HOLOMORPHIC SOLUTIONS OF SEMILINEAR HEAT EQUATIONS

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1. Introduction

Let us consider the following semilinear heat equation:

\[ \partial_t u - \Delta u = P(u, \overline{u}) \quad \text{for} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \]

\[ u(0, x) = \varphi(x) \quad \text{for} \quad x \in \mathbb{R}^n \]

with \( \varphi \in L^p(\mathbb{R}^n) \), where \( P \) is a polynomial vanishing at the origin and \( \Delta \) stands for the Laplacian with respect to \( x \). The analyticity in time of the solutions of a semilinear heat equation has been considered by many authors. For example Ōuchi [2] treated the analyticity in time of the solutions of certain initial boundary value problems with bounded continuous initial functions, which include (1) if \( P(u, \overline{u}) \) is a monotone polynomial of \( u \) with real coefficients.

The main aim of the present paper is to prove that the solution of (1) local in time with the initial function in \( L^p(\mathbb{R}^n) \), \( 1 \leq p < \infty \), \( n \geq 1 \), is holomorphic not only in time but also in space variables. We shall, in fact, show that the solution \( u(t, \cdot) \) of (1) extends analytically to a strip of which width grows in proportion to \( \sqrt{t} \). Hence, we shall regard the solution \( u(t, x) \) as a function of \( t \) and \( x \) than as a Banach space valued function of \( t \). This is a sharp contrast with the treatment in [2].

Let us introduce a domain \( \Omega \) in \( \mathbb{C}^{n+1} \), to which the solution \( u \) will extend holomorphically, as well as some function spaces of holomorphic functions defined on \( \Omega \). For \( 0 < \alpha < \pi/2, \beta > 0 \) and \( T > 0 \) we let

\[ \Omega = \{ (\tau, x + iy); 0 < |\tau| < T, |\arg \tau| < \alpha, |y| < \beta \sqrt{\tau} \}. \]

We can write \( \Omega \) as \( D \times \mathbb{R}^n \) with \( D = \cup_{0 < t < T} D(t) \) and \( D(t) = \{ (te^{i\theta}, y); |\theta| < \alpha, |y| < \beta \sqrt{t} \} \). Let us define norms for holomorphic functions \( f \) on \( \Omega \). Let

\[ \| f \|_{H^p(t)} = \sup_{(\tau, y) \in D(t)} \left( \int_{\mathbb{R}^n} |f(\tau, x + iy)|^p dx \right)^{1/p}, \]

\[ \| f \|_{H^m_p(t)} = \sum_{j=0}^m \| (\sqrt{t} \nabla)^j f \|_{H^p(t)}. \]
for $0 < t < T$, where $\nabla$ stands for the nabla with respect to $x$. Then we put

$$H^p_m(\Omega) = \{ f; f \text{ is holomorphic on } \Omega \text{ and } \| f \|_{H^p_m(t)} \text{ is locally bounded on } (0,T) \},$$

$$BH^p_m(\Omega) = \{ f; f \text{ is holomorphic on } \Omega \text{ and } \| f \|_{BH^p_m(\Omega)} = \sup_{0 < t < T} \| f \|_{H^p_m(t)} < \infty \}.$$ 

It is easy to see that $BH^p_m(\Omega)$ is a Banach space. The above function spaces are kinds of Hardy spaces on a tube domain (cf. [4; Chapter III]). Let $BH^p_\infty(\Omega) = \cap_{m=0}^\infty BH^p_m(\Omega)$. For simplicity we shall drop the subscript $m$ if $m = 0$.

We shall show

**Theorem 1.** Let $1 \leq p < \infty$ and $\varphi \in L^p(\mathbb{R}^n)$. Suppose the degree of $P$ is smaller than $1 + \frac{2p}{n}$. Then there exists $T > 0$ for which (1) has a unique solution $u(t,x)$ for $0 < t < T$ extensible holomorphically to $\Omega$ and the extension belongs to $BH^p_\infty(\Omega)$.

The limiting case of Theorem 1 as $p \to \infty$ is

**Theorem 2.** Let $h(z,w)$ be a holomorphic function on $\mathbb{C}^2$ vanishing at the origin. If $\varphi$ is a bounded continuous function on $\mathbb{R}^n$, then there exists $T > 0$ for which

$$\partial_t u - \Delta u = h(u, \overline{u}) \quad \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

(2)

$$u(0,x) = \varphi(x) \quad \text{for } x \in \mathbb{R}^n$$

has a unique solution $u(t,x)$ for $0 < t < T$ extensible holomorphically to $\Omega$ and the extension belongs to $BH^\infty_\infty(\Omega)$.

Let us remark that one may obtain, in the same way as in [2], the solution of (1) global in time and its analyticity in time and space variables under an additional assumption on the polynomial $P$ which guarantees an a priori estimate, involving only the $L^p$-norm of $\varphi$, for the $L^p$-norm of the solution.

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2. Preliminaries

We observe that if \( f \) belongs to \( H^p_0(\Omega) \), then so does the function \( f^*(\tau, x + iy) = \overline{f(\tau, x - iy)} \). Obviously, \( f^*(t, x) = \overline{f(t, x)} \) and \( \|f^*\|_{H^p_0(t)} = \|f\|_{H^p_0(t)} \). Throughout this section we let \( 1 \leq p < \infty \). For simplicity we write const. for a positive constant independent of functions \( f, g, \ldots, \) and variables \( t, \tau, x, y, \ldots \). Since the \( p \)-th power of the modulus of a holomorphic function is a subharmonic function, we have from the submean value property

**Lemma 1.** Let \( f \in H^p(\Omega) \). If \( K \) is a compact subset of \( D \), then \( \{f(\tau, x + iy)\}_{(\tau, y) \in K} \) are uniformly bounded functions of \( x \in \mathbb{R}^n \). Moreover

\[
\sup_{(\tau, y) \in K, |x|=r} |f(\tau, x + iy)| \to 0 \quad \text{as } r \to \infty.
\]

**Proof.** Suppose \( K \) is a compact subset of \( D \), and let \( \rho = \frac{1}{2} \text{dist}(K, \partial D) > 0 \), \( K_\rho = \{(\tau, y); \text{dist}((\tau, y), K) < \rho\} \). We observe that

\[
(3) \quad \int_{K_\rho} dtdu \int_{\mathbb{R}^n} |f(\tau, x + iy)|^p dx \leq \int_{K_\rho} \|f\|_{H^p([\tau])}^p dtdu \leq \text{const.} \sup_{t_1 \leq \tau \leq t_2} \|f\|_{H^p(t)}^p < \infty,
\]

where \( \tau = t + iu, t_1 = \inf\{|\tau|; (\tau, y) \in K_\rho \} > 0 \) and \( t_2 = \sup\{|\tau|; (\tau, y) \in K_\rho \} < \tau \). Since \( |f|^p \) is a subharmonic function of \( 2(n + 1) \) real variables, it follows from the submean value property that

\[
\sup_{(\tau, y) \in K, |x|=r} |f(\tau, x + iy)|^p \leq \text{const.} \rho^{2(n+1)} \int_{K_\rho} dtdu \int_{|x|>r-\rho} |f(\tau, x + iy)|^p dx.
\]

The last term tends to zero as \( r \to \infty \) by (3) and the dominated convergence theorem. The lemma is proved.

Let \( f \in H^p(\Omega) \) and \( (\tau, y) \in D \). Since \( \psi(x) = f(\tau, x + iy) \) is a smooth function of \( x \) vanishing at \( \infty \) by Lemma 1, the Gagliardo-Nirenberg inequality (see e.g. [1; Theorem 9.3 in p.24]) applies to \( \psi \). We have

**Lemma 2.** Let \( f \in H^p_0(\Omega) \) and \( 0 < t < T \). Then

\[
\|f\|_{H^p_0(t)} \leq \text{const.} \|f\|_{H^p(t)} + \|\nabla \psi\|_{H^{p/n}_0(t)}^n f \|_{H^p(t)},
\]

\[
\|\nabla \psi\|_{H^{p/n}_0(t)} \leq \text{const.} \|\nabla \psi\|_{H^{p/n}_0(t)}^{1-j/n} \|\psi\|_{H^p(t)}^{j-1/n} \quad \text{for } 0 \leq j \leq n.
\]

**Proof.** Let \( \psi \) be as before the lemma with \( \tau = t e^{i\theta} \). Suppose \( 1 \leq j \leq n - 1 \). Then the Gagliardo-Nirenberg inequality says \( \|\nabla \psi\|_{H^{p/n}_0(t)} \leq \text{const.} \|\nabla \psi\|_{H^{p/n}_0(t)}^{j/n} \|\psi\|_{H^p(t)}^{j-1/n} \), and hence

\[
\|\nabla \psi\|_{H^{p/n}_0(t)} \leq \text{const.} \|\nabla \psi\|_{H^{p/n}_0(t)}^{j/n} \|\psi\|_{H^p(t)}^{j-1/n} \leq \text{const.} \|\nabla \psi\|_{H^{p/n}_0(t)} + \|\psi\|_{H^p(t)}.
\]

Taking the supremum with respect to \( \theta \) and \( y \), we obtain

\[
\|\nabla \psi\|_{H^{p/n}_0(t)} \leq \text{const.} \|\psi\|_{H^{p/n}_0(t)} + \|\nabla \psi\|_{H^p(t)} + \|\nabla \psi\|_{H^{p/n}_0(t)} + \|\psi\|_{H^p(t)},
\]

which proves the first assertion. The second can be proved similarly.

In order to handle the nonlinear term, we need a multiplicative property of the norm \( \|f\|_{H^p_0(t)} \).
Lemma 3. If \( f \) and \( g \) belong to \( H^p_n(\Omega) \), then so does \( fg \); for \( 0 < t < T \)

\[
\| fg \|_{H^p_n(t)} \leq \text{const.} t^{-n/(2p)} \| f \|_{H^p_n(t)} \| g \|_{H^p_n(t)}.
\]

Proof. Let \( f \) and \( g \) belong to \( H^p_n(\Omega) \). Leibniz’s formula shows \( |(\sqrt{t\nabla})^n (fg)| \leq \text{const.} \sum_{j=0}^n |(\sqrt{t\nabla})^j f| |(\sqrt{t\nabla})^{n-j} g| \). Taking the \( p \)-th power, applying Hölder’s inequality and Lemma 2, and then taking the \( p \)-th root, we obtain

\[
\| (\sqrt{t\nabla})^n (fg) \|_{H^p_n(t)} \leq \text{const.} \sum_{j=0}^n \| (\sqrt{t\nabla})^j f \|_{H^{p/(n-j)}_n(t)} \| (\sqrt{t\nabla})^{n-j} g \|_{H^{n/(n-j)}_n(t)}
\]

\[
\leq \text{const.} t^{-n/(2p)} \| f \|_{H^p_n(t)} \| g \|_{H^p_n(t)}.
\]

Similarly

\[
\| fg \|_{H^p_n(t)} \leq \| f \|_{H^{\infty}(t)} \| g \|_{H^p_n(t)} \leq \text{const.} t^{-n/(2p)} \| f \|_{H^p_n(t)} \| g \|_{H^p_n(t)}
\]

by Lemma 2. Therefore the first assertion of Lemma 2 completes the proof.

Corollary. Let \( P(w_1, \ldots, w_k) \) be a polynomial of degree \( m \geq 1 \) vanishing at the origin. If \( f_1, \ldots, f_k \) belong to \( H^p_n(\Omega) \), then so does \( P(f_1, \ldots, f_k) \); for \( 0 < t < T \)

\[
\| P(f_1, \ldots, f_k) \|_{H^p_n(t)} \leq \text{const.} \left\{ \sum_{j=1}^k \| f_j \|_{H^p_n(t)} + t^{-n(m-1)/2p} \left( \sum_{j=1}^k \| f_j \|_{H^p_n(t)} \right)^m \right\}.
\]

Now let us state the main estimate in this section.

Lemma 4. Let \( P(w_1, w_2) \) be a polynomial of degree \( m \) vanishing at the origin. If \( f_1, f_2 \in H^p_n(\Omega) \), then for \( 0 < t < T \)

\[
\| P(f_1, f_1^*) - P(f_2, f_2^*) \|_{H^p_n(t)} \leq \text{const.} \left\{ 1 + t^{-n(m-1)/2p} \left( \sum_{j=1}^2 \| f_j \|_{H^p_n(t)} \right)^{m-1} \right\} \| f_1 - f_2 \|_{H^p_n(t)}.
\]

In particular, letting \( f_1 = f \in H^p_n(\Omega) \) and \( f_2 = 0 \), we obtain for \( 0 < t < T \)

\[
\| P(f, f^*) \|_{H^p_n(t)} \leq \text{const.} (1 + t^{-n(m-1)/2p} \| f \|_{H^p_n(t)}^{m-1}) \| f \|_{H^p_n(t)}.
\]

Proof. Observe that the polynomial \( P(w_1, w_2) - P(w_3, w_2) \) can be written as \( (w_1 - w_3)(Q(w_1, w_2, w_3) + c) \) with a polynomial \( Q \) of degree \( m - 1 \) vanishing at the origin and a constant \( c \). Hence Lemma 3 and its corollary yield

\[
\| P(f_1, f_1^*) - P(f_2, f_2^*) \|_{H^p_n(t)} \leq \text{const.} \| f_1 - f_2 \|_{H^p_n(t)} \left( 1 + t^{-n/(2p)} \| Q(f_1, f_2, f_1^*) \|_{H^p_n(t)} \right)
\]

\[
\leq \text{const.} \| f_1 - f_2 \|_{H^p_n(t)} \left\{ 1 + t^{-n(m-1)/(2p)} \left( \sum_{j=1}^2 \| f_j \|_{H^p_n(t)} \right)^{m-1} \right\}.
\]

The norm \( \| P(f_2, f_2^*) - P(f_2, f_2^*) \|_{H^p_n(t)} \) is similarly estimated, and hence the lemma follows.

For the proof of Theorem 2 we prepare
Lemma 5. Suppose $h(z, w) = \sum_{j+k \geq 1} c_{jk} z^j w^k$ be as in Theorem 2. Let $\hat{h}(\rho) = \sum_{m=1}^{\infty} d_m \rho^{m-1}$ with $d_m = m \sum_{j+k=m} |c_{jk}|$. If $f_1, f_2 \in BH^\infty(\Omega)$, then

$$\|h(f_1, f_1^*) - h(f_2, f_2^*)\|_{BH^\infty(\Omega)} \leq \hat{h}(\rho)\|f_1 - f_2\|_{BH^\infty(\Omega)},$$

where $\rho = \max\{\|f_1\|_{BH^\infty(\Omega)}, \|f_2\|_{BH^\infty(\Omega)}\}$. In particular, letting $f_1 = f \in BH^\infty(\Omega)$ and $f_2 = 0$, we obtain

$$\|h(f, f^*)\|_{BH^\infty(\Omega)} \leq \hat{h}(\|f\|_{BH^\infty(\Omega)})\|f\|_{BH^\infty(\Omega)}.$$

Proof. We observe that

$$\|h(f_1, f_1^*) - h(f_2, f_2^*)\|_{BH^\infty(\Omega)} \leq \sum_{j+k \geq 1} |c_{jk}|\|f_1^j f_1^k - f_2^j f_1^k\|_{BH^\infty(\Omega)}$$

$$\leq \sum_{j+k \geq 1} |c_{jk}|\|f_1\|_{BH^\infty(\Omega)}^j \|f_1\|_{BH^\infty(\Omega)}^{j-1} + \cdots + \|f_2\|_{BH^\infty(\Omega)}^{j-1} \|f_1 - f_2\|_{BH^\infty(\Omega)}$$

$$\leq \sum_{j+k \geq 1} j|c_{jk}|\rho^{j-1} \|f_1 - f_2\|_{BH^\infty(\Omega)}.$$

Similarly, $\|h(f_2, f_2^*) - h(f_2, f_2^*)\|_{BH^\infty(\Omega)}$ is dominated by $\sum_{j+k \geq 1} k|c_{jk}|\rho^{j+k-1} \|f_1 - f_2\|_{BH^\infty(\Omega)}$, and hence the lemma follows.

3. Main estimates

In this section we shall deal with the following linear equation:

$$\partial_t u - \Delta u = g \quad \text{for} \ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

(4)

$$u(0, x) = \varphi(x) \quad \text{for} \ x \in \mathbb{R}^n$$

It is well known that the solution of (4) is written as

(5) \[ u(t, x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(\frac{-(x-\xi)^2}{4t}\right)\varphi(\xi) d\xi \]

$$+ \int_0^t ds \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} \exp\left(\frac{-(x-\xi)^2}{4(t-s)}\right) g(s, \xi) d\xi,$$

where $(x-\xi)^2 = (x-\xi) \cdot (x-\xi) = \sum_{j=1}^n (x_j - \xi_j)^2$. Note that we avoid the usual notation $|x-\xi|^2$ for $(x-\xi)^2$ because $(x-\xi)^2$ need not be nonnegative when $x$ is replaced by $x + iy \in \mathbb{C}^n$. Writing the last integral of (5) as

$$\int_0^1 d\sigma \int_{\mathbb{R}^n} \frac{t}{(4\pi t(1-\sigma))^{n/2}} \exp\left(\frac{-(x-\xi)^2}{4t(1-\sigma)}\right) g(\sigma t, \xi) d\xi,$$
we extend (5) to Ω by
\begin{equation}
(6) \quad u(\tau, x + iy) = \frac{1}{(4\pi\tau)^{n/2}} \int_{\mathbb{R}^n} \exp(-\frac{(x + iy - \xi)^2}{4\tau}) \varphi(\xi) d\xi \\
+ \int_0^1 d\sigma \int_{\mathbb{R}^n} \frac{\tau}{(4\pi\tau(1 - \sigma))^{n/2}} \exp(-\frac{(x + iy - \xi)^2}{4\tau(1 - \sigma)}) g(\sigma\tau, \xi) d\xi
\end{equation}
for (\tau, x + iy) \in \Omega, where (4\pi\tau)^{n/2} and (4\pi\tau(1 - \sigma))^{n/2} stand for the single valued branches in the sector \{\tau; |\arg \tau| < \alpha\} which assume positive values on the positive real axis.

Throughout this section we let 1 \leq p \leq \infty. The main estimate in this section is

**Lemma 6.** Let \( g \in H^p_R(\Omega) \) and suppose \( \int_0^T \| g \|_{H^p_R(t)} dt < \infty \). If \( \varphi \in L^p(\mathbb{R}^n) \), then the function \( u \) given by (6) belongs to \( BH^p_R(\Omega) \):

\[ \| u \|_{BH^p_R(\Omega)} \leq \text{const.}(\| \varphi \|_p + \int_0^T \| g \|_{H^p_R(t)} dt). \]

We shall divide the proof into three steps.

**Lemma 7.** Let \( g \in H^p(\Omega) \) and suppose \( \int_0^T \| g \|_{H^p(t)} dt < \infty \). Then the function
\begin{equation}
(7) \quad u(\tau, x + iy) = \int_0^1 d\sigma \int_{\mathbb{R}^n} \frac{\tau}{(4\pi\tau(1 - \sigma))^{n/2}} \exp(-\frac{(x + iy - \xi)^2}{4\tau(1 - \sigma)}) g(\sigma\tau, \xi) d\xi
\end{equation}
belongs to \( BH^p(\Omega) \) and for \( 0 < t < T \)

\[ \| u \|_{H^p(t)} \leq \exp\left(\frac{\beta^2 \sec \alpha}{4}\right) \sec^{n/2} \alpha \int_0^t \| g \|_{H^p(s)} ds. \]

**Proof.** Let us prove first the norm estimate. Cauchy’s theorem (together with Lemma 1 if \( 1 \leq p < \infty \)) implies that \( u(\tau, x + iy) = \int_0^1 v(\tau, x + iy; \sigma) d\sigma \) with \( v \) defined by
\begin{equation}
(8) \quad \int_{\mathbb{R}^n} \frac{\tau}{(4\pi\tau(1 - \sigma))^{n/2}} \exp(-\frac{(x + iy - \xi - iy\sqrt{\sigma})^2}{4\tau(1 - \sigma)}) g(\sigma\tau, \xi + iy\sqrt{\sigma}) d\xi.
\end{equation}

By an elementary calculation

\[ |\exp(-\frac{(x + iy - \xi - iy\sqrt{\sigma})^2}{4\tau(1 - \sigma)})| \leq \exp\left(\frac{\beta^2 \sec \alpha}{4}\right) \exp\left(-\frac{(x - \xi + y(1 - \sqrt{\sigma}) \tan \theta)^2 \cos \alpha}{4t(1 - \sigma)}\right) \]

for \( (\tau, x + iy) = (te^{i\theta}, x + iy) \in \Omega \). Hence a change of variable shows that \( v(te^{i\theta}, x + iy; \sigma) \) is dominated in modulus by

\[ \exp\left(\frac{\beta^2 \sec \alpha}{4}\right) \times \int_{\mathbb{R}^n} \frac{t}{(4\pi t(1 - \sigma))^{n/2}} \exp(-\frac{\xi^2 \cos \alpha}{4t(1 - \sigma)}) |g(\sigma te^{i\theta}, x + y(1 - \sqrt{\sigma}) \tan \theta - \xi + iy\sqrt{\sigma})| d\xi. \]

\(-6\)
Therefore Minkowski’s inequality for integrals (see e.g. [3; p.271]) yields
\[ \|u(te^{i\theta}, \cdot + iy)\|_p \leq \exp\left(\frac{\beta^2 \sec \alpha}{4} \right) \int_0^1 \|v\|_{H^p(t)} dt, \]
which implies the required inequality.

In order to show that \( u \) is holomorphic, we put \( u_\varepsilon(\tau, x + iy) = \int_\varepsilon^1 v(\tau, x + iy; \sigma) d\sigma \) for \( \varepsilon > 0 \). Let \( K \) be a compact subset of \( D \). Then by Lemma 1
\[ \sup_{(\tau, y) \in K, \xi \in \mathbb{R}^n, \varepsilon \leq \sigma \leq 1} |g(\sigma \tau, \xi + iy\sqrt{\sigma})| < \infty. \]
Hence the dominated convergence theorem shows the continuity of \( u_\varepsilon \), and then Fubini’s theorem and Morera’s theorem yield that \( u_\varepsilon \) is holomorphic. The norm estimate implies that \( \{u_\varepsilon\}_{\varepsilon > 0} \) forms a Cauchy sequence in \( BH^p(\Omega) \) and that the limit \( u \) must be holomorphic in \( \Omega \) and belong to \( BH^p(\Omega) \).

**Lemma 8.** Let \( g \in H^p_\alpha(\Omega) \) and suppose \( \int_0^T \|g\|_{H^p(t)} dt < \infty \). Then the function \( u(\tau, x + iy) \) given by (7) belongs to \( BH^p_\alpha(\Omega) \) and for \( 0 < t < T \)
\[ \|u\|_{H^p(\sigma)} \leq \text{const.} \int_0^t \|g\|_{H^p(s)} ds. \]

**Proof.** Let \( (\tau, x + iy) = (te^{i\theta}, x + iy) \in \Omega \). In view of Lemmas 2 and 7, it suffices to estimate \( \| (\sqrt{t} \nabla)^n u \|_{H^{p}(t)} \). We split \( u \) into \( u_1(\tau, x + iy) = \int_0^{1/2} v(\tau, x + iy; \sigma) d\sigma + \int_{1/2}^1 v(\tau, x + iy; \sigma) d\sigma \) with \( v(\tau, x + iy; \sigma) \) given by (8). Differentiation under the integral sign yields that if \( 0 < \sigma < 1/2 \), then \( (\sqrt{t} \nabla)^n v(te^{i\theta}, x + iy; \sigma) \) is bounded in modulus by
\[
\int_{\mathbb{R}^n} tQ_n(|x - \xi|, |y|(1 - \sqrt{\sigma}), \sqrt{t(1 - \sigma)}) \exp\left(-\frac{(x - \xi)^2 \sec \alpha}{4t(1 - \sigma)}\right) |g(\sigma te^{i\theta}, \xi + iy\sqrt{\sigma})| d\xi,
\]
\[
\leq \int_{\mathbb{R}^n} 2^{n/2} tQ_n(|\eta|, \beta, 1) \exp\left(-\frac{\eta^2 \sec \alpha}{2}\right) |g(\sigma te^{i\theta}, x - \sqrt{\eta} + iy\sqrt{\sigma})| d\eta,
\]
where \( Q_n \) is a homogeneous polynomial of degree \( n \) whose coefficients are all positive. Hence Minkowski’s inequality for integrals shows that
\[ \| (\sqrt{t} \nabla)^n u_1(te^{i\theta}, \cdot + iy) \|_p \leq \text{const.} \int_0^{1/2} t\|g\|_{H^p(t)} d\sigma \leq \text{const.} \int_0^{t/2} \|g\|_{H^p(s)} ds. \]

Writing \( v(\tau, x + iy; \sigma) \) as
\[ \int_{\mathbb{R}^n} \frac{\tau}{(4\pi \tau(1 - \sigma))^{n/2}} \exp\left(-\frac{(\xi + iy - iy\sqrt{\sigma})^2}{4\tau(1 - \sigma)}\right) g(\sigma \tau, x - \xi + iy\sqrt{\sigma}) d\xi, \]
and differentiating under the integral sign, we can estimate \( (\sqrt{t} \nabla)^n u_2 \) in the same way as in Lemma 7. We have
\[ \| (\sqrt{t} \nabla)^n u_2(te^{i\theta}, \cdot + iy) \|_p \leq \text{const.} \int_0^{1/2} \frac{t}{\sqrt{\tau}} \| (\sqrt{\tau} \nabla)^n g \|_{H^p(t)} d\sigma \leq \text{const.} \int_0^{t/2} \|g\|_{H^p(s)} ds. \]

The lemma follows.

Now Lemma 6 follows from Lemma 8 and
Lemma 9. Let $\varphi \in L^p(\mathbb{R}^n)$. Then the function

$$u(\tau, x + iy) = \frac{1}{(4\pi \tau)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{(x + iy - \xi)^2}{4\tau}\right) \varphi(\xi) d\xi$$

belongs to $BH^p(\Omega)$ and $\|u\|_{BH^p(\Omega)} \leq \text{const} \cdot \|\varphi\|_p$.

Proof. Let $(te^{i\theta}, x + iy) \in \Omega$. In the same way as in the proof of Lemma 7, we see that $u(te^{i\theta}, x + iy)$ is bounded in modulus by

$$\frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{(x - \xi)^2 \cos \theta + y^2 \sin \theta}{4t}\right) |\varphi(\xi)| d\xi$$

$$\leq \exp\left(\frac{\beta^2 \sec \alpha}{4}\right) \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{\xi^2 \cos \alpha}{4t}\right) |\varphi(x - \xi)| d\xi.$$

Hence Minkowski’s inequality for integrals yields

$$\|u\|_{H^p(t)} \leq \exp\left(\frac{\beta^2 \sec \alpha}{4}\right) \text{sec}^{n/2} \alpha \|\varphi\|_p.$$

By differentiation under the integral sign we have

$$|(\sqrt{t}\nabla)^n u(te^{i\theta}, x + iy)| \leq \int_{\mathbb{R}^n} Q_n(|x - \xi|, |y|, \sqrt{t}) \exp\left(-\frac{(x - \xi)^2 \cos \alpha}{4t}\right) |\varphi(\xi)| d\xi,$$

where $Q_n$ is a homogeneous polynomial of degree $n$ whose coefficients are all positive, whence by a change of variable and Minkowski’s inequality for integrals

$$\|(\sqrt{t}\nabla)^n u\|_{H^p(t)} \leq \|\varphi\|_p \int_{\mathbb{R}^n} Q_n(|\eta|, \beta, 1) \exp\left(-\frac{\eta^2 \cos \alpha}{4}\right) d\eta = \text{const} \cdot \|\varphi\|_p.$$

Therefore Lemma 2 completes the proof.
4. Proof of Theorems

Let us first note that it is sufficient to show that (1) has a solution \( u \) in \( BH^p_n(\Omega) \); and that (2) has a solution \( u \) in \( BH^\infty(\Omega) \).

**Lemma 10.** Let \( \beta' > \beta \) and \( \Omega' = \{ (\tau, x + iy); 0 < |\tau| < T, |\arg \tau| < \alpha, |y| < \beta' \sqrt{|\tau|} \} \). Then

\[
BH^p_\infty(\Omega) \subset BH^p(\Omega').
\]

In particular, if (1) (resp. (2)) has a solution in \( BH^p_n(\Omega') \) (resp. \( BH^\infty(\Omega') \)), then it belongs to \( BH^p_\infty(\Omega) \) (resp. \( BH^\infty_\infty(\Omega) \)).

**Proof.** Let \( f \in BH^p(\Omega') \) and \( (\tau, y) \in D \). Applying Cauchy’s integral formula to \( f(\tau, x + iy) \) with fixed \( x_2, \ldots, x_n \), we obtain that if \( 0 < r < (\beta' - \beta) \sqrt{|\tau|} \), then

\[
\left| \frac{\partial}{\partial x_1} f(\tau, x + iy) \right| \leq \frac{1}{2\pi r} \int_0^{2\pi} |f(\tau, x + iy + (re^{i\theta}, 0, \ldots, 0))| d\theta.
\]

Hence Minkowski’s inequality for integrals yields \( \| \sqrt{r} \frac{\partial f}{\partial x_1} \|_{H^p(\Omega)} \leq \frac{1}{\beta' - \beta} \| f \|_{BH^p(\Omega')} \). Therefore, in general, \( \| f \|_{BH^p_\infty(\Omega)} \leq \text{const.} \| f \|_{BH^p(\Omega')} \) for \( m \geq 1 \), and the implication follows.

**Proof of Theorem 1.** Define the mapping \( M \) on \( BH^p_n(\Omega) \) by \( Mv = u \), where \( u \) is the function defined by (6) with \( g = P(v, v^*) \). Let \( m \) be the degree of \( P \). Then Lemmas 4 and 6 yield that \( Mv \in BH^p_n(\Omega) \) and

\[
\|Mv\|_{BH^p_n(\Omega)} \leq C_1 \|\varphi\|_p + C_2(T + T^{1-n(m-1)/(2p)} \|v\|_{BH^p_n(\Omega)})^2 \|v\|_{BH^p_n(\Omega)}.
\]

If \( v_1 \) and \( v_2 \) belong to \( BH^p_n(\Omega) \), then \( Mv_1 - Mv_2 \) is the function defined by (6) with \( g = P(v_1, v_1^*) - P(v_2, v_2^*) \) and \( \varphi = 0 \), and hence Lemmas 4 and 6 again yield

\[
\|Mv_1 - Mv_2\|_{BH^p_n(\Omega)} \leq C_3 \left\{ T + T^{1-n(m-1)/(2p)} \left( \sum_{j=1}^2 \|v_j\|_{BH^p_n(\Omega)}^{m-1} \right) \right\} \|v_1 - v_2\|_{BH^p_n(\Omega)},
\]

where \( C_1, C_2 \) and \( C_3 \) are positive constants independent of \( v, v_1, v_2 \) and \( T \). Let \( R = 2C_1 \|\varphi\|_p \) and take \( T > 0 \) so small that

\[
C_2(T + T^{1-n(m-1)/(2p)} R^{m-1}) R < \frac{R}{2},
\]

\[
C_3(T + T^{1-n(m-1)/(2p)} (2R)^{m-1}) < 1.
\]

This is possible by the hypothesis on the degree of \( P \). Then \( M \) is a contraction mapping from the closed ball \( \{ f \in BH^p_n(\Omega); \|f\|_{BH^p_n(\Omega)} \leq R \} \) to itself, and hence has a unique fixed point \( u \). The restriction of this function \( u \) on \( \mathbb{R}^+ \times \mathbb{R}^n \) is the solution of (1). The theorem is proved.

**Proof of Theorem 2.** Define the mapping \( M \) on \( BH^\infty(\Omega) \) by \( Mv = u \), where \( u \) is the function defined by (6) with \( g = h(v, v^*) \). Let \( \tilde{h} \) be as in Lemma 5. Then Lemmas 5 and 7 yield that \( Mv \in BH^\infty(\Omega) \) and

\[
\|Mv\|_{BH^\infty(\Omega)} \leq C_4 \|\varphi\|_\infty + C_5 T \tilde{h}(\|v\|_{BH^\infty(\Omega)}) \|v\|_{BH^\infty(\Omega)};
\]
if \( v_1 \) and \( v_2 \) belong to \( BH^\infty(\Omega) \), then

\[
\|Mv_1 - Mv_2\|_{BH^\infty(\Omega)} \leq C_6 T \tilde{h}(\rho) \|v_1 - v_2\|_{BH^\infty(\Omega)},
\]

where \( \rho = \max\{\|v_1\|_{BH^\infty(\Omega)}, \|v_2\|_{BH^\infty(\Omega)}\} \); \( C_4, C_5 \) and \( C_6 \) are positive constants independent of \( v, v_1, v_2 \) and \( T \). Let \( R = 2C_4\|\varphi\|_{\infty} \) and take \( T > 0 \) so small that

\[
C_5 T \tilde{h}(R) R < \frac{R}{2},
\]

\[
C_6 T \tilde{h}(R) < 1.
\]

Then \( M \) is a contraction mapping from the closed ball \( \{f \in BH^\infty(\Omega); \|f\|_{BH^\infty(\Omega)} \leq R\} \) to itself, and hence has a unique fixed point \( u \). The restriction of this function \( u \) on \( \mathbb{R}^+ \times \mathbb{R}^n \) is the solution of (2). The theorem is proved.

**Remark.** Let \( u \) be the extension of the solution of (1) or (2). Then

\[
\lim_{(\tau, y) \to (0, 0), (\tau, y) \in D} \|u(\tau, \cdot + iy) - \varphi\|_p = 0.
\]

This may be considered to be a complex extension of the initial condition.

References