QUASIADDITIVITY OF CAPACITY AND MINIMAL THINNESS

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Abstract. It is well known that a capacity C is countably subadditive, i.e.

$$C(E) \le \sum C(E_j), \quad E = \bigcup E_j.$$

We shall prove that the reverse inequality, up to a multiplicative constant, holds for some decomposition of E, provided there is a measure *comparable* to C. Such a property will be referred to as quasiadditivity.

As an application, we shall show that the Green energy for a uniformly Δ-regular domain is quasiadditive with respect to the Whitney decomposition of the domain. The Hardy inequality due to Ancona [3, (1)] will be a main tool. We shall apply the quasiadditivity of the Green energy to obtain a *refined* Wiener criterion for minimal thinness in an NTA domain.

1. Introduction

Let X be a locally compact Hausdorff space and let k be a nonnegative lower semicontinuous function on $X \times X$. We refer to k as a kernel and define the capacity C_k by

$$C_k(E) = \inf\{\|\mu\| : k(\cdot, \mu) \ge 1 \text{ on } E\},$$

where $k(\cdot, \mu) = \int_X k(\cdot, y) d\mu(y)$. It is well known that C_k is countably subadditive, i.e.

$$C_k(E) \le \sum C_k(E_j), \quad E = \bigcup E_j.$$

In this article we shall show that the reverse inequality, up to a multiplicative constant, holds for some decomposition of E.

1991 *Mathematics Subject Classification.* Primary 31B15; Secondary 31B25.

Definition. Let $\{Q_j\}$ and $\{Q_j^*\}$ be families of Borel subsets of X such that:

(i) $Q_j \subset Q_j^*$,
(ii) $X = \bigcup Q_j$,
(iii) Q_j^* do not overlap so often, i.e., $\sum \chi_{Q_j^*} \le N$.

Then we say that $\{Q_j, Q_j^*\}$ is a quasidisjoint decomposition of X. We sometimes suppress Q_j^* and simply write $\{Q_j\}$.

We write M for a positive constant whose value may change from one occurrence to the next. If $M^{-1}f \le g \le Mf$ for two positive quantities f and g, then we write $f \approx g$. This constant is referred to as the constant of comparison.

Definition. Let σ be a (Borel) measure on X. We say that σ is comparable to C_k with respect to $\{Q_j\}$ if

$$\sigma(Q_j) \approx C_k(Q_j) \text{ for every } Q_j, \tag{1}$$

$$\sigma(E) \le MC_k(E) \text{ for every Borel set } E. \tag{2}$$

Definition. We say that the kernel k has the Harnack property with respect to $\{Q_j, Q_j^*\}$ if

$$k(x, y) \approx k(x', y)$$

for $x, x' \in Q_j$ and $y \in X \setminus Q_j^*$ with the constant of comparison independent of Q_j.

Our main theorem is as follows.

Theorem 1. *Let $\{Q_j, Q_j^*\}$ be a quasidisjoint decomposition of X. Suppose the kernel k has the Harnack property with respect to $\{Q_j, Q_j^*\}$. If there is a measure σ comparable to C_k with respect to $\{Q_j, Q_j^*\}$, then for every $E \subset X$*

$$C_k(E) \approx \sum C_k(E \cap Q_j). \tag{3}$$

We shall say that C_k is quasiadditive with respect to $\{Q_j, Q_j^*\}$ if (3) holds. The proof of the theorem will be carried out in §2 in the same spirit as in [2].

It is, in general, difficult to find a measure comparable to a given capacity. In [2] we found such a measure for the Riesz capacity with the aid of a certain weighted norm inequality. In this paper we shall employ a different device—Hardy's inequality— to obtain a measure comparable to a generalized Green energy. The most general Hardy inequality is given for a uniformly Δ-regular domain by Ancona [3]. By $B(x, r)$ we denote the open ball with radius r and center at x. We shall say that $D \subset \mathbb{R}^d$ is uniformly Δ-regular if there are constants $r_0 > 0$ and ε_1, $0 < \varepsilon_1 < 1$, such that, for all $x \in \partial D$ and all $0 < r < r_0$,

$$w_{x,r} \le 1 - \varepsilon_1 \quad \text{on } B(x, r/2) \cap D,$$

where $w_{x,r}$ is the harmonic measure of $\partial B(x, r) \cap D$ in the region $B(x, r) \cap D$ ([3, Definition 2]). The following lemma is a consequence of [3, Theorem 1 and Proposition 1].

Lemma A (Hardy's inequality)**.** *Assume that D is uniformly Δ-regular. Then, there is a positive constant M depending only on D such that*

$$\int_D \left|\frac{\psi(x)}{\delta(x)}\right|^2 dx \le M \int_D |\nabla\psi(x)|^2 dx \qquad \text{for all } \psi \in W_0^{1,2}(D),$$

where $\delta(x) = \operatorname{dist}(x, \partial D)$ and $W_0^{1,2}(D)$ stands for the usual Sobolev space, namely the completion of $C_0^\infty(D)$ with norm $\left(\int_D(|\psi|^2 + |\nabla\psi|^2)dx\right)^{1/2}$.

We let $G(x,y)$ be the Green function normalized by

$$\Delta G(\cdot, y) = -\delta_y,$$

where the left hand side denotes the distributional Laplacian of $G(\cdot, y)$ and δ_y the point mass at y. Let u be a nonnegative superharmonic function on D. For a compact subset K of D we let $\widehat{R}_u^K$ be the regularized reduced function of u with respect to K. Observe that $\widehat{R}_u^K$ is a Green potential, $G(\cdot, \lambda_u^K)$. The energy

$$\gamma_u(K) = \iint G(x,y) d\lambda_u^K(x) d\lambda_u^K(y)$$

is called the Green energy of K relative to u. For an open subset V we let

$$\gamma_u(V) = \sup\{\gamma_u(K) : K \text{ is compact, } K \subset V\},$$

and then for a general subset E

$$\gamma_u(E) = \inf\{\gamma_u(V) : V \text{ is open, } E \subset V\}.$$

The quantity $\gamma_u(E)$ is also called the Green energy relative to u. If $u \equiv 1$, then γ_u is the usual Green capacity C_G (see [15, pp.174–177]). If $D = \{x = (x_1, \dots, x_d) : x_1 > 0\}$ and $u(x) = x_1$, then γ_u is the Green energy defined by Essén and Jackson [11, Definition 2.2]. Let $k(x,y) = G(x,y)/(u(x)u(y))$. Then it is not so difficult to see that $\gamma_u(E) = C_k(E)$ (see [12]). Thus Theorem 1 applies to γ_u. Let $\delta(x) = \operatorname{dist}(x, \partial D)$. We define the measure σ_u on D by

$$\sigma_u(E) = \int_E \left(\frac{u(x)}{\delta(x)}\right)^2 dx.$$

Theorem 2. *Let D be a uniformly Δ-regular domain and let $\{Q_j\}$ be the Whitney decomposition of D. Suppose a positive superharmonic function u satisfies*

$$\sup_{Q_j} u \le M_0 \inf_{Q_j} u \tag{4}$$

with M_0 independent of Q_j. Then σ_u is comparable to γ_u with respect to $\{Q_j\}$ and γ_u is quasiadditive with respect to $\{Q_j\}$, i.e. $\gamma_u(E) \approx \sum \gamma_u(E \cap Q_j)$.

For each Whitney cube Q_j we let x_j be the center of Q_j, r_j the diameter of Q_j and $t_j = \operatorname{dist}(Q_j, \partial D)$. We put $u_j = u(x_j)$. By cap we denote the Newton capacity if $d \ge 3$; the logarithmic capacity if $d = 2$. If $E \subset Q_j$, then $\gamma_u(E) \approx u_j^2 \operatorname{cap}(E)$ for $d \ge 3$, and $\gamma_u(E) \approx u_j^2 / \log(4r_j / \operatorname{cap}(E))$ for $d = 2$ (see Lemma 1 below). Hence Theorem 2 shows that the Green energy γ_u is estimated by the summation of the ordinary capacities of $E \cap Q_j$.

Let h be a positive harmonic function. Then the Harnack principle shows that $u(x) = \min\{h(x)^a, b\}$ with $0 < a \le 1$ and $b > 0$ satisfies (4). Hence we have the following corollaries.

Corollary 1. *Let D and $\{Q_j\}$ be as in Theorem 2. Then the Green capacity $C_G = \gamma_1$ is quasiadditive with respect to $\{Q_j\}$, i.e.*

$$C_G(E) \approx \sum C_G(E \cap Q_j) \approx \begin{cases} \sum \operatorname{cap}(E \cap Q_j) & \text{if } d \ge 3, \\ \sum \dfrac{1}{\log(4r_j / \operatorname{cap}(E \cap Q_j))} & \text{if } d = 2. \end{cases}$$

Corollary 2. *Let D and $\{Q_j\}$ be as in Theorem 2. Let $x_0 \in D$ and let $g(x) = \min\{G(x, x_0), 1\}$. Then γ_g is quasiadditive with respect to $\{Q_j\}$, i.e. with $g_j = g(x_j)$*

$$\gamma_g(E) \approx \sum \gamma_g(E \cap Q_j) \approx \begin{cases} \sum g_j^2 \operatorname{cap}(E \cap Q_j) & \text{if } d \ge 3, \\ \sum \dfrac{g_j^2}{\log(4r_j / \operatorname{cap}(E \cap Q_j))} & \text{if } d = 2. \end{cases}$$

In [14] Jerison and Kenig introduced the notion of NTA domains. A bounded domain D is called NTA when there exist positive constants M and r_1 such that

- (a) Corkscrew condition. For any $z \in \partial D$, $r < r_1$ there exists a point $A_r(z) \in D$ such that $M^{-1}r < |A_r(z) - z| < r$ and $\delta(A_r(z)) > M^{-1}r$.
- (b) The complement of D satisfies the corkscrew condition.
- (c) Harnack chain condition. If $\varepsilon > 0$ and x_1 and x_2 belong to to D, $\delta(x_j) > \varepsilon$ and $|x_1 - x_2| < C\varepsilon$, then there exists a Harnack chain from x_1 and x_2 whose length depends on C, but not ε.

In view of (b) we see that an NTA domain D is uniformly Δ-regular. It is known that the Martin boundary of D is homeomorphic to the Euclidean boundary ∂D and every boundary point is minimal ([14]). We shall apply Corollary 2 to a characterization of minimally thin sets in an NTA domain. For details see Section 4.

Acknowledgment. I would like to thank Professor Stephen Gardiner for bringing Ancona's paper [4] to my attention. I would like to thank Professor Alano Ancona for the comments on Section 4.

2. Proof of Theorem 1

Proof of Theorem 1. Let $E \subset X$. We may assume that $C_k(E) < \infty$. By definition we can find a measure μ such that $k(\cdot, \mu) \geq 1$ on E and $\|\mu\| \leq 2C_k(E)$. For each Q_j we let

$$\mu_j = \mu|_{Q_j^*}, \qquad \mu_j' = \mu|_{X \setminus Q_j^*}.$$

We have the following two cases:

(a) $k(x, \mu_j) \geq \dfrac{1}{2}$ for all $x \in E \cap Q_j$.

(b) $k(x, \mu_j') \geq \dfrac{1}{2}$ for some $x \in E \cap Q_j$.

If (a) holds, then $C_k(E \cap Q_j) \leq 2\|\mu_j\|$ by definition. Since Q_j^* do not overlap so often, we obtain

$$\sum{}' C_k(E \cap Q_j) \leq 2 \sum{}' \|\mu_j\| \leq M\|\mu\| \leq MC_k(E), \tag{5}$$

where $\sum'$ denotes the summation over all Q_j for which (a) holds. If (b) holds, then the Harnack property of k yields that $k(\cdot, \mu) \geq k(\cdot, \mu_j') \geq M$ on Q_j, so that

$$k(\cdot, \mu) \geq M \text{ on } \cup'' \, Q_j,$$

where $\cup''$ denotes the union over all Q_j for which (b) holds. Hence

$$C_k(\cup'' Q_j) \leq M\|\mu\|.$$

Since σ is comparable to C_k, it follows from (1), (2) and the countable additivity of σ that

$$\begin{aligned} \sum{}'' C_k(E \cap Q_j) &\leq \sum{}'' C_k(Q_j) \leq M \sum{}'' \sigma(Q_j) \\ &\leq M\sigma(\cup'' Q_j) \leq MC_k(\cup'' Q_j) \leq M\|\mu\| \leq MC_k(E). \end{aligned}$$

This, together with (5), completes the proof.

Corollary 3. *Let k, $\{Q_j, Q_j^*\}$ and σ be as in Theorem 1. If E is a union of Q_j, then $C_k(E) \approx \sigma(E)$.*

Remark. It is not so difficult to give an L^p capacity version of Theorem 1. For $1 < p < \infty$ and a fixed measure ν on X we let

$$C_{k,p}(E) = \inf\{\int_X f^p d\nu : k(\cdot, f\nu) \geq 1 \text{ on } E,\ f \geq 0\}.$$

Then the same argument yields that if k has the Harnack property with respect to $\{Q_j, Q_j^*\}$ and there is a measure σ comparable to C_p with respect to $\{Q_j, Q_j^*\}$, then

$$C_{k,p}(E) \approx \sum C_{k,p}(E \cap Q_j).$$

See [2, Theorem 1].

3. Proof of Theorem 2

In this section we let D be a uniformly Δ-regular domain. Let $\{Q_j\}$ be the Whitney decomposition of D and let Q_j^* be the double of Q_j (see [20, Chapter VI]). Then $\{Q_j, Q_j^*\}$ is a quasidisjoint decomposition of D. Throughout this section we suppose that a positive superharmonic function u satisfies (4).

We shall show that σ_u is comparable to γ_u with respect to $\{Q_j\}$. We can easily verify the first condition (1) by recalling the identity $\gamma_u(E) = C_k(E)$, where $k(x, y) = G(x, y)/(u(x)u(y))$.

Lemma 1. *Let Q_j be a Whitney cube and let r_j and u_j be as in the introduction. If $E \subset Q_j$, then*

$$\gamma_u(E) \approx \begin{cases} u_j^2 \operatorname{cap}(E) & \text{if } d \geq 3, \\ \dfrac{u_j^2}{\log(4r_j/\operatorname{cap}(E))} & \text{if } d = 2. \end{cases}$$

In particular, $\gamma_u(Q_j) \approx \sigma_u(Q_j) \approx u_j^2 r_j^{d-2}$.

Proof. From (4), we deduce the estimate

$$k(x, y) \approx \begin{cases} u_j^{-2}|x - y|^{2-d} & \text{if } d \geq 3, \\ u_j^{-2} \log \dfrac{4r_j}{|x - y|} & \text{if } d = 2, \end{cases}$$

for $x, y \in Q_j$, which proves the lemma.

Let K be a compact subset of D. Then (4) yields that u is bounded on K. Hence $\widehat{R}_u^K$ is a bounded Green potential of a measure supported on K. By the next lemma we can apply Lemma A to $\widehat{R}_u^K$.

Lemma 2. *Let K be a compact subset of D. Suppose that μ is a measure on K and that $v = G(\cdot, \mu)$ is a bounded Green potential. Then $v \in W_0^{1,2}(D)$ and*

$$\iint G(x,y)d\mu(x)d\mu(y) = \int_D |\nabla v|^2 dx.$$

In fact, Lemma 2 can be extended to more general situations (see e.g. [7, Satz 7.2]); but the above form is sufficient for our purpose. Let us now prove (2).

Lemma 3. *Let E be a Borel subset of D. Then*

$$\sigma_u(E) \le M\gamma_u(E).$$

Proof. Let K be a compact subset of E and write $v_K = \widehat{R}_u^K = G(\cdot, \lambda_u^K)$. Then

$$\gamma_u(K) = \iint G(x,y)d\lambda_u^K(x)d\lambda_u^K(y) = \int_D |\nabla v_K|^2 dx$$

by Lemma 2. Since $v_K = u$ q.e. on K and hence a.e. on K, it follows from Lemmas A and 2 that

$$\gamma_u(E) \ge \gamma_u(K) = \int_D |\nabla v_K|^2 dx \ge M \int_D \left(\frac{v_K}{\delta}\right)^2 dx \ge M \int_K \left(\frac{u}{\delta}\right)^2 dx = M\sigma_u(K).$$

Since $K \subset E$ is an arbitrary compact set, we have the required inequality.

Proof of Theorem 2. We have seen in Lemmas 1 and 3 that the measure σ_u is comparable to γ_u with respect to $\{Q_j\}$. Suppose $y \in D \setminus Q_j^*$. Then $G(\cdot, y)$ is a positive harmonic function in the interior of Q_j^*. Hence the Harnack principle yields that, for $x, x' \in Q_j$, $G(x,y) \approx G(x',y)$, so that $k(x,y) \approx k(x',y)$ by (4). Thus the kernel k has the Harnack property with respect to $\{Q_j, Q_j^*\}$. Therefore, the quasiadditivity of the Green energy γ_u follows from Theorem 1.

Remark. If D is a Liapunov domain, then $g(x) \approx \delta(x)$ and Corollary 2 follows from the weak L^1 estimate of Naïm's Θ kernel (cf. [2], [18], [19] and [21]). For a general uniformly Δ-regular domain, however, the Θ kernel is not necessarily weak $(1,1)$; it may not be even a standard kernel.

4. Minimally thin sets in an NTA domain

In this section we let D be an NTA domain in $\mathbb{R}^d$. Let $D_1 = \{x \in D : G(x, x_0) > 1\}$. By definition $g(x) = G(x, x_0)$ for $x \in D \setminus D_1$. Since the Martin boundary of D is homeomorphic to the Euclidean boundary ∂D and every boundary point is minimal, it follows that the ratio

$$K(x, y) = \frac{G(x, y)}{g(y)}$$

becomes continuous in the extended sense on $D \times \overline{D}$. By the same symbol we denote the continuous extension on $D \times \overline{D}$. Sometimes we write K_y for $K(\cdot, y)$. By definition $K_y(x_0) = 1$ for $y \in \overline{D} \setminus D_1$. If $y \in \partial D$, then K_y is a minimal harmonic function on D. Let $E \subset D$ and let $y \in \partial D$. We say that E is minimally thin at y if the regularized reduced function $\widehat{R}^E_{K_y}$ is a Green potential, or equivalently there is a finite measure μ on $\overline{D}$ such that $\mu(\{y\}) = 0$ and

$$K_y \le K(\cdot, \mu) \text{ on } E. \tag{6}$$

By the definition of Green energy we have

$$\widehat{R}^E_g(x_0) = \gamma_g(E) \tag{7}$$

for any set E. This, together with Corollary 2, yields a refined Wiener criterion for minimal thinness in terms of usual capacity. For each Whitney cube Q_j we let we put $R_j(y) = |x_j - y|$ and $A_j(y) = A_{R_j(y)}(y)$, where we recall x_j is the center of Q_j and that $A_r(y)$ is the point appearing in the corkscrew condition for $y \in \partial D$.

Theorem 3. *Suppose $y \in \partial D$ and $E \subset D$. Then E is minimally thin at y if and only if*

$$\sum \left(\frac{g_j}{g(A_j(y))} \right)^2 R_j(y)^{2-d} \operatorname{cap}(E \cap Q_j) < \infty \qquad \text{if } d \ge 3,$$

$$\sum \left(\frac{g_j}{g(A_j(y))} \right)^2 \frac{1}{\log(4r_j / \operatorname{cap}(E \cap Q_j))} < \infty \qquad \text{if } d = 2.$$

Corollary 4. *Let $y \in \partial D$. If a measurable set E is minimally thin at y, then*

$$\int_E \left(\frac{g(x)}{g(A_{|x-y|}(y))} \right)^2 \frac{|x - y|^{2-d}}{\delta(x)^2} dx < \infty. \tag{8}$$

Remark. For a half space Essén [9] has first introduced the refined Wiener criterion. (See also [2].) He used the weak L^1 estimate due to Sjögren [19]. As remarked before, the weak L^1 estimate needs not hold for an NTA domain.

Definition. Suppose $y \in \partial D$ and $E \subset D$. We say that E is minimally thin at y for harmonic functions if there is a finite measure μ concentrated on ∂D such that $\mu(\{y\}) = 0$ and (6) holds.

Theorem 4. *Suppose $y \in \partial D$ and $E \subset D$. Then E is minimally thin at y for harmonic functions if and only if*

$$\sum_{E \cap Q_j \neq \emptyset} \left(\frac{g_j}{g(A_j(y))} \right)^2 \left(\frac{r_j}{R_j(y)} \right)^{d-2} < \infty.$$

Corollary 5. *Let $y \in \partial D$ and let $0 < \rho < 1$. If E is minimally thin at y for harmonic functions, then $E_\rho = \bigcup_{x \in E} B(x, \rho\delta(x))$ satisfies (8) with E_ρ replacing E.*

Remark. A set E is said to determine the point measure at $y \in \partial D$ if, for every finite measure μ concentrated on ∂D, (6) implies that $\mu(\{y\}) > 0$ ([6], [8], [17] and [19]). We note that E is minimally thin at y for harmonic functions if and only if E does not determine the point measure at y. Obviously if E is minimally thin at y for harmonic functions, then it is minimally thin; but the converse is not necessarily true.

Let us remark that Hayman [13, p.481 and Theorem 7.37] defined sets "rarefied for harmonic functions". We note that the term "rarefied sets" was introduced by Essén and Jackson [11]. A set is "rarefied" in the terminology of Lelong-Ferrand [16] if and only if it is "semirarefied" in the terminology of Essén and Jackson. A set rarefied for harmonic functions corresponds to a rarefied set defined by Essén and Jackson. See [10] for further information.

Remark. A sequence $\{z_i\} \subset D$ is said to be *separated* if $|z_i - z_{i'}| \geq M\delta(z_i)$ for $i \neq i'$ with some positive constant M. It is easy to see that $\{z_i\}$ is separated if and only if the number of points z_i included in a Whitney cube Q_j is bounded by a positive constant independent of Q_j. Thus Theorem 4 implies Ancona's criterion for a set to determine the point measure ([4, Theorem 7.4]). Ancona first indicated that the Hardy inequality can be used. His short description of the proof was stated only for Lipschitz domains, but it seems to work for NTA domains. He also gave further discussions for hyperbolic graphs [4] and Denjoy domains [5].

Remark. If D is a Liapunov domain, then $g(x) \approx \delta(x)$ ([21]), so that Theorems 3, 4 and their Corollaries are generalization of the results of Beurling [6], Maz'ya [17], Dahlberg [8], Sjögren [19] and Essén [10, Section 2].

5. Proofs of Theorems 3 and 4

In this section let D be an NTA domain. Let $\Theta(x,y) = G(x,y)/(g(x)g(y))$. It is known that $\Theta(x,y)$ has a continuous extension on $\overline{D}\times\overline{D}$. By the same symbol we denote the continuous extension. The kernel Θ is referred to as the Naïm's Θ kernel for D ([18]). By definition Θ is symmetric.

Lemma 4. *For $0 < R < r_1$ and $y \in \partial D$ we let $\theta_y(R) = \Theta(A_R(y), y)$ with $A_R(y)$ appearing in the corkscrew condition. Then*

$$\theta_y(R) \approx R^{2-d} g(A_R(y))^{-2}. \tag{9}$$

If $x \in D$ and $|x-y| = R$, then $\Theta(x,y) \approx \theta_y(R)$. Moreover, if $r \approx R$, then $\theta_y(r) \approx \theta_y(R)$.

Proof. This is essentially proved in [1, Lemma 3.3]; but for the sake of convenience we give a proof. We shall use the same notation as in [14]; $\Delta(y,r)$ stands for the surface ball $B(y,r)\cap\partial D$ and $\omega^x(\cdot)$ for the harmonic measure of D evaluated at x. If $x = x_0$, then we write simply $\omega(\cdot)$ for $\omega^{x_0}(\cdot)$.

It is well known that

$$K(x,y) = \lim_{r\to 0} \frac{\omega^x(\Delta(y,r))}{\omega(\Delta(y,r))}$$

(cf. [14, Theorem 5.5]). If we carefully consider the above convergence, then we have, for $0 < r \le R$,

$$K(A_R(y), y) \approx \frac{\omega^{A_R(y)}(\Delta(y,r))}{\omega(\Delta(y,r))}.$$

On the other hand, if $r \approx R$, then $\omega^{A_R(y)}(\Delta(y,r)) \approx 1$. Therefore, letting $\frac{1}{2}R \le r \le R$, we obtain from [14, Lemma 4.8] and the Harnack principle that

$$K(A_R(y), y) \approx \frac{1}{\omega(\Delta(y,r))} \approx \frac{1}{r^{d-2} g(A_r(y))} \approx R^{2-d} g(A_R(y))^{-1}.$$

By definition (9) follows. Moreover the boundary Harnack principle (cf. [14, Lemma 4.10]) says that if $x \in D$ and $|x-y| = R$, then

$$\frac{K(x,y)}{K(A_R(y),y)} \approx \frac{g(x)}{g(A_R(y))},$$

which implies that $\Theta(x,y) \approx \Theta(A_R(y), y) = \theta_y(R)$. The last assertion follows from the Harnack principle. The lemma is proved.

Proof of Theorem 3. As proved in the same way as in [1], we can show that E is minimally thin at y if and only if

$$\sum_i \widehat{R}_{K_y}^{E_i}(x_0) < \infty,$$

where $E_i = \{x \in E : 2^{-i} \le |x-y| < 2^{1-i}\}$. By Lemma 4 we see that $K_y \approx \theta_y(2^{-i})g$ on $\{x \in D : 2^{-i} \le |x-y| < 2^{1-i}\}$. Hence it follows from (7) that the above condition is equivalent to

$$\sum_i \theta_y(2^{-i})\gamma_g(E_i) < \infty.$$

Invoking Corollary 2, we can rewrite the above condition as

$$\sum_j \theta_y(R_j(y))\gamma_g(E \cap Q_j) < \infty, \tag{10}$$

where we recall $R_j(y) = |x_j - y|$ and x_j is the center of Q_j. Hence Lemmas 1 and 4 assert that (10) leads to the condition of Theorem 3. Thus the theorem is proved.

Proof of Corollary 4. In view of Lemmas 3 and 4, we see that (10) implies that

$$\sum R_j(y)^{2-d} g(A_{R_j(y)}(y))^{-2} \frac{g_j^2}{t_j^2} \int_{E \cap Q_j} dx < \infty,$$

which is equivalent to (8).

Proof of Theorem 4. It is not so difficult to see that E is minimally thin at y for harmonic functions if and only if $\widetilde{E} = \cup_{E \cap Q_j \ne \emptyset} Q_j$ is minimally thin at y ([10]). Hence Lemma 1 and Theorem 3 readily imply the theorem.

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