

POTENTIAL THEORETIC CHARACTERIZATIONS OF NONSMOOTH DOMAINS

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Dedicated to the Memory of Professor Hajime Nishimura

A . We give necessary and sufficient conditions in terms of potential theoretic properties for a bounded domain satisfying the capacity density condition to be a John domain, a uniform domain and an inner uniform domain, respectively.

1. I

A large number of works have been devoted to the study of potential theory for nonsmooth domains, such as Lipschitz domains, NTA domains, uniform domains, John domains and Hölder domains, since the pioneering works of Carleson [12] and Hunt-Wheeden [18, 19] for Lipschitz domains. Among them, the boundary Harnack principle (abbreviated to BHP) and the identification of the Martin boundary played a central role. See Ancona [5], Dahlberg [13] and Wu [27] for Lipschitz domains, Jerison-Kenig [20] for NTA domains, Bass-Burdzy [10] for Hölder domains, [1] and [4] for uniform domains and uniformly John domains. See [3] for a survey. These studies are motivated by the desire to generalize the geometric conditions imposed on a domain to guarantee potential theoretic properties, such as the boundary Harnack principle.

This paper is in the opposite direction. Namely, we shall give some potential theoretic properties which yield geometric properties of the domain. Our conditions will be necessary and sufficient, provided the domain satisfies the capacity density condition. To be more precise, we use the following notation. Throughout the paper we let D be a bounded domain in \mathbb{R}^n with $n \geq 2$ and let $\delta_D(x) = \text{dist}(x, \partial D)$. We write $B(x, r)$ and $S(x, r)$ for the open ball and the sphere of center at x and radius r , respectively. By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use A_0, A_1, \dots , to specify them. We shall say that two positive quantities f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \geq 1$ such that $A^{-1}f_1 \leq f_2 \leq Af_1$. The constant A will be called the constant of comparison.

Let us recall the definitions of some nonsmooth domains. We say that D is a *John domain* with John constant $c_J > 0$ and John center $x_0 \in D$ if each $x \in D$ can be joined to x_0 by a

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rectifiable curve γ such that

$$(1) \quad \delta_D(y) \geq c_J \ell(\gamma(x, y)) \quad \text{for all } y \in \gamma,$$

where $\gamma(x, y)$ and $\ell(\gamma(x, y))$ stand for the subarc of γ connecting x and y and its length, respectively. In general, $0 < c_J < 1$. A John domain may be visualized as a domain satisfying a twisted cone condition. We say that D is a *uniform domain* if there exist constants A and A' such that each pair of points $x, y \in D$ can be joined by a rectifiable curve $\gamma \subset D$ such that $\ell(\gamma) \leq A|x - y|$ and

$$(2) \quad \min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\} \leq A\delta_D(z) \quad \text{for all } z \in \gamma.$$

See Gehring-Martio [15], Gehring-Osgood [16] and Väisälä [25]. Let $0 < c < 1$. For $x, y \in D$, a *Harnack chain* from x to y in D is a sequence $\{B(x_j, c\delta_D(x_j))\}_{j=0}^N$ such that $x_0 = x$, $x_N = y$ and $B(x_{j-1}, c\delta_D(x_{j-1})) \cap B(x_j, c\delta_D(x_j)) \neq \emptyset$ for $j = 1, \dots, N$. The number N is referred to as the length of the Harnack chain. The value of c is not important and we may take $c = 1/2$. We say that D is a *nontangentially accessible domain* or an *NTA domain* if there exist constants A and r_0 with the following property:

- (i) Corkscrew condition. For any $\xi \in \partial D$ and $0 < r < r_0$, there exists a point $x \in D \cap B(\xi, r)$ such that $\delta_D(x) \approx r$.
- (ii) The complement of D satisfies the corkscrew condition.
- (iii) Harnack chain condition. If $\varepsilon > 0$, and $x, y \in D$ with $\delta_D(x) > \varepsilon$, $\delta_D(y) > \varepsilon$ and $|x - y| < C\varepsilon$, then there exists a Harnack chain from x to y whose length depends on C , but not ε .

See Jerison-Kenig [20]. It is easy to see that if the complement of a uniform domain D satisfies the corkscrew condition, then D becomes an NTA domain. We define the *inner length distance* $\lambda_D(x, y)$ by

$$\lambda_D(x, y) = \inf\{\ell(\gamma) : \gamma \text{ is a curve connecting } x \text{ and } y \text{ in } D\}$$

for $x, y \in D$. We say that D is an *inner uniform domain* if there exists a constant $A > 1$ such that each pair of points $x, y \in D$ can be connected by a curve $\gamma \subset D$ with $\ell(\gamma) \leq A\lambda_D(x, y)$ and (2). This definition is due to Bonk-Heinonen-Koskela [11]. They showed that an inner uniform domain is a Gromov hyperbolic domain. The *inner diameter metric* or *internal metric* $\rho_D(x, y)$ is similarly defined by

$$\rho_D(x, y) = \inf\{\text{diam}(\gamma) : \gamma \text{ is a curve connecting } x \text{ and } y \text{ in } D\},$$

where $\text{diam}(\gamma)$ denotes the diameter of γ . We say that D is a *uniformly John domain* if there exists a constant $A > 1$ such that each pair of points $x, y \in D$ can be connected by a curve $\gamma \subset D$ with $\ell(\gamma) \leq A\rho_D(x, y)$ and (2). This definition is due to Balogh-Volberg [8, 9]. Obviously $|x - y| \leq \rho_D(x, y) \leq \lambda_D(x, y)$. It turns out that they $\rho_D(x, y) \approx \lambda_D(x, y)$ for a John domain (Väisälä [26, Theorem 3.4]). Hence an inner uniform domain is a uniformly John domain and vice versa. The relationships among above domains are summarized as

$$\text{NTA} \subsetneq \text{Uniform} \subsetneq \text{Inner uniform} = \text{Uniformly John} \subsetneq \text{John}.$$

Though we used the terminology “uniformly John domain” in the previous paper [4], we shall use “inner uniform domain” in this paper, since it is closely linked with uniform domains than with John domains (see Theorems 2 and 3 below).

Now we consider potential theoretic properties and their connections to the above nonsmooth domains.

Definition 1. We say that a domain D enjoys the *uniform BHP* if there exist constants $A_0, A_1 > 1$ and $r_0 > 0$ depending only on D with the following property: Let $\xi \in \partial D$ and let $0 < r < r_0$. Suppose u and v are positive harmonic functions on $D \cap B(\xi, A_0 r)$, bounded on $D \cap B(\xi, A_0 r)$ and vanishing on $\partial D \cap B(\xi, A_0 r)$ except for a polar set. Then

$$(3) \quad \frac{u(x)/u(y)}{v(x)/v(y)} \leq A_1, \quad \text{whenever } x, y \in D \cap B(\xi, r).$$

In [1, Theorem 1], we proved the following.

Theorem A. *A uniform domain satisfies the uniform BHP.*

We shall give a converse of Theorem A. The definition of a uniform domain is very sensitive about the boundary; if we remove a closed polar set from the domain D , then D may not be a uniform domain, whereas the uniform BHP remains to hold. So, we need some additional assumption on the boundary in order to obtain a converse. The following *capacity density condition* (abbreviated to CDC) is reasonable and widely known.

Definition 2. By Cap we denote the logarithmic capacity if $n = 2$, and the Newtonian capacity if $n \geq 3$. We say that the CDC holds if there exist constants $A > 1$ and $r_0 > 0$ such that

$$\text{Cap}(B(\xi, r) \setminus D) \geq \begin{cases} Ar & \text{if } n = 2, \\ Ar^{n-2} & \text{if } n \geq 3, \end{cases}$$

whenever $\xi \in \partial D$ and $0 < r < r_0$.

See Armitage-Gardiner [7, Chapter 5] for the definition and the properties of capacities. Since $\text{Cap}(B(\xi, r)) = Ar$ if $n = 2$ and $\text{Cap}(B(\xi, r)) = Ar^{n-2}$ if $n \geq 3$, we can rewrite the above condition as

$$\frac{\text{Cap}(B(\xi, r) \setminus D)}{\text{Cap}(B(\xi, r))} \geq \frac{1}{A}.$$

It is easy to see that if the volume of $B(\xi, r) \setminus D$ is comparable to r^n , then the CDC holds.

Remark 1. The CDC can be also stated in terms of Green capacity. Let U be an open set with Green function G_U , i.e., for each fixed $y \in U$, $-\Delta G(\cdot, y) = \delta_y$ in U in the distribution sense and $G(\cdot, y)$ vanishes on ∂U except for a polar set. The Green capacity $\text{Cap}_U(E)$ for a Borel set $E \subset U$ is defined by

$$\text{Cap}_U(E) = \sup\{\mu(E) : G_U \mu \leq 1 \text{ on } U, \mu \text{ is a Borel measure supported on } E\}.$$

In the usual way $\text{Cap}_U(E)$ extends to a general set $E \subset U$. Then the CDC holds if and only if

$$\frac{\text{Cap}_{B(\xi, 2r)}(B(\xi, r) \setminus D)}{\text{Cap}_{B(\xi, 2r)}(B(\xi, r))} \geq \frac{1}{A}.$$

In the previous paper [1, 2, 4] we adopted this characterization. This has the advantage of unified treatments of the cases $n = 2$ and $n \geq 3$. For example $\text{Cap}_{B(\xi, 2r)}(B(\xi, r)) = Ar^{n-2}$ for $n \geq 2$ with A depending only on the dimension n . See also the capacity width in Lemma 2 below.

Remark 2. If D satisfies the CDC, then D is regular and the assumption on u and v for the BHP becomes that u and v are positive and harmonic on $D \cap B(\xi, A_0r)$ and continuously vanish on $\partial D \cap B(\xi, A_0r)$. In the sequel, we assume the CDC and use this simplified assumption.

Ancona [6, Lemma 3] showed that the CDC has an equivalent formulation in terms of harmonic measure. We write $\omega(\cdot, E, U)$ for the harmonic measure over an open set U of $E \subset \partial U$, i.e., $\omega(\cdot, E, U)$ is the Dirichlet solution in U of the boundary function χ_E (see e.g. [7, Chapter 6]). The CDC holds if and only if there exist constants $\beta > 0$, $A > 1$ and $r_0 > 0$ such that

$$(4) \quad \omega(x, D \cap S(\xi, r), D \cap B(\xi, r)) \leq A \left(\frac{|x - \xi|}{r} \right)^\beta \quad \text{for } x \in D \cap B(\xi, r),$$

whenever $\xi \in \partial D$ and $0 < r < r_0$. See also [2] for the connection to the Dirichlet problem. Under the assumption of the CDC, we can characterize a John domain, a uniform domain and a uniformly John domain in terms of potential theoretical properties.

Theorem 1. *Let D satisfy the CDC. Then D is a John domain if and only if there exist constants $\alpha > 0$, $A > 1$ and $r_0 > 0$ such that*

$$(5) \quad \omega(x, D \cap S(\xi, r), D \cap B(\xi, r)) \geq \frac{1}{A} \left(\frac{\delta_D(x)}{r} \right)^\alpha \quad \text{for } x \in D \cap B(\xi, \frac{r}{A}),$$

whenever $\xi \in \partial D$ and $0 < r < r_0$.

Remark 3. In general, $0 < \beta < 1 < \alpha$. If D is smooth, then we may let $\alpha = \beta = 1$.

The following theorem includes a converse of Theorem A.

Theorem 2. *Let D satisfy the CDC. Then D is a uniform domain if and only if the uniform BHP and (5) hold.*

Remark 4. Jerison-Kenig [20] developed a fruitful potential theory on an NTA domain, including the BHP and the Carleson estimate (See Definition 4 below). An NTA domain can be regarded as a uniform domain with the corkscrew condition of the complement. The requirement of the complement, needed for the construction of a uniform barrier, may be replaced by a more general condition, CDC. Thus the arguments of Jerison-Kenig remain to hold for a uniform domain with CDC. Our Theorem 2 asserts its converse, i.e., if a domain enjoys the CDC, the BHP and (5), then it must be a uniform domain. On the other hand, Theorem A does not rely on a uniform barrier and, as a result, it is valid for a uniform domain without CDC. Theorem A was derived in a spirit of Bass-Burdzy [10] rather than by the method of Jerison-Kenig. Characterization of a uniform domain without CDC remains open.

Remark 5. Uniform domains are known to be *nice* domains of analysis appearing in many contexts (Jones [21, 22], Gehring [14], Gotoh [17], Bonk-Heinonen-Koskela [11]), although their boundaries may be very complicated. Our characterization provides another example of this nature. See also Smith-Stegenga [23] and Stegenga-Ullrich [24] for characterizations of a Hölder domain, a domain satisfying the quasihyperbolic metric condition.

A ball of center at a boundary point with respect to the internal metric becomes a connected component of the intersection of a Euclidean ball and the domain ([4, Lemma 2.2]). So, we arrive at the following version of the uniform BHP, which is a property weaker than the uniform BHP.

Definition 3. We say that a domain D enjoys the *uniform BHP* with respect to the internal metric if there exist constants $A_2, A_3 > 1$ and $r_0 > 0$ depending only on D with the following property: Let $\xi \in \partial D$, $0 < r < r_0$, U a connected component of $D \cap B(\xi, r)$ and V a connected component of $D \cap B(\xi, A_2 r)$ including U . Suppose u and v are positive harmonic functions on V vanishing on $\partial D \cap \partial V$. Then

$$\frac{u(x)/u(y)}{v(x)/v(y)} \leq A_3, \quad \text{whenever } x, y \in U.$$

In [4, Theorem 3.1], we proved the following.

Theorem B. *An inner uniform domain satisfies the uniform BHP with respect to the internal metric.*

The following characterization includes a converse of Theorem B.

Theorem 3. *Let D satisfy the CDC. Then D is an inner uniform domain if and only if the uniform BHP with respect to the internal metric and (5) hold.*

It is known that a finitely connected planar domain without singleton boundary components satisfies the CDC. Hence we have the following corollaries.

Corollary 1. *Let D be a bounded finitely connected planar domain. Then the following statements hold:*

- (i) *D is a John domain if and only if (5) holds.*
- (ii) *D is a uniform domain if and only if the uniform BHP and (5) hold.*
- (iii) *D is an inner uniform domain if and only if the uniform BHP with respect to the internal metric and (5) hold.*

2. P T 1

We define the quasihyperbolic metric $k_D(x, y)$ by

$$k_D(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable curves γ connecting x to y in D . We observe that the shortest length of a Harnack chain connecting x and y is comparable to $k_D(x, y)$. Hence, in

view of the Harnack inequality, there is a positive constant A_4 depending only on the dimension n such that

$$(6) \quad \exp(-A_4 k_D(x, y)) \leq \frac{h(x)}{h(y)} \leq \exp(A_4 k_D(x, y))$$

for every positive harmonic function h on D .

Proof of Theorem 1. Necessity. Suppose D is a John domain with John constant c_J and John center x_0 . Let $\xi \in \partial D$ and $0 < r < r_0$ and let $\Omega = \omega(\cdot, D \cap S(\xi, r), D \cap B(\xi, r))$. Take $x \in D \cap B(\xi, r/A_5)$ with $A_5 > 1$. By definition we find a rectifiable curve γ connecting x and x_0 such that $\delta_D(y) \geq c_J \ell(\gamma(x, y))$ for any $y \in \gamma$. Let x^* be the first hit of γ to $S(\xi, r)$. We observe that $\delta_D(x^*) \geq c_J(1 - A_5^{-1})r$ and $k_D(x, x^*) \leq A \log(r/\delta_D(x)) + A$. We find $x' \in \gamma(x, x^*) \cap B(x^*, \delta_D(x^*)/2)$ with $k_{D \cap B(\xi, r)}(x, x') \leq A \log(r/\delta_D(x)) + A$. Then $\Omega(x') \approx 1$ and

$$\frac{\Omega(x')}{\Omega(x)} \leq \exp(A_4 k_{D \cap B(\xi, r)}(x, x')) \leq A \left(\frac{r}{\delta_D(x)} \right)^\alpha$$

with $\alpha > 0$ depending on D by (6). Hence (5) follows. \square

For the sufficiency we prepare the following two lemmas.

Lemma 1. *Suppose D satisfies the CDC. Let $\xi \in \partial D$ and $0 < r < r_0$ and let $\Omega = \omega(\cdot, D \cap S(\xi, r), D \cap B(\xi, r))$. Then*

$$\Omega(x) \leq 3^\beta A \left(\frac{\delta_D(x)}{r} \right)^\beta \quad \text{for } x \in D \cap B(\xi, r/3).$$

Proof. Let $x \in D \cap B(\xi, r/3)$. Then $\delta_D(x) < r/3$ and there is $\zeta \in \partial D$ with $\delta_D(x) = |x - \zeta|$. We observe that $B(\zeta, r/3) \subset B(\xi, r)$. The maximum principle over $D \cap B(\zeta, r/3)$ gives

$$\Omega(x) \leq \omega(x, D \cap S(\zeta, r/3), D \cap B(\zeta, r/3)) \leq A \left(\frac{\delta_D(x)}{r/3} \right)^\beta = 3^\beta A \left(\frac{\delta_D(x)}{r} \right)^\beta,$$

where the second inequality follows from the CDC applied to ζ and $r/3$ in place of ξ and r . \square

The following lemma ([1, Lemma 1]) includes a crucial estimate.

Lemma 2. *Let $0 < \eta < 1$. For an open set $U \subset \mathbb{R}^n$ we define the capacity width $w_\eta(U)$ by*

$$w_\eta(U) = \inf \left\{ r > 0 : \frac{\text{Cap}(B(x, r) \setminus U)}{\text{Cap}(B(x, r))} \geq \eta \quad \text{for all } x \in U \right\},$$

where Cap stands for the logarithmic capacity if $n = 2$, and the Newtonian capacity if $n \geq 3$. Then there is a positive constant A depending only on η and n with the following property: if $x \in U$ and $R > 0$, then

$$\omega(x, U \cap S(x, R), U \cap B(x, R)) \leq \exp \left(2 - A \frac{R}{w_\eta(U)} \right).$$

Proof of Theorem 1. Sufficiency. It is sufficient to show that there is a constant $A_6 \geq 2$ with the following property: Let $x \in D$ and $\xi \in \partial D$ be such that $|x - \xi| = \delta_D(x) < r_0/2$. Then there are a point $y \in D \cap B(\xi, A_6\delta_D(x))$ with $\delta_D(y) \geq 2\delta_D(x)$ and a curve $\tilde{x}\tilde{y}$ connecting x and y in $D \cap B(\xi, A_6\delta_D(x))$ such that $\delta_D(z) \geq \delta_D(x)/A_6$ for every $z \in \tilde{x}\tilde{y}$.

Let $A_6 \geq 2$ and put $\Omega = \omega(\cdot, D \cap S(\xi, 3A_6\delta_D(x)), D \cap B(\xi, 3A_6\delta_D(x)))$. Then (5) gives

$$\Omega(x) \geq \frac{1}{A} \left(\frac{\delta_D(x)}{3A_6\delta_D(x)} \right)^\alpha = \frac{1}{A} \left(\frac{1}{3A_6} \right)^\alpha.$$

Let 2ε be the right hand side and let U be the connected component of $\{z \in D \cap B(\xi, A_6\delta_D(x)) : \Omega(z) > \varepsilon\}$ containing x . Observe that $h = (\Omega - \varepsilon)/(1 - \varepsilon)$ is a harmonic function on U such that $0 \leq h \leq 1$ on U and $h = 0$ on $\partial U \cap B(\xi, A_6\delta_D(x))$. Hence, the maximum principle yields

$$h(x) \leq \omega(x, U \cap S(x, (A_6 - 1)\delta_D(x)), U \cap B(x, (A_6 - 1)\delta_D(x))).$$

We claim that there is a point $y \in U$ with $\delta_D(y) \geq 2\delta_D(x)$, provided A_6 is large. Suppose, to the contrary,

$$\delta_D(y) < 2\delta_D(x) \quad \text{for all } y \in U.$$

Then the CDC implies $w_\eta(U) \leq 4\delta_D(x)$ for some $\eta > 0$. In fact, for an arbitrary point $y \in U$, there is $\zeta \in \partial D \cap B(y, 2\delta_D(x))$ by assumption. Since $B(\zeta, 2\delta_D(x)) \setminus D \subset B(y, 4\delta_D(x)) \setminus D$, it follows from the homogeneity of Cap and the definition of the CDC that

$$\frac{\text{Cap}(B(y, 4\delta_D(x)) \setminus D)}{\text{Cap}(B(y, 4\delta_D(x)))} \geq \frac{\text{Cap}(B(\zeta, 2\delta_D(x)) \setminus D)}{A\text{Cap}(B(\zeta, 2\delta_D(x)))} \geq \eta$$

for some $\eta > 0$ independent of y . Hence $w_\eta(U) \leq 4\delta_D(x)$. Now, Lemma 2 yields that

$$h(x) \leq \exp\left(2 - A \frac{(A_6 - 1)\delta_D(x)}{4\delta_D(x)}\right) = \exp\left(2 - A \frac{(A_6 - 1)}{4}\right).$$

On the other hand

$$h(x) = \frac{\Omega(x) - \varepsilon}{1 - \varepsilon} \geq \frac{\varepsilon}{1 - \varepsilon} \geq \frac{1}{2A} \left(\frac{1}{3A_6} \right)^\alpha.$$

Hence, A_6 cannot be so large. In other words, the claim follows with sufficiently large A_6 . Since U is connected, we find a curve $\tilde{x}\tilde{y} \subset U$ connecting x and y . Let $z \in \tilde{x}\tilde{y}$. Then Lemma 1 implies that

$$\varepsilon \leq \Omega(z) \leq A \left(\frac{\delta_D(z)}{3A_6\delta_D(x)} \right)^\beta,$$

so that $\delta_D(z) \geq \delta_D(x)/A$ for some $A > 1$. Thus the theorem is proved. \square

3. P T 2

The BHP is closely related to the so-called Carleson estimate.

Definition 4. We say that the Carleson estimate (abbreviated to CE) holds if there is a positive constant A_7 with the following property: Suppose $\xi \in \partial D$, $0 < r < r_0$ and u is a positive

harmonic function in $D \cap B(\xi, A_7 r)$ vanishing on $\partial D \cap B(\xi, A_7 r)$. If $y \in D \cap B(\xi, r)$ and $\delta_D(y) \geq \varepsilon r$ with $0 < \varepsilon < 1$, then

$$u(x) \leq A_\varepsilon u(y) \quad \text{for } x \in D \cap B(\xi, r),$$

where $A_\varepsilon > 1$ depends only on D and ε .

This CE follows from the uniform BHP and (5).

Lemma 3. *Suppose that the uniform BHP and (5) hold. Then the CE holds with $A_7 = A_0$.*

Proof. Suppose $\xi \in \partial D$, $0 < r < r_0$ and u is a positive harmonic function in $D \cap B(\xi, A_0 r)$ vanishing on $\partial D \cap B(\xi, A_0 r)$. Let $\Omega = \omega(\cdot, D \cap S(\xi, A_0 r), D \cap B(\xi, A_0 r))$ and apply the BHP to u and Ω . Since $\Omega(x) \leq 1$, it follows from the BHP and (5) that

$$u(x) \leq Au(y) \left(\frac{A_0 r}{\delta_D(y)} \right)^\alpha \quad \text{for } x, y \in D \cap B(\xi, r).$$

Hence, if $\delta_D(y) \geq \varepsilon r$, then $u(x) \leq A\varepsilon^{-\alpha} u(y)$. Thus the lemma follows. \square

In the next lemma we give an upper estimate of the Green function $G_D(x, y)$ for D . This is a crucial step of the proof of Theorem 2. The main idea, a repeated application of the mean value property, comes from [2, Lemma 2]. From now on we suppress the subscript D and write G for the Green function for D for simplicity.

Lemma 4. *Suppose that the CE and the CDC hold. Then there is a positive constant τ such that*

$$G(x, y) \leq A \min\{\delta_D(x), \delta_D(y)\}^{2-n} \exp(-\tau k_D(x, y))$$

whenever $x, y \in D$ and $|x - y| \geq \frac{1}{2} \min\{\delta_D(x), \delta_D(y)\}$.

Proof. We may assume that $\delta_D(x) \leq \delta_D(y)$. Let $D_j = \{z \in D : k_D(x, z) < j\}$ and $M_j = \sup_{D \setminus D_j} G(x, \cdot)$. Obviously, $\{M_j\}$ is a nonincreasing sequence. We claim that there are positive constant $\kappa < 1$ and positive integers k and j_0 such that $M_j \leq \kappa M_{j-k}$ for $j \geq j_0$. Then we have $M_j \leq AM_{j_0} \kappa^{(j-j_0)/k}$ for $j \geq j_0$ by induction. Take j such that $y \in D_j \setminus D_{j-1}$; or in other words $j-1 \leq k_D(x, y) < j$. Since $M_{j_0} \approx \delta_D(x)^{2-n}$ with constant comparison depending only on j_0 , it follows from the claim that

$$G(x, y) \leq A \delta_D(x)^{2-n} \exp(-\tau k_D(x, y)) \quad \text{with } \tau = -k^{-1} \log \kappa,$$

so that the lemma follows.

Now we prove the claim. Let $u = G(x, \cdot)$. Take $z \in D \setminus D_j$ with sufficiently large j . It is easy to see that if k is sufficiently large (depending only on A_7), then $B(z, (1 - (16A_7)^{-1})\delta_D(z)) \cap D_{j-k} = \emptyset$. Hence

$$(7) \quad u \leq M_{j-k} \quad \text{on } B(z, (1 - (16A_7)^{-1})\delta_D(z)).$$

Take $\xi \in \partial D$ such that $|z - \xi| = \delta_D(z)$.

Case 1. $x \notin B(\xi, 4^{-1}\delta_D(z))$. In this case $u = G(x, \cdot)$ is a positive harmonic function on $D \cap B(\xi, 4^{-1}\delta_D(z))$. Hence the CE implies

$$u \leq Au(z^*) \leq AM_{j-k} \quad \text{on } D \cap B(\xi, (4A_7)^{-1}\delta_D(z)),$$

where z^* is the point on the line segment $\overline{z\xi}$ such that $|z^* - \xi| = (8A_7)^{-1}\delta_D(z)$. Here the second inequality follows from (7), since $z^* \in B(z, (1 - (16A_7)^{-1})\delta_D(z))$. By the maximum principle

$$u \leq AM_{j-k}\omega(\cdot, D \cap S(\xi, (4A_7)^{-1}\delta_D(z)), D \cap B(\xi, (4A_7)^{-1}\delta_D(z))) \quad \text{on } D \cap B(\xi, (4A_7)^{-1}\delta_D(z)),$$

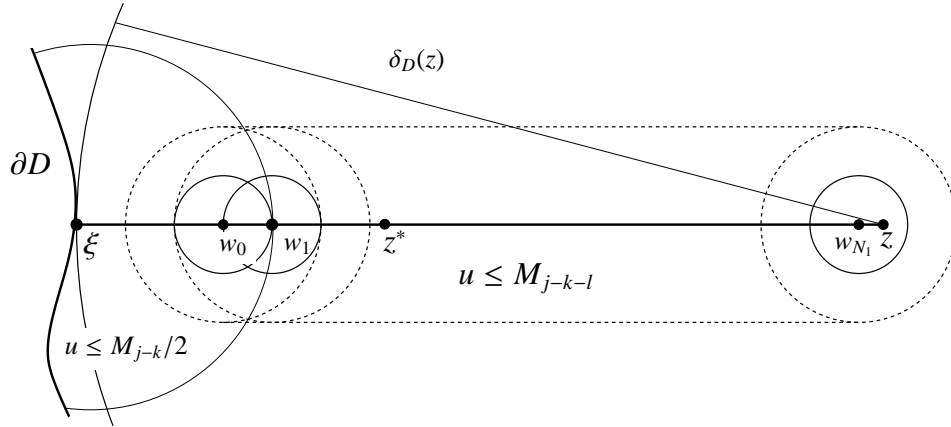
so that the CDC implies that

$$u \leq AM_{j-k} \left(\frac{4\varepsilon\delta_D(z)}{(4A_7)^{-1}\delta_D(z)} \right)^\beta = AM_{j-k}(16A_7\varepsilon)^\beta \quad \text{on } D \cap B(\xi, 4\varepsilon\delta_D(z))$$

for $0 < 4\varepsilon < (4A_7)^{-1}$. Letting ε be sufficiently small, we may assume that

$$u \leq \frac{1}{2}M_{j-k} \quad \text{on } D \cap B(\xi, 4\varepsilon\delta_D(z)).$$

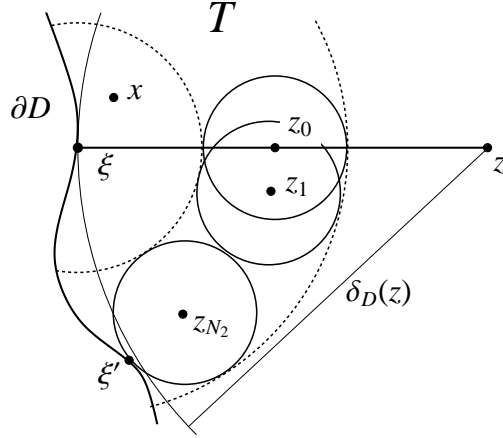
Let $w_i \in \overline{z\xi}$ be the point such that $|w_i - \xi| = (3 + i)\varepsilon\delta_D(z)$ for $i = 0, \dots, N_1$ with N_1 being the greatest integer such that $(3 + N_1)\varepsilon \leq 1$. Then we observe that there is an integer $l \geq 1$ such that $D \setminus D_{j-k-l}$ includes $\cup_{i=0}^{N_1} B(w_i, 2\varepsilon\delta_D(z))$; and hence $u \leq M_{j-k-l}$ there. Moreover, $u \leq \frac{1}{2}M_{j-k} \leq \frac{1}{2}M_{j-k-l}$ on $B(w_0, \varepsilon\delta_D(z)) \subset D \cap B(\xi, 4\varepsilon\delta_D(z)) \cap B(w_1, 2\varepsilon\delta_D(z))$. Hence the mean value property for harmonic functions yields $u \leq \kappa_1 M_{j-k-l}$ on $B(w_1, \varepsilon\delta_D(z))$, where $\kappa_1 < 1$ depends only on ε and the dimension n . Repeating this procedure, we obtain $u(z) \leq \kappa M_{j-k-l}$ with $\kappa < 1$ depending only on N_1, ε and n . Since $z \in D \setminus D_j$ is arbitrary, it follows that $M_j \leq \kappa M_{j-k-l}$. Thus the claim with $k + l$ in place of k follows in this case. See Figure 1.



F 1. Case 1.

Case 2. $x \in B(\xi, 4^{-1}\delta_D(z))$. In this case $u = G(x, \cdot)$ is a positive harmonic function on $D \setminus B(\xi, 4^{-1}\delta_D(z))$. Observe that $T = B(\xi, \frac{3}{4}\delta_D(z)) \setminus B(\xi, 4^{-1}\delta_D(z))$ can be represented as the union of open balls with centers on $S(\xi, 2^{-1}\delta_D(z))$ and the common radius $4^{-1}\delta_D(z)$. Let $z_0 \in \overline{z\xi}$ with $|z_0 - \xi| = 2^{-1}\delta_D(z)$. Observe that $B(z_0, 4^{-1}\delta_D(z)) \subset B(z, \delta_D(z)) \subset D$. By elementary geometry, we can find consecutive balls $B(z_i, 4^{-1}\delta_D(z)) \subset T$ with $|z_{i-1} - z_i| \leq 8^{-1}\delta_D(z)$ for $i = 1, \dots, N_2$ such that $S(z_{N_2}, 4^{-1}\delta_D(z))$ has a point $\xi' \in \partial D$. Here the number N_2 of balls depends only on n . Then

we observe that there is an integer $l \geq 1$ (possibly larger than that in Case 1) such that $D \setminus D_{j-k-l}$ includes $\cup_{i=0}^{N_2} B(z_i, 8^{-1}\delta_D(z))$ and hence $u \leq M_{j-k-l}$ there. We apply the same argument as in Case 1 for ξ' in place of ξ to obtain $u(z_{N_2}) \leq \kappa M_{j-k-l}$ and then $u(z) \leq \kappa M_{j-k-l}$ by the repeated application of the mean value property, where $\kappa < 1$ and $l \geq 1$ are possibly larger than those in Case 1. Thus the claim follows in this case, too. See Figure 2. \square



F 2. Case 2.

The constant A_0 in the definition of the BHP is, in general, very large. If we replace A_0 by a smaller constant, then (3) may not hold. The problem arises from the disconnectedness of $D \cap B(\xi, r)$. Although it is disconnected in general, it is included in one connected component of $D \cap B(\xi, A_0 r)$ by the BHP. This observation is important in the following modified BHP.

Lemma 5. *Let the uniform BHP hold. Then for each $0 < \varepsilon < 1/2$ there is a positive constant A_ε depending only on ε, A_0, A_1 and n with the following property: Let $\xi \in \partial D$ and $0 < r < r_0$. Suppose that there is $x^* \in D \cap B(\xi, r)$ with $\delta_D(x^*) \geq 2\varepsilon r$ and let U be a connected component of $D \cap B(\xi, r)$ containing x^* . If u and v are positive harmonic functions on $D \cap B(\xi, 2r) \setminus \{x^*\}$ vanishing on $\partial D \cap B(\xi, 2r)$, then*

$$(8) \quad \frac{u(x)/u(y)}{v(x)/v(y)} \leq A_\varepsilon$$

whenever $x, y \in U \setminus B(x^*, \varepsilon r)$.

Proof. We claim that if $\zeta \in \overline{U \setminus B(x^*, \varepsilon r)}$, then (8) holds for $x, y \in D \cap B(\zeta, (6A_0)^{-1}\varepsilon r)$. If $B(\zeta, (3A_0)^{-1}\varepsilon r) \subset D$, then this follows from the usual Harnack inequality. Otherwise, we find $\zeta' \in \partial D$ such that $|\zeta' - \zeta| < (3A_0)^{-1}\varepsilon r$. Observe that

$$B(\zeta', \frac{\varepsilon}{2}r) \subset B(\xi, (1 + \frac{\varepsilon}{3A_0} + \frac{\varepsilon}{2})r) \subset B(\xi, 2r)$$

and

$$|\zeta' - x^*| \geq \varepsilon r - \frac{\varepsilon}{3A_0} r > \frac{\varepsilon}{2} r.$$

Hence, u and v are positive harmonic functions on $D \cap B(\zeta', \varepsilon r/2)$ vanishing on $\partial D \cap B(\zeta', \varepsilon r/2)$, so that the BHP implies (8) for $x, y \in D \cap B(\zeta', (2A_0)^{-1} \varepsilon r)$, which includes $D \cap B(\zeta, (6A_0)^{-1} \varepsilon r)$. Thus the claim follows. Now we find points $\zeta_1, \dots, \zeta_N \in \overline{U \setminus B(x^*, \varepsilon r)}$ such that $\cup_{j=1}^N B(\zeta_j, (7A_0)^{-1} \varepsilon r)$ is connected and covers $U \setminus B(x^*, \varepsilon r)$, where N depends only on ε, A_0 and n . Applying the claim to ζ_j repeatedly, we obtain (8) for $x, y \in U \setminus B(x^*, \varepsilon r)$. The lemma is proved. \square

Remark 6. If $D \cap B(\xi, r)$ is disconnected, then we may not be able to find a *connected* covering $\cup_{j=1}^N B(\zeta_j, (7A_0)^{-1} \varepsilon r)$ of $D \cap B(\xi, r) \setminus B(x^*, \varepsilon r)$. Thus we have to restrict our attention to a connected component of $D \cap B(\xi, r)$. See Lemma 8 below.

Let us observe that a uniform domain is characterized in terms of the quasihyperbolic metric as follows: D is a uniform domain if and only if there are positive constants A and A' such that

$$(9) \quad k_D(x, y) \leq A \log \frac{\delta_D(x) + \delta_D(y) + |x - y|}{\min\{\delta_D(x), \delta_D(y)\}} + A' \quad \text{whenever } x, y \in D$$

(Gehring-Osgood [16]).

Proof of Theorem 2. The necessity follows from Theorems A and 1. Let us prove the sufficiency. Suppose the uniform BHP and (5) hold. Let $x, y \in D$. We shall prove (9). Without loss of generality we may assume that $\delta_D(y) \leq \delta_D(x)$ and $\delta_D(x)$ is sufficiently small. Suppose first $|x - y| \leq \frac{1}{2} \delta_D(x)$. Then x and y can be connected by the line segment $\overline{xy} \subset B(x, \frac{1}{2} \delta_D(x))$ and hence $k_D(x, y) \leq A$. Thus (9) follows. Suppose then $|x - y| \geq \frac{1}{2} \delta_D(x)$. Let $r = |x - y|$ and take $\xi \in \partial D$ with $\delta_D(x) = |x - \xi|$. Since $x, y \in D \cap B(\xi, 3r)$, it follows from the observation before Lemma 5 that x and y belong to the same connected component of $D \cap B(\xi, 3A_0 r)$. Theorem 1 yields that D is a John domain with John constant c_J and John center x_0 . Without loss of generality, we may assume that r is so small that $x_0 \notin B(\xi, 4A_0 r)$. Hence x is connected to x_0 by γ satisfying (1), so that we find $x^* \in \gamma \cap B(\xi, 4A_0 r)$ such that $\delta_D(x^*) \geq c_J A_0 r$ and x, y and x^* altogether belong to the same connected component U of $D \cap B(\xi, 4A_0 r)$. Let $R = 4A_0 r$. Apply Lemma 5 to $u = G(x^*, \cdot)$ and $v = \omega(\cdot, D \cap S(\xi, 2R), D \cap B(\xi, 2R))$ with $\varepsilon = c_J/8$ and R in place of r . We obtain

$$\frac{G(x^*, x')}{G(x^*, x)} \Big/ \frac{v(x')}{v(x)} \leq A \quad \text{for } x' \in S(x^*, c_J A_0 r/2).$$

Hence, (5), Lemmas 3 and 4 yield

$$r^{2-n} \approx G(x^*, x') \leq A \frac{v(x')}{v(x)} G(x^*, x) \leq A \left(\frac{2R}{\delta_D(x)} \right)^\alpha \delta_D(x)^{2-n} \exp(-\tau k_D(x, x')).$$

Since $R \approx r$, it follows that

$$k_D(x, x^*) \leq \frac{n-2+\alpha}{\tau} \log \frac{r}{\delta_D(x)} + A.$$

The same inequality holds for y in place of x , and hence

$$k_D(x, y) \leq k_D(x, x^*) + k_D(x^*, y) \leq \frac{2(n-2+\alpha)}{\tau} \log \frac{|x-y|}{\delta_D(y)} + A.$$

Thus (9) follows. The theorem is proved. \square

4. P T 3

The necessity is given in Theorem B. So, we have only to show the sufficiency. Since the proof can be carried out almost analogous way to that of Theorem 2, we give only a sketch. The same arguments as for Lemmas 3 and 4 give

Lemma 6. *Suppose that the CDC, the uniform BHP with respect to the internal metric and (5) hold. Then there is a positive constant τ such that*

$$G(x, y) \leq A \min\{\delta_D(x), \delta_D(y)\}^{2-n} \exp(-\tau k_D(x, y))$$

whenever $x, y \in D$ and $|x - y| \geq \frac{1}{2} \min\{\delta_D(x), \delta_D(y)\}$.

We need an estimate of the number of components of the intersection of an open ball and a John domain.

Lemma 7. *Suppose D is a John domain with John constant c_J and John center x_0 . Let $A > 1$. If $\zeta \in \overline{D}$ and $0 < r < r_0$ with small $r_0 > 0$, then the number of connected components of $D \cap B(\zeta, Ar)$ intersecting $D \cap B(\zeta, r)$ is bounded by an integer depending only on A , n and the John constant c_J of D .*

Proof. Let U be a connected component of $D \cap B(\zeta, Ar)$ intersecting $D \cap B(\zeta, r)$. If $\delta_D(\zeta) \geq r$, then $D \cap B(\zeta, r) = B(\zeta, r)$ and U includes $B(\zeta, r)$, so that there is just one such connected component. Suppose, otherwise, $0 \leq \delta_D(\zeta) < r < r_0$. Making r_0 small, if necessary, we may assume that x_0 does not lie in $B(\zeta, Ar)$. Take $x \in U \cap D \cap B(\zeta, r)$. Then x can be connected to x_0 by a curve γ satisfying (1). So, we find $y \in \gamma \cap U$ with $\delta_D(y) \geq c_J(A-1)r$, and hence the volume of U is comparable to r^n . Therefore, the number of such U is bounded by a constant depending only on A , c_J and n . \square

Remark 7. Note that $D \cap B(\zeta, Ar)$ may have infinitely many connected components, which accumulate to $S(\zeta, Ar)$.

Using Lemma 7, we obtain a counterpart of Lemma 5.

Lemma 8. *Let the uniform BHP with respect to the internal metric, the CDC and (5) hold. Then for each $0 < \varepsilon < 1/2$ there is a positive constant A_ε depending only on ε , A_2 , A_3 and n with the following property: Let $\xi \in \partial D$, $0 < r < r_0$, U a connected component of $D \cap B(\xi, r)$ and V a connected component of $D \cap B(\xi, 2r)$ including U . Suppose that there is $x^* \in U$ with $\delta_D(x^*) \geq 2\varepsilon r$. If u and v are positive harmonic functions on $V \setminus \{x^*\}$ vanishing on $\partial D \cap \partial V$, then (8) holds for $x, y \in U \setminus B(x^*, \varepsilon r)$.*

Proof. Let $\zeta \in \overline{U \setminus B(x^*, \varepsilon r)}$ and V a connected component of $D \cap B(\zeta, (6A_2)^{-1}\varepsilon r)$. Then the same argument as in the proof of Lemma 5 shows

$$(10) \quad \frac{u(x)/u(y)}{v(x)/v(y)} \leq A \quad \text{for } x, y \in V.$$

Now we find $\zeta_1, \dots, \zeta_N \in \overline{U \setminus B(x^*, \varepsilon r)}$ such that $\cup_{j=1}^N B(\zeta_j, (7A_2)^{-1}\varepsilon r)$ covers $U \setminus B(x^*, \varepsilon r)$, where N depends only on ε, A_2 and n . Let W be a connected component of $D \cap B(\zeta_j, (7A_2)^{-1}\varepsilon r)$ intersecting $U \setminus B(x^*, \varepsilon r)$. Then W is included in some connected component V of $D \cap B(\zeta_j, (6A_2)^{-1}\varepsilon r)$. Lemma 7 implies that the number of such components V is bounded, since D is a John domain by Theorem 1. This applies to each $D \cap B(\zeta_j, (7A_2)^{-1}\varepsilon r)$ intersecting $U \setminus B(x^*, \varepsilon r)$ for $j = 1, \dots, N$. Hence we enumerate these V as V_1, \dots, V_M and obtain a connected covering $\cup_{i=1}^M V_i$ of $U \setminus B(x^*, \varepsilon r)$, where M depends only on ε, A_2 and n . This, together with (10), yields the lemma. \square

Proof of Theorem 3. The necessity follows from Theorems B and 1. We have only to prove the sufficiency. Suppose the uniform BHP with respect to the internal metric and (5) hold. Let $x, y \in D$. We may assume that $r = \rho_D(x, y) \geq \frac{1}{2}\delta_D(x)$ and r is sufficiently small. Take $\xi \in \partial D$ with $\delta_D(x) = |x - \xi|$. Then x and y belong to the same connected component of $D \cap B(\xi, 3r)$. Since D is a John domain with John constant c_J for some $0 < c_J < 1$ by Theorem 1, we find $x^* \in \gamma \cap B(\xi, 4A_0r)$ such that $\delta_D(x^*) \geq c_J A_0 r$ and x, y and x^* altogether belong to the same connected component of $D \cap B(\xi, 4A_0r)$. Use Lemmas 6 and 8 instead of Lemmas 4 and 5. Then the rest of the proof is verbatim as in the proof of Theorem 2. \square

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