alpha,p}(E_k) \text{ with } E_k = E \cap Q_k.$$

We need to prove $\widetilde{R}_{\alpha,p}(E) \approx R_{\alpha,p}(E)$. Let us begin with comparing $\widetilde{R}_{\alpha,p}$ with a Hausdorff type outer measure. For $\beta > 0$ define the Hausdorff type outer measure H_β by

$$H_\beta(E) = \inf\{\sum_i r_i^\beta : E \subset \bigcup_i C(z_i, r_i), z_i \in F\}.$$

One should note that a point x has positive H_β measure unless it lies on F. In fact, $H_\beta(\{x\}) = \delta(x)^\beta$.

Lemma 2. *Let $\alpha p + d(F) < n$. Then*

$$\widetilde{R}_{\alpha,p}(E) \le NH_{n-\alpha p}(E).$$

Proof. Let us prove first

$$\widetilde{R}_{\alpha,p}(C(z,r)) \le Nr^{n-\alpha p} \text{ for } z \in F. \tag{2.4}$$

Observe that if Q_k meets $C(z,r)$, then $Q_k \subset C(z, Nr)$. Hence

$$\begin{aligned}\widetilde{R}_{\alpha,p}(C(z,r)) \le \sum_{Q_k \cap C(z,r) \ne \emptyset} R_{\alpha,p}(Q_k) \le N \sum_{Q_k \cap C(z,r) \ne \emptyset} m_{\alpha p}(Q_k) \\ \le N m_{\alpha p}(C(z,Nr)) \le Nr^{n-\alpha p}\end{aligned}$$

by (2.3) and Lemma 1. Thus (2.4) follows.

Take an arbitrary positive number ε. By definition we can find $z_i \in F$ and $r_i > 0$ such that

$$E \subset \bigcup_i C(z_i, r_i),$$
$$\sum_i r_i^{n-\alpha p} \le H_{n-\alpha p}(E) + \varepsilon.$$

By (2.4)

$$\widetilde{R}_{\alpha,p}(E) \le \sum_i \widetilde{R}_{\alpha,p}(C(z_i, r_i)) \le N \sum_i r_i^{n-\alpha p} \le N(H_{n-\alpha p}(E) + \varepsilon).$$

Since ε is arbitrary, we have the desired inequality. The lemma is proved.

Proof of Theorem 1. Let us prove the inequality only for $1 < p < \infty$. The $p = 1$ case is similar. It suffices to prove that $\widetilde{R}_{\alpha,p}(E) \le N R_{\alpha,p}(E)$ for $R_{\alpha,p}(E) < \infty$. Take an arbitrary positive number ε. We can find a nonnegative function f such that $k_\alpha * f \ge 1$ on E and $\|f\|_p^p < R_{\alpha,p}(E) + \varepsilon$. We split $k_\alpha * f(x)$ into

$$I(x) = \int_{\widetilde{Q}_k} k_\alpha(x-y) f(y) dy, \qquad \text{for } x \in Q_k,$$
$$J(x) = \int_{\mathbb{R}^n \setminus \widetilde{Q}_k} k_\alpha(x-y) f(y) dy, \qquad \text{for } x \in Q_k.$$

Put $E' = \{x : I(x) \ge \frac{1}{2}\}$ and $E'' = \{x : J(x) \ge \frac{1}{2}\}$. Then $E \subset E' \cup E''$. Since the multiplicity of $\widetilde{Q}_k$ is bounded by a constant depending only on the dimension, it follows that

$$\widetilde{R}_{\alpha,p}(E') = \sum_k R_{\alpha,p}(E' \cap Q_k) \le 2^p \sum_k \int_{\widetilde{Q}_k} f^p dx \le N \|f\|_p^p \le N(R_{\alpha,p}(E) + \varepsilon). \tag{2.5}$$

By an elementary calculation we see that if $E'' \cap Q_k \ne \emptyset$, then $J(x) \ge N$ on Q_k. Hence $k_\alpha * f(x) \ge J(x) \ge N$ on

$$\widetilde{E''} = \bigcup_{Q_k \cap E'' \ne \emptyset} Q_k,$$

whence $R_{\alpha,p}(\widetilde{E''}) \le N \|f\|_p^p$. Since $H_{n-\alpha p}(Q_k) \approx r_k^{n-\alpha p}$, it follows from (2.3) and Theorem 2

$$\begin{aligned} H_{n-\alpha p}(E'') \le H_{n-\alpha p}(\widetilde{E''}) &\le \sum_{Q_k \cap E'' \ne \emptyset} H_{n-\alpha p}(Q_k) \le N \sum_{Q_k \cap E'' \ne \emptyset} r_k^{n-\alpha p} \\ &\le N m_{\alpha p}(\widetilde{E''}) \le N R_{\alpha,p}(\widetilde{E''}) \le N \|f\|_p^p \le N(R_{\alpha,p}(E) + \varepsilon). \end{aligned}$$

Hence $\widetilde{R}_{\alpha,p}(E'') \le N(R_{\alpha,p}(E) + \varepsilon)$ by Lemma 2. This, together with (2.5), completes the proof, since ε is arbitrary.

3. Proof of Theorem 2

We shall show Theorem 2 as a corollary to a certain weighted norm inequality. For future reference we shall state the result in a slightly general form. Let $K_\alpha(x,y) = |x-y|^{\alpha-n}$. Define

$$T_\alpha f(x) = \int_{\mathbb{R}^n} K_\alpha(x,y) f(y) dm_\alpha(y).$$

Let us prove

Theorem 3. *Let* $\alpha + d(F) < n$.

(i) Let $1 < p < \infty$. *Then* $\|T_\alpha f\|_{p,\alpha} \le N \|f\|_{p,\alpha}$, *where* $\|f\|_{p,\alpha} = (\int |f|^p dm_\alpha)^{1/p}$. *Moreover, if* w *satisfies the Muckenhoupt* A_p *condition with respect to* m_α, *i.e.*

$$(A_p) \qquad \sup_Q \left(\frac{1}{m_\alpha(Q)} \int_Q w dm_\alpha \right) \left(\frac{1}{m_\alpha(Q)} \int_Q w^{1/1-p} dm_\alpha \right)^{p-1} < \infty,$$

then

$$\int_{\mathbb{R}^n} |T_\alpha f|^p w dm_\alpha \le N \int_{\mathbb{R}^n} |f|^p w dm_\alpha.$$

(ii) If $\lambda > 0$, *then*

$$m_\alpha(\{x \in \mathbb{R}^n : \int_{\mathbb{R}^n} K_\alpha(x,y) d\mu(y) > \lambda\}) \le N \|\mu\| / \lambda.$$

First we prove that Theorem 2 follows from Theorem 3.

Proof of Theorem 2. Suppose $p = 1$. Then the conclusion readily follows from Theorem 3 (ii). Suppose $1 < p < \infty$. In view of Lemma 1, we see that the weight $w(x) = \delta(x)^{(1-p)\alpha}$ satisfies (A_p). Hence, Theorem 3 (i) implies that

$$\int_{\mathbb{R}^n} |k_\alpha * g|^p dm_{\alpha p}(x) \le N \int_{\mathbb{R}^n} |g|^p dx,$$

where we put $g(x) = f(x)\delta(x)^{-\alpha}$. This immediately yields Theorem 2.

Now let us prove Theorem 3. Although the proof is carried out in a standard way (cf. [3]), we give it for the completeness. In the rest of this section we let

$$\alpha + d(F) < n.$$

First we note

Lemma 3. *Let* $1 < p < \min\{\frac{n}{n-\alpha}, \frac{n-\alpha}{d(F)}\}$ *and let* $1/p + 1/q = 1$. *If* Q *is a cube in* $\mathbb{R}^n$, *then*

$$\int_Q |T_\alpha f(x)| dm_\alpha(x) \le N \|f\|_{q,\alpha} m_\alpha(Q)^{1/p}.$$

Proof. Let us prove first

$$\int_{\mathbb{R}^n} K_\alpha(x,y)^p dm_\alpha(y) \le N \delta(x)^{(\alpha-n)(p-1)}. \tag{3.1}$$

Since $p < n/(n-\alpha)$, it follows that

$$\int_{C(x,\delta(x)/2)} K_\alpha(x,y)^p dm_\alpha(y) \le N\delta(x)^{-\alpha} \int_{C(x,\delta(x)/2)} |x-y|^{(\alpha-n)p} dy \le N\delta(x)^{(\alpha-n)(p-1)}.$$

In view of Lemma 1, we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus C(x,\delta(x)/2)} K_\alpha(x,y)^p dm_\alpha(y) &= \sum_{j=0}^{\infty} \int_{2^{j-1}\delta(x)<|x-y|\le 2^j\delta(x)} |x-y|^{(\alpha-n)p} dm_\alpha(y) \\ &\le N\sum_{j=0}^{\infty} (2^{j-1}\delta(x))^{(\alpha-n)p}(2^j\delta(x))^{n-\alpha} \le N\delta(x)^{(\alpha-n)(p-1)},\end{aligned}$$

since $(\alpha-n)(p-1)<0$. Thus (3.1) holds. Hölder's inequality and (3.1) yield

$$|T_\alpha f(x)| \le \|f\|_{q,\alpha}(\int_Q K_\alpha(x,y)^p dm_\alpha(y))^{1/p} \le N\|f\|_{q,\alpha}\delta(x)^{(\alpha-n)/q}. \tag{3.2}$$

Observe that $p < (n-\alpha)/d(F)$ implies that $\alpha + (n-\alpha)/q + d(F) < n$. Hence Lemma 1 yields that

$$m_{\alpha+(n-\alpha)/q}(Q) \le Nr^{n-\alpha-(n-\alpha)/q} \le Nm_\alpha(Q)^{1/p}.$$

This, together with (3.2), completes the proof.

Let $\mathcal{M}_\alpha f(x)$ be the maximal function defined by

$$\mathcal{M}_\alpha f(x) = \sup_Q \frac{1}{m_\alpha(Q)} \int_Q |f| dm_\alpha,$$

where the supremum is taken over all cubes containing x. Observe

$$\sup_{x,x'\in B(x_0,1)} \int_{\mathbb{R}^n\setminus B(x_0,2)} |K_\alpha(x,y) - K_\alpha(x',y)| f(y) dm_\alpha(y) \le N\mathcal{M}_\alpha f(x_0). \tag{3.3}$$

As a result we have the following

Lemma 4. *Let Q be a cube and $\widetilde{Q}$ the double of Q. Then*

$$\sup_{x,x'\in Q} \int_{y\notin\widetilde{Q}} |K_\alpha(x,y) - K_\alpha(x',y)| dm_\alpha(y) \le N.$$

We observe that if $\gamma = (p+1)^{-1}$, then $p = \dfrac{1}{\gamma} - 1$ and

$$1 < p < \min\{\frac{n}{n-\alpha}, \frac{n-\alpha}{d(F)}\} \iff \max\{\frac{n-\alpha}{2n-\alpha}, \frac{d(F)}{n-\alpha+d(F)}\} < \gamma < \frac{1}{2}.$$

Lemma 5. *Let* $\max\{\frac{n-\alpha}{2n-\alpha}, \frac{d(F)}{n-\alpha+d(F)}\} < \gamma < \frac{1}{2}$*. Let* Q *be a cube and* $\widetilde{Q}$ *the double of* Q*. If* $f \geq 0$*,* $\operatorname{supp} f \subset \widetilde{Q}$ *and* $\|f\|_{1,\alpha} \leq \varepsilon m_\alpha(\widetilde{Q})$ *with* $0 < \varepsilon < 1$*, then*

$$m_\alpha(\{x \in Q : T_\alpha f(x) > 1\}) \leq N\varepsilon^{1-\gamma} m_\alpha(Q).$$

Proof. By the Calderón-Zygmund lemma (e.g. [14, p.17]) we have a family of mutually disjoint cubes Q_j such that

(i) $f(x) \leq \varepsilon^\gamma$ a.e. on $\mathbb{R}^n \setminus \Omega$ with $\Omega = \cup_j Q_j$;

(ii) For each cube Q_j

$$\varepsilon^\gamma < \frac{1}{m_\alpha(Q_j)} \int_{Q_j} f dm_\alpha(x) \leq N\varepsilon^\gamma.$$

Let

$$g(x) = \begin{cases} f(x) & \text{on } \mathbb{R}^n \setminus \Omega, \\ \dfrac{1}{m_\alpha(Q_j)} \displaystyle\int_{Q_j} f dm_\alpha(x) & \text{on } Q_j. \end{cases}$$

and $b = f - g$. Obviously, $\|g\|_{1,\alpha} \leq \|f\|_{1,\alpha}$ and $\|b\|_{1,\alpha} \leq 2\|f\|_{1,\alpha}$. Let $p = \frac{1}{\gamma} - 1$ and $1/p + 1/q = 1$. Since $0 \leq g \leq N\varepsilon^\gamma$, it follows that

$$\|g\|_{q,\alpha}^q \leq N\varepsilon^{\gamma(q-1)} \|f\|_{1,\alpha} \leq N\varepsilon^{\gamma(q-1)+1} m_\alpha(Q).$$

We have from Lemma 3

$$m_\alpha(\{x \in Q : T_\alpha g(x) \geq 1/2\}) \leq N\|g\|_{q,\alpha} m_\alpha(Q)^{1/p} \leq N\varepsilon^{1-\gamma} m_\alpha(Q). \tag{3.4}$$

Let y_j be the center of Q_j and $\widetilde{Q}_j$ the double of Q_j. We put $\widetilde{\Omega} = \cup_j \widetilde{Q}_j$. It follows from Lemma 4 and the symmetry of K_α that

$$\begin{aligned} \int_{\mathbb{R}^n \setminus \widetilde{\Omega}} |T_\alpha b| dm_\alpha &\leq \sum_j \int_{x \notin \widetilde{Q}_j} \int_{y \in Q_j} |K_\alpha(x,y) - K_\alpha(x,y_j)||b(y)| dm_\alpha(y) dm_\alpha(x) \\ &\leq N \sum_j \int_{y \in Q_j} |b(y)| dm_\alpha(y) \leq N\|f\|_{1,\alpha}. \end{aligned}$$

Therefore

$$m_\alpha(\{x \in \mathbb{R}^n : |T_\alpha b(x)| > 1/2\}) \leq N(\|f\|_{1,\alpha} + \varepsilon^{1-\gamma} m_\alpha(Q)) \leq N\varepsilon^{1-\gamma} m_\alpha(Q),$$

since

$$m_\alpha(\widetilde{\Omega}) \leq N \sum_j m_\alpha(Q_j) \leq \frac{N}{\varepsilon^\gamma} \sum_j \int_{Q_j} f dm_\alpha(x) \leq \frac{N}{\varepsilon^\gamma} \|f\|_{1,\alpha} \leq N\varepsilon^{1-\gamma} m_\alpha(Q).$$

This, together with (3.4), implies

$$m_\alpha(\{x \in Q : T_\alpha f(x) > 1\}) \leq N\varepsilon^{1-\gamma} m_\alpha(Q).$$

The proof is complete.

Lemma 6. *Let* $\max\{\frac{n-\alpha}{2n-\alpha}, \frac{d(F)}{n-\alpha+d(F)}\} < \gamma < \frac{1}{2}$*. Then there is a positive constant* B *such that if* $\lambda > 0$*,* $0 < \varepsilon < 1$*,* $f \geq 0$*, and a cube* Q *has a point* x' *satisfying* $T_\alpha f(x') \leq \lambda$*, then*

$$m_\alpha(\{x \in Q : T_\alpha f(x) > B\lambda, \mathcal{M}_\alpha f(x) \leq \varepsilon\lambda\}) \leq N\varepsilon^{1-\gamma} m_\alpha(Q).$$

Proof. We may assume that there is a point x_0 in Q such that $\mathcal{M}_\alpha f(x_0) \leq \varepsilon\lambda$. Let $\widetilde{Q}$ be the double of Q. In view of (3.3), we have for $x \in Q$

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\widetilde{Q}} K_\alpha(x,y)f(y)dm_\alpha(y) &\leq \int_{\mathbb{R}^n\setminus\widetilde{Q}} |K_\alpha(x,y) - K_\alpha(x',y)|f(y)dm_\alpha(y) + T_\alpha f(x') \\ &\leq N\varepsilon\lambda + \lambda \leq N_1\lambda.\end{aligned}$$

Let $h = f/\lambda$ on $\widetilde{Q}$ and $h = 0$ elsewhere. Since $\|h\|_{1,\alpha} \leq \varepsilon m_\alpha(\widetilde{Q})$, it follows from Lemma 5 that

$$m_\alpha(\{x \in Q : T_\alpha h(x) > 1\}) \leq N\varepsilon^{1-\gamma} m_\alpha(Q).$$

Let $B = N_1 + 1$. Then

$$m_\alpha(\{x \in Q : T_\alpha f(x) > B\lambda\}) \leq N\varepsilon^{1-\gamma} m_\alpha(Q).$$

The lemma follows.

Proof of Theorem 3. Suppose that w satisfies (A_p). In the same way as in [3, Theorem I], we see that

$$\int_{\mathbb{R}^n} (\mathcal{M}_\alpha f)^p w dm_\alpha \leq N \int_{\mathbb{R}^n} |f|^p w dm_\alpha. \tag{3.5}$$

Hence it is sufficient to show that

$$\int_{\mathbb{R}^n} |T_\alpha f|^p w dm_\alpha \leq N \int_{\mathbb{R}^n} (\mathcal{M}_\alpha f)^p w dm_\alpha. \tag{3.6}$$

In the proof of (3.6) we may assume that f is bounded and has compact support. Since $|T_\alpha f(x)| \leq N|x|^{\alpha-n} \leq N\mathcal{M}_\alpha f(x)$ as $|x| \to \infty$, it follows from (3.5) that

$$\int_{\mathbb{R}^n} |T_\alpha f|^p w dm_\alpha < \infty.$$

Let $\lambda > 0$ and let $\{Q_j\}$ be the Whitney decomposition of the set $\{T_\alpha f > \lambda\}$. Observe that there is a constant $N_2 > 1$ such that the cube Q_j^* with the same center as Q_j but expanded N_2 times meets the set $\{T_\alpha f \leq \lambda\}$. Hence it follows from Lemma 6 that if γ is as in Lemma 6, then

$$m_\alpha(\{x \in Q_j : T_\alpha f(x) > B\lambda, \mathcal{M}_\alpha f(x) \leq \varepsilon\lambda\}) \leq N\varepsilon^{1-\gamma} m_\alpha(Q_j^*) \leq N\varepsilon^{1-\gamma} m_\alpha(Q_j)$$

for $0 < \varepsilon < 1$. It is well known ([3, Lemma 3]) that w satisfies (A_∞), that is, there exists $\delta > 0$ such that for given any cube Q and any measurable set $E \subset Q$

$$\frac{w(E)}{w(Q)} \leq N \left(\frac{m_\alpha(E)}{m_\alpha(Q)}\right)^\delta,$$

where $w(E) = \int_E w dm_\alpha$. Hence

$$w(\{x \in Q_j : T_\alpha f(x) > B\lambda, \mathcal{M}_\alpha f(x) \le \varepsilon\lambda\}) \le N\varepsilon^{\delta(1-\gamma)} w(Q_j).$$

Summing over j, we obtain

$$w(\{T_\alpha f > B\lambda, \mathcal{M}_\alpha f \le \varepsilon\lambda\}) \le N\varepsilon^{\delta(1-\gamma)} w(\{T_\alpha f > \lambda\}),$$

which implies that

$$\int_{\mathbb{R}^n} |T_\alpha f|^p w dm_\alpha + N(\varepsilon) \int_{\mathbb{R}^n} (\mathcal{M}_\alpha f)^p w dm_\alpha \le N_3 \varepsilon^{\delta(1-\gamma)} \int_{\mathbb{R}^n} |T_\alpha f|^p w dm_\alpha.$$

Letting $\varepsilon > 0$ be so small that $N_3\varepsilon^{\delta(1-\gamma)} < 1/2$, we obtain (3.6). Thus (i) follows.

REFERENCES

1. D. R. Adams, *Sets and functions of finite L^p-capacity*, Indiana Univ. Math. J. **27** (1978), 611–627.
2. M. Brelot, *On topologies and boundaries in potential theory, Lecture Notes in Math. 174*, Springer Verlag, 1971.
3. R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
4. B. E. J. Dahlberg, *A minimum principle for positive harmonic functions*, Proc. London Math. Soc. (3) **33** (1976), 238–250.
5. M. Essén, *On Wiener conditions for minimally thin sets and rarefied sets*, Complex Analysis, Articles dedicated to A. Pfluger on the occasion of his 80th birthday, Birkhäuser, 1988, pp. 41–50.
6. M. Essén, *On minimal thinness, reduced functions and Green potentials*, To appear (1990).
7. B. Fuglede, *Le théorème du minimax et la théorie fine du potentiel*, Ann. Inst. Fourier **15** (1965), 65–87.
8. J.-L. Journé, *Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*, Lecture Notes in Math. 994, Springer, 1983.
9. N. S. Landkof, *Foundations of modern potential theory*, Springer, 1972.
10. N. G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. **26** (1970), 255–292.
11. Y. Mizuta, *On semi-fine limits of potentials*, Analysis **2** (1982), 115–139.
12. L. Naïm, *Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel*, Ann. Inst. Fourier (Grenoble) **7** (1957), 183–281.
13. P. Sjögren, *Weak L_1 characterization of Poisson integrals, Green potentials and H^p spaces*, Trans. Amer. Math. Soc. **233** (1977), 179–196.
14. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.
15. K.-O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations*, Math. Scand. **21** (1967), 17–37.

DEPARTMENT OF MATHEMATICS, GUNMA UNIVERSITY, KIRYU 376, JAPAN

Current address: Department of Mathematics, Faculty of Science, Kumamoto University, Kumamoto 860, Japan

E-mail address: `aikawa@kumamoto-u.ac.jp`