Harmonic functions and Green potentials having no tangential limits

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§1. Introduction

In connection with Littlewood's work [3], Barth [2; p.551] raised the following problem: Let C_0 be a tangential curve in $\{|z| < 1\}$ which ends at $z = 1$ and let C_θ be any rotation of C_0 about the origin through an angle θ. Does there exist a positive harmonic function h in $\{|z| < 1\}$ such that

$$\lim_{|z|\to 1,\, z\in C_\theta} h(z)$$

does not exist for *all* $\theta, 0 \le \theta \le 2\pi$? In our previous paper [1] we showed

THEOREM A. *There exists a* bounded *harmonic function h in $\{|z| < 1\}$ such that*

$$\lim_{|z|\to 1,\, z\in C_\theta} h(z)$$

does not exist for all $\theta, 0 \le \theta \le 2\pi$.

This theorem answers Barth's question in the affirmative. The main aim of this paper is to improve Theorem A and consider its analogue for Green potentials.

In order to facilitate the succeeding argument, we reformulate Theorem A as follows: Let $R_+^{n+1} = \{(x, y); x \in R^n, y > 0\}$ be the upper half space, $n \ge 1$. Let γ be a curve in R_+^{n+1} which ends at a point $(x_0, 0)$ on the boundary. We say that γ is tangential if γ lies eventually outside any nontangential cone with vertex at $(x_0, 0)$, or equivalently, for any $m > 0$ there is $\varepsilon > 0$ such that

$$\{(x, y) \in \gamma; 0 \le y \le \varepsilon\} \subset \{(x, y); my \le |x - x_0|\}. \tag{1}$$

By $\gamma + \xi$ we denote the translation $\{(x + \xi, y); (x, y) \in \gamma\}$. Then an immediate analogue of Theorem A is

THEOREM B. *Let γ be a tangential curve in R_+^{n+1} which ends at a point on the boundary. Then there exists a bounded harmonic function u in R_+^{n+1} such that*

$$\lim_{y\to 0,\, (x,y)\in\gamma+\xi} u(x, y)$$

does not exist for all $\xi \in R^n$*.*

In case $n = 1$, Theorem B is essentially the same as Theorem A and can be proved with minor modifications. In case $n > 1$, we can consider a further extentison of Theorem B as far as curve families are concerned. The curve family considered in Theorem B is just the family $\{\gamma + \xi; \xi \in R^n\}$, which is generated by translation from a given tangential curve γ. M. Sakai orally asked whether the conclusion of Theorem B holds if the family of curves rotated around the y-axis is taken instead of $\{\gamma+\xi; \xi \in R^n\}$. We shall, in fact, show that this is true. More generally, we consider curves having tangency equivalent with γ.

Definition. Let γ and γ' be curves in R^{n+1}_+ which end at points on the boundary R^n. We say that γ' has tangency equivalent with γ if γ is mapped onto γ' by a bi-Lipschitz mapping T preserving y coordinate. More precisely, γ' has tangency equivalent with γ if there is T such that $T\gamma = \gamma'$, T is written as $T(x, y) = (T_y x, y)$ and there is a positive constant c_T such that

$$c_T^{-1}(|x - x'| + |y - y'|) \le |T(x, y) - T(x', y')| \le c_T(|x - x'| + |y - y'|). \tag{2}$$

Obviously, γ, $\gamma + \xi$ ($\xi \in R^n$) and their curves rotated around the y-axis have equivalent tangency. We can state a generalization of Theorem B.

Theorem 1. *Let γ be a tangential curve in R^{n+1}_+ which ends at a point on the boundary. Then there exists a bounded harmonic function u in R^{n+1}_+ such that*

$$\lim_{y \to 0,\, (x,y) \in \gamma'} u(x, y)$$

does not exist for all curves γ' having tangency equivalent with γ. Moreover, we can make u satisfy

$$\liminf_{y \to 0,\, (x,y) \in \gamma'} u(x, y) = \inf_{(x,y) \in R^{n+1}_+} u(x, y)$$

$$\limsup_{y \to 0,\, (x,y) \in \gamma'} u(x, y) = \sup_{(x,y) \in R^{n+1}_+} u(x, y)$$

for all curves γ' having tangency equivalent with γ.

We shall prove the theorem along a way similar to [1]. However, since the codimension of cuves in R^{n+1}_+ is greater than 1 for $n > 1$, we shall employ a new technique, a grid argument (see §2).

We now consider the boundary behavior of Green potentials. Roughly speaking, a Green potential vanishes on the boundary. Littlewood [4] showed that if v is a Green potential on $\{|z| < 1\}$, then

$$\lim_{r \to 1} v(re^{i\theta}) = 0$$

for almost all θ, $0 \le \theta \le 2\pi$. For the upper half space R^{n+1}_+, this result reads as

THEOREM C. *If v is a Green potential on R^{n+1}_+, then for a.e. $\xi \in R^n$*

$$\lim_{y \to 0} v(x, y) = 0.$$

As another illustration of the vanishing property, we mention a theorem due to Rippon [5; Theorem 3], who also gave an extension of Theorem C to limits of Green potentials along uniformly separated nontangential curves ([5; Theorem 1]). Let $V_\alpha(x) = \{(\xi, \eta); |x - \xi| < \alpha\eta\}$ be the nontangential cone with vertex at $(x, 0)$ and aperture $\alpha > 0$. A set $A \subset R^{n+1}_+$ is said to be minimally thin at $(x, 0)$ if there is a Green potential which majorizes on A the Martin kernel $\eta/|\xi - x|^{n+1}$ at $(x, 0)$.

THEOREM D. *Let v be a Green potential on R^{n+1}_+. Then there exists a set $A \subset R^{n+1}_+$ such that for a.e. $\xi \in R^n$ and all $\alpha > 0$, the set $A \cap V_\alpha(\xi)$ is minimally thin at ξ and*

$$\lim_{y \to 0,\, (x,y) \in V_\alpha(\xi) \setminus A} v(x, y) = 0.$$

On the contrary, the tangential boundary behavior of Green potentials is fairly complicated. Let γ be an arbitrary tangential curve in R^{n+1}_+ ending at a point on the boundary. We shall construct a Green potential v such that the exceptional set A in Theorem D is empty and so v has nontangential limit zero at almost every boundary point, and yet it falis to have tangential limit zero along every curve γ' having tangency equivalent with γ.

THEOREM 2. *Let γ be a tangential curve in R^{n+1}_+ which ends at a point on the boundary. Then there exists a Green potential v in R^{n+1}_+ with the property that*

$$\liminf_{y \to 0,\, (x,y) \in \gamma'} v(x, y) = 0$$

for all curves γ' tending to the boundary,

$$\lim_{y \to 0,\, (x,y) \in V_\alpha(\xi)} v(x, y) = 0$$

for a.e. $\xi \in R^n$ *and all* $\alpha > 0$*, and yet*

$$\limsup_{y \to 0,\, (x,y) \in \gamma'} v(x,y) = +\infty$$

for all curves γ' *having tangency equivalent with* γ*.*

§2. Preliminaries

For simplicity we write $\{a \le y \le b\}$ for the strip $\{(x,y); a \le y \le b\}$. By π we denote the projection mapping from R^{n+1}_+ onto the boundary R^n.

LEMMA 1 (cf. [1; Lemma 5]). *Let* γ *be a tangential curve in* R^{n+1}_+ *which ends at a point on the boundary. Then there exist decreasing sequences* $\{a_j\}$ *and* $\{b_j\}$

$$b_0 > a_0 > b_1 > a_1 > \cdots \to 0$$

with the property that one of the connected components of $\gamma \cap \{a_j \le y \le b_j\}$*, say* γ_j*, satisfies*

$$\lim_{j \to \infty} \frac{1}{b_j} \operatorname{diam} \pi(\gamma_j) = \infty, \tag{3}$$

$$\operatorname{diam} \pi(\gamma_j) \le a_{j-1} \text{ for } j \ge 1. \tag{4}$$

Proof. Without loss of generality we may assume that γ ends at the origin. We shall choose $\{a_j\}$ and $\{b_j\}$ inductively. Let c_j be an increasing sequence diverging to ∞. By (1) we find $b_0 > 0$ such that $b_0 < \sup_{(x,y)\in\gamma} y$ and

$$\gamma \cap \{0 \le y \le b_0\} \subset \{2c_0 y \le |x|\}.$$

Let γ_0' be the connected component of $\gamma \cap \{0 \le y \le b_0\}$ which includes the origin. Since γ_0' meets the hyperplane $\{y = b_0\}$, it follows that

$$\operatorname{diam} \pi(\gamma_0') \ge \sup_{(x,y)\in\gamma_0'} |x| \ge 2c_0 b_0.$$

Let γ_0'' be a subcurve of γ_0' which connects the hyperplane $\{y = b_0\}$ and a point near the origin such that

$$\operatorname{diam} \pi(\gamma_0'') \ge \frac{1}{2} \operatorname{diam} \pi(\gamma_0').$$

Let $0 < a_0 < \inf_{(x,y)\in\gamma_0''} y$. Then one of the connected components of $\gamma \cap \{a_0 \le y \le b_0\}$ contains γ_0''. Hence this component γ_0 satisfies $\operatorname{diam} \pi(\gamma_0) \ge c_0 b_0$.

Now, by (1) and the continuity, we can take b_1, $0 < b_1 < a_0$, such that

$$\gamma \cap \{0 \le y \le b_1\} \subset \{2c_1 y \le |x| \le a_0\}.$$

In the same way as above we can choose $a_1, 0 < a_1 < b_1$, with the property that one of the connected components of $\gamma \cap \{a_1 \le y \le b_1\}$, say γ_1, satisfies $\operatorname{diam} \pi(\gamma_1) \ge c_1 b_1$. Obviously, $\operatorname{diam} \pi(\gamma_1) \le a_0$. Continuing this procedure, we obtain decreasing sequences $\{a_j\}$ and $\{b_j\}$ with the required property. The proof is complete.

Let γ, $\{a_j\}$ and $\{b_j\}$ be as in Lemma 1. Let γ_j be a connected component of $\gamma \cap \{a_j \le y \le b_j\}$ for which (3) and (4) hold. Put $\ell_j = \sqrt{b_j}\sqrt{\operatorname{diam} \pi(\gamma_j)}$ and let M_j be a mesh with side length ℓ_j in R^n, viz.,

$$M_j = \bigcup_{k=1}^{n} \bigcup_{\nu=-\infty}^{\infty} \{(x_1, ..., x_n); x_k = \nu \ell_j\}.$$

Let $G_j = M_j \times [a_j, b_j]$. This is a set of grid shape.

Lemma 2. *Let G_j be as above. If γ' is a curve with tangency equivalent with γ, then γ' meets G_j for all sufficiently large j.*

Proof. Since γ is mapped onto γ' by a bi-Lipschitz mapping T preserving y coordinate, it follows from (2) that $\gamma_j' = T\gamma_j$ is a subcurve of γ' such that $\gamma_j' \subset \gamma' \cap \{a_j \le y \le b_j\}$ and

$$c_T^{-1} \operatorname{diam} \pi(\gamma_j) - (1 - c_T^{-1})(b_j - a_j) \le \operatorname{diam} \pi(\gamma_j') \le c_T\{\operatorname{diam} \pi(\gamma_j) + (b_j - a_j)\},$$

where $c_T \ge 1$ depends only on T. We observe that $R^n \setminus M_j$ consists of cubes of side length ℓ_j, and hence of diameter $\sqrt{n}\ell_j$. We have from (3) that there is $j_0 = j_0(T)$ such that if $j \ge j_0$, then

$$\frac{\operatorname{diam} \pi(\gamma_j')}{\sqrt{n}\ell_j} \ge \frac{c_T^{-1}}{\sqrt{n}} \sqrt{\frac{\operatorname{diam} \pi(\gamma_j)}{b_j}} - \frac{1 - c_T^{-1}}{\sqrt{n}} \sqrt{\frac{b_j}{\operatorname{diam} \pi(\gamma_j)}} > 1.$$

This implies that if $j \ge j_0$, then $\pi(\gamma_j')$ and M_j intersect, and so do γ' and G_j. The lemma is proved.

We close this section with some further notation. Let $B(x,r)$ stand for the open ball in R^n with center at x and radius r. We shall write const. for a positive constant depending only on the dimension, whose value may change from one occurrence to the next.

§3. Proof of Theorem 1

For a locally integrable function f on R^n we define the Poisson integral $u(x,y)$ of f by

$$u(x,y) = f * P_y(x),$$

where $P_y(x) = y^{-n}P(x/y)$, $P(x) = \Gamma(\frac{n+1}{2})\pi^{-(n+1)/2}(1+|x|^2)^{-(n+1)/2}$. Let us show two estimates of Poisson integrals. The first one is easy.

LEMMA 3 (cf. [3; Lemma 2] and [1; Lemma 2]). *There exists a positive constant m_1 depending only on the dimension such that if $|f| \le 1$ on R^n and $f = 1$ on $B(x, \kappa r)$, then the Poisson integral u of f satisfies*

$$u(x,y) \ge 1 - \frac{m_1}{\kappa} \quad \text{for } 0 < y < r.$$

The second one may be of some independent interest. Let $M_y f(x)$ be the truncated maximal function of f defined by

$$M_y f(x) = \sup_{r \ge y} r^{-n} \int_{B(x,r)} |f| d\xi.$$

LEMMA 4 (cf. [1; Lemma 3]). *There exists a positive constant m_2 depending only on the dimension such that*

$$|f * P_y(x)| \le m_2 \|f\|_\infty^{n/n+1} (M_y f(x))^{1/n+1}.$$

Proof. Since $|f * P_y(x)| \le \|f\|_\infty$, there is nothing to prove if $M_y f(x) \ge \|f\|_\infty$. Suppose that $M_y f(x) \le \|f\|_\infty$. Let

$$\ell = \left(\frac{\|f\|_\infty}{M_y f(x)}\right)^{1/n+1},$$

and split f into $f_1 = f\chi_{B(x,\ell y)}$ and $f_2 = f - f_1$. Since $\ell \geq 1$, it follows from the definition of $M_y f(x)$ that

$$|f_1 * P_y(x)| \leq \text{const.} y^{-n} \int_{B(x,\ell y)} |f| d\xi$$
$$\leq \text{const.} \ell^n M_y f(x) = \text{const.} \|f\|_\infty^{n/n+1} (M_y f(x))^{1/n+1}.$$

We estimate the Poisson integral of f_2 as

$$|f_2 * P_y(x)| \leq \text{const.} y \|f\|_\infty \int_{|\xi - x| > \ell y} \frac{d\xi}{|\xi - x|^{n+1}}$$
$$= \text{const.} \|f\|_\infty \ell^{-1} = \text{const.} \|f\|_\infty^{n/n+1} (M_y f(x))^{1/n+1}.$$

The lemma is proved.

COROLLARY. *There exists a positive constant m_3 depending only on the dimension such that*

$$\sup_{y \geq r, x \in R^n} |f * P_y(x)| \leq m_3 \|f\|_\infty^{n/n+1} \sup_{x \in R^n} \left(r^{-n} \int_{B(x,r)} |f| d\xi \right)^{1/n+1}.$$

Proof of Theorem 1. Let $a_j, b_j, \gamma_j, \ell_j, M_j$ and G_j be as in Lemma 2. Let

$$\kappa_j = \left(\frac{1}{b_j} \operatorname{diam} \pi(\gamma_j) \right)^{1/4},$$
$$E_j = \bigcup_{\xi \in M_j} B(\xi, \kappa_j b_j)$$

and $g_j = \chi_{E_j}$. We observe that

$$E_j = \bigcup_{k=1}^{n} \bigcup_{\nu=-\infty}^{\infty} \{(x_1, ..., x_n); |x_k - \nu \ell_j| < \kappa_j b_j\}.$$

We claim

$$\sup_{\xi \in R^n} \frac{1}{a_{j-1}^n} \int_{B(\xi, a_{j-1})} g_j dx \to 0$$

as j tends to infinity. For this purpose we write

$$R^n \setminus M_j = \bigcup_k Q_k,$$

where Q_k are cubes of side length ℓ_j. Since $\ell_j \leq a_{j-1}$ by (4), an elementary geometrical observation shows that the number of cubes Q_k that meet an open ball with radius a_{j-1} is not greater than const.$(a_{j-1}/\ell_j)^n$, and that

$$|E_j \cap Q_k| \leq 2n\kappa_j b_j \ell_j^{n-1}.$$

Therefore

$$\begin{aligned} (5) \qquad & \frac{1}{a_{j-1}^n} \int_{B(\xi,a_{j-1})} g_j dx \\ &= \frac{1}{a_{j-1}^n} \sum_k |B(\xi, a_{j-1}) \cap Q_k \cap E_j| \\ &\leq \text{const.} \frac{1}{a_{j-1}^n} (a_{j-1}/\ell_j)^n \kappa_j b_j \ell_j^{n-1} = \text{const.} \left(\frac{b_j}{\operatorname{diam} \pi(\gamma_j)} \right)^{1/4} \to 0. \end{aligned}$$

We infer from Corollary to Lemma 4 that

$$\sup_{x \in R^n, y \geq a_{j-1}} g_j * P_y(x) \leq \text{const.} \left(\frac{b_j}{\operatorname{diam} \pi(\gamma_j)} \right)^{1/4(n+1)},$$

so that taking subsequences of $\{a_j\}$ and $\{b_j\}$, if necessary, we may assume that

$$(6) \qquad \sup_{x \in R^n, y \geq a_{j-1}} g_j * P_y(x) \leq 2^{-j}.$$

As another consequence of (5), we obtain that if $R > a_{j-1}$, then

$$\begin{aligned} |E_j \cap B(\xi, R)| &\leq \text{const.} \left(\frac{R}{a_{j-1}} \right)^n a_{j-1}^n \left(\frac{b_j}{\operatorname{diam} \pi(\gamma_j)} \right)^{1/4} \\ &= \text{const.} R^n \left(\frac{b_j}{\operatorname{diam} \pi(\gamma_j)} \right)^{1/4}. \end{aligned}$$

Hence, taking subsequences of $\{a_j\}$ and $\{b_j\}$, if necessary, we may assume that

$$(7) \quad \sum_j |E_j \cap B(\xi, R)| \leq \sum_{a_{j-1} \leq R} |E_j \cap B(\xi, R)| + \text{const.} R^n \sum_j \left(\frac{b_j}{\operatorname{diam} \pi(\gamma_j)} \right)^{1/4} < \infty$$

for any open ball $B(\xi, R)$.

Since the rest of the proof can be carried out in the same way as in [1], we give only a sketch. The second step is an alternative argument. We inductively form two sequences $\{E_j^+\}$ and $\{E_j^-\}$ of sets in R^n. Let $E_1^+ = \emptyset, E_1^- = E_1$ and let

$$\begin{aligned} E_{j+1}^+ &= E_j^+ \cup E_{j+1}, \quad & E_{j+1}^- &= E_j^- \setminus E_{j+1} \quad & &\text{if } j \text{ is odd} \\ E_{j+1}^+ &= E_j^+ \setminus E_{j+1}, \quad & E_{j+1}^- &= E_j^- \cup E_{j+1} \quad & &\text{if } j \text{ is even} \end{aligned}$$

for $j \geq 1$. Observe that the function

$$f_j = \begin{cases} 1 & \text{on } E_j^+ \\ -1 & \text{on } E_j^- \\ 0 & \text{elsewhere} \end{cases}$$

converges to

$$f(x) = \begin{cases} 1 & \text{if } x \in E_j \text{ for the last time with even } j \\ -1 & \text{if } x \in E_j \text{ for the last time with odd } j \\ 0 & \text{if } x \notin E_j \text{ for any } j \end{cases}$$

whenever $x \notin \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j$. By (7) $|\cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j| = 0$, and hence f_j converges to f a.e. on R^n. We note that

(8) $$|f_j| \leq 1, \quad |f_j - f_{j+1}| \leq 2g_{j+1} \text{ on } R^n; \qquad f_j = (-1)^j \text{ on } E_j.$$

The last step is to show that $u(x, y) = f * P_y(x)$ is a bounded harmonic function with the required property. By the aid of Lemma 3, (6) and (8) we see that if j is even, then

$$\begin{aligned} u(x, y) &= f_j * P_y(x) + (f_{j+1} - f_j) * P_y(x) + \cdots \\ &\geq 1 - \frac{m_1}{\kappa_j} - 2 \sum_{k=j+1}^{\infty} 2^{-k} \quad \text{for } (x, y) \in G_j; \end{aligned}$$

if j is odd, then

$$u(x, y) \leq -1 + \frac{m_1}{\kappa_j} + 2 \sum_{k=j+1}^{\infty} 2^{-k} \quad \text{for } (x, y) \in G_j.$$

It follows from Lemma 1 that for all cuvers γ' in R_+^{n+1} having tangency equivalent with γ

$$\liminf_{y \to 0,\, (x,y) \in \gamma'} u(x, y) = -1 < 1 = \limsup_{y \to 0,\, (x,y) \in \gamma'} u(x, y),$$

which, in particular, implies that $\lim_{y \to 0} u(x, y)$ along γ' does not exist. The theorem is proved.

§4. Proof of Theorem 2

Let $G(x,y;\xi,\eta)$ be the Green function for R_+^{n+1} with argument (ξ,η) and pole at (x,y) associated with the Laplacian. For a measurable function f on R_+^{n+1} we write

$$G(x,y;f) = \iint_{R_+^{n+1}} G(x,y;\xi,\eta) f(\xi,\eta) d\xi d\eta.$$

Observe that the Green function is homogeneous, i.e.

$$G(tx,ty;t\xi,t\eta) = t^{1-n} G(x,y;\xi,\eta) \quad \text{for } t>0 \tag{9}$$

and satisfies

$$G(x,y;\xi,\eta) \le \text{const.} y\eta |(x,y)-(\xi,\eta)|^{-1-n}. \tag{10}$$

We have from (9)

Lemma 5. *There is a positive constant m_3 depending only on the dimension such that if $(x,y) \in R_+^{n+1}$ and a nonnegative measurable function f satisfies $f(\xi,\eta) \ge \eta^{-2}$ on $B(x,y/2) \times (y/2,y)$, then $G(x,y;f) \ge m_3$.*

Let us consider some estimates analogous to Lemma 4. To this end we let

$$F(\xi) = \int_0^\infty \eta |f(\xi,\eta)| d\eta$$

for a measurable function f on R_+^{n+1}. We recall that $M_r F(\xi)$ stands for the truncated maximal function of F.

Lemma 6. *There exists a positive constant m_4 depending only on the dimension such that if $\ell > 0$ and $f(\xi,\eta) = 0$ for $|\xi - x| < \ell$, then*

$$G(x,y;f) \le m_4 \frac{y}{\ell} M_\ell F(x) \quad \textit{for } y > 0.$$

Proof. We may assume that $f \ge 0$. We observe from (10) that

$$\begin{aligned} G(x,y;f) &\le \text{const.} y \iint_{|x-\xi|>\ell} \eta |x-\xi|^{-n-1} f(\xi,\eta) d\xi d\eta \\ &= \text{const.} y \sum_{j=0}^\infty \int_{2^j\ell<|x-\xi|\le 2^{j+1}\ell} |x-\xi|^{-n-1} F(\xi) d\xi \\ &\le \text{const.} y \sum_{j=0}^\infty (2^j\ell)^{-n-1} (2^{j+1}\ell)^n M_\ell F(x) \le \text{const.} \frac{y}{\ell} M_\ell F(x). \end{aligned}$$

The lemma follows.

Lemma 7. *There exists a positive constant m_5 depending only on the dimension such that if $b > 0$ and $f(\xi, \eta) = 0$ for $\eta > b$, then*

$$G(x, y; f) \le m_5 M_y F(x) \quad \textit{for } y \ge 2b.$$

Proof. We may assume that $f \ge 0$. Split $G(x, y; f)$ into

$$\iint_{|x-\xi|\le 2y} G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta + \iint_{|x-\xi|>2y} G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta.$$

We observe from (10) and Lemma 6 that the first integral is bounded by

$$\begin{aligned} \text{const.} y^{-n} \iint_{|x-\xi|\le 2y} \eta f(\xi, \eta) d\xi d\eta &= \text{const.} y^{-n} \int_{|x-\xi|\le 2y} F(\xi) d\xi \\ &\le \text{const.} M_y F(x); \end{aligned}$$

and the second by $\text{const.} M_{2y} F(x) \le \text{const.} M_y F(x)$. The lemma follows.

Lemma 8. *There exists a positive constant m_6 depending only on the dimension such that if $a > 0$ and $f(\xi, \eta) = 0$ for $0 < \eta < a$, then*

$$G(x, y; f) \le m_6 \frac{y}{a} M_a F(x) \quad \textit{for } 0 < y \le a/2.$$

Proof. We may assume that $f \ge 0$. Split $G(x, y; f)$ into

$$\iint_{|x-\xi|\le a} G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta + \iint_{|x-\xi|>a} G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta.$$

We observe from (10) and Lemma 6 that the first integral is bounded by

$$\begin{aligned} \text{const.} \iint_{|x-\xi|\le a} y \eta^{-n} f(\xi, \eta) d\xi d\eta &\le \text{const.} y a^{-n-1} \int_{|x-\xi|\le a} F(\xi) d\xi \\ &\le \text{const.} \frac{y}{a} M_a F(x); \end{aligned}$$

and the second by $\text{const.} \frac{y}{a} M_a F(x)$. The lemma follows.

Proof of Theorem 2. Let $a_j, b_j, \gamma_j, \ell_j, M_j$ and G_j be as in Lemma 2. Since $\ell_j/b_j \to \infty$, we may, without loss of generality, assume that

$$\left(\frac{\ell_j}{b_j}\right)^{1-1/2n} \geq 2. \tag{11}$$

Recall that $V_\alpha(\xi)$ stands for the nontangential cone with vertex at $(\xi, 0)$ and aperture α. Put

$$U_j = \bigcup_{x \in M_j} V_1(x) \cap \{\frac{a_j}{2} < y < b_j\}$$

and $f_j(\xi, \eta) = \eta^{-2}\chi_{U_j}$. We observe that if $a_j/2 < y < b_j$, then

$$\{x; (x,y) \in U_j\} = \bigcup_{k=1}^{n} \bigcup_{\nu=-\infty}^{\infty} \{(x_1, \ldots, x_n); |x_k - \nu\ell_j| < y\}.$$

Let us estimate the truncated maximal fucntion of

$$F_j(\xi) = \int_0^\infty \eta f_j(\xi, \eta) d\eta.$$

Writing $R^n \setminus M_j = \bigcup_k Q_k$, where Q_k are cubes of side length ℓ_j, we obtain that

$$\int_{Q_k} F_j d\xi \leq \int_0^{b_j} \eta^{-1} |\{\xi \in Q_k; (\xi, \eta) \in U_j\}| d\eta \leq \text{const.} b_j \ell_j^{n-1}$$

for each cube Q_k. Let $r \geq \ell_j$. Observing that the number of cubes Q_k that meet $B(\xi_0, r)$ is bounded by const.$(r/\ell_j)^n$, we obtain

$$\int_{B(\xi_0, r)} F_j d\xi \leq \text{const.} \left(\frac{r}{\ell_j}\right)^n b_j \ell_j^{n-1} = \text{const.} r^n \frac{b_j}{\ell_j}.$$

Hence

$$\|M_{\ell_j} F_j\|_\infty \leq \text{const.} \frac{b_j}{\ell_j}.$$

By definition if $0 < \mu \leq 1$, then

$$\|M_{\mu\ell_j} F_j\|_\infty \leq \mu^{-n} \|M_{\ell_j} F_j\|_\infty \leq \text{const.} \mu^{-n} \frac{b_j}{\ell_j}.$$

In particular,

$$\|M_{a_j/2}F_j\|_\infty \le \text{const.}\left(\frac{\ell_j}{a_j}\right)^n \frac{b_j}{\ell_j},$$
$$\|M_{\mu_j\ell_j}F_j\|_\infty \le \text{const.}\sqrt{\frac{b_j}{\ell_j}},$$

where $\mu_j = (b_j/\ell_j)^{1/2n} \le 1$ by (11). Therefore Lemmas 7, 8 and (11) yield that

$$G(x,y;f_j) \le \text{const.}\sqrt{\frac{b_j}{\ell_j}} \quad \text{for } y \ge \mu_j\ell_j = \left(\frac{\ell_j}{b_j}\right)^{1-1/2n} b_j, \tag{12}$$

$$G(x,y;f_j) \le \text{const.}\left(\frac{\ell_j}{a_j}\right)^n \frac{b_j}{\ell_j}\frac{y}{a_j} \quad \text{for } 0 < y \le a_j/4. \tag{13}$$

Now we extract subsequences $\{a_{j_k}\}, \{b_{j_k}\}$ and $\{\ell_{j_k}\}$, and choose a decreasing sequence $\{c_{j_k}\}$ which satisfy

$$k\sqrt{\frac{b_{j_k}}{\ell_{j_k}}} < 2^{-k} \text{ and } \frac{b_{j_k}}{\ell_{j_k}} + \mu_{j_k} < 2^{-k}, \tag{14}$$

$$k\left(\frac{\ell_{j_k}}{a_{j_k}}\right)^n \frac{b_{j_k}}{\ell_{j_k}}\frac{c_{j_k}}{a_{j_k}} < 2^{-k}, \tag{15}$$

$$\cdots < \mu_{j_{k+1}}\ell_{j_{k+1}} < c_{j_k} < \frac{a_{j_k}}{4} < a_{j_k} < b_{j_k} < \mu_{j_k}\ell_{j_k} < c_{j_{k-1}} < \cdots. \tag{16}$$

This is possible since $b_j/\ell_j \to 0$, $\ell_j \to 0$ and $\mu_j \to 0$ as j tends to infinity. In fact, we can find j_1 for which (14) with $k=1$ holds. Then choose c_{j_1}, $0 < c_{j_1} < a_{j_1}/4$, so that (15) holds. Next we can find j_2 such that $\mu_{j_2}\ell_{j_2} < c_{j_1}$ and (14) with $k=2$ holds. Continuing this procedure, we can extract subsequences $\{a_{j_k}\}$, $\{b_{j_k}\}$, $\{\ell_{j_k}\}$ and choose $\{c_{j_k}\}$ with the required property. For simplicity we renumber the suffix j_k just as j.

We shall show that

$$v(x,y) = G(x,y;\sum_{j=1}^{\infty} jf_j)$$

is a Green potential with the required property. In view of Lemma 5

$$v(x,y) \ge jG(x,y;f_j) \ge m_3 j \quad \text{for } (x,y) \in G_j,$$

which, together with Lemma 2, implies that

$$\limsup_{y\to 0,\,(x,y)\in\gamma'} v(x,y) = +\infty$$

for every curve γ' having tangency equivalent with γ.

Next we give upper estimates of v. We infer from (12) and (14) that

$$\sum_{k=j+1}^{\infty} G(x,y;kf_k) \le \text{const.}2^{-j} \quad \text{for } y \ge \mu_{j+1}\ell_{j+1}. \tag{17}$$

In view of (16) we have

$$\frac{c_j}{c_k} < 4^{k-j} \quad \text{for } k \le j.$$

Hence it follows from (13), (15) and (16) that

$$\begin{aligned} \sum_{k=1}^{j} G(x,y;kf_k) &\le \text{const.} \sum_{k=1}^{j} k \left(\frac{\ell_k}{a_k}\right)^n \frac{b_k}{\ell_k} \frac{c_k}{a_k} \frac{c_j}{c_k} \\ &\le \text{const.} \sum_{k=1}^{j} 2^{-k} 4^{k-j} \le \text{const.}2^{-j} \quad \text{for } 0 < y \le c_j. \end{aligned} \tag{18}$$

Adding (17) and (18), we obtain that

$$\sup_{\mu_{j+1}\ell_{j+1} \le y \le c_j} v(x,y) \le \text{const.}2^{-j} \to 0 \tag{19}$$

as j tends to infinity, which implies that

$$\liminf_{y\to 0,\, (x,y)\in\gamma'} v(x,y) = 0$$

for all curves γ' ending at a point on the bounday.

Finally we shall show that v has nontangential limit zero at almost every boundary point. Let

$$v_j'(x,y) = \sum_{k\ne j} G(x,y;kf_k).$$

Adding (17) and (18) with $j-1$ replacing j, we obtain from (16) that

$$\sup_{c_j \le y \le \mu_j \ell_j} v_j'(x,y) \le \sup_{\mu_{j+1}\ell_{j+1} \le y \le c_{j-1}} v_j'(x,y) \le \text{const.}2^{-j} \to 0 \tag{20}$$

Observe that $f_j(\xi,\eta)$ vanishes if $\xi \notin \cup_{\xi\in M_j} B(\xi, b_j)$. We apply Lemma 6 directly to $f = jf_j$ and $\ell = \mu_j\ell_j$ to obtain from (14) that for $0 < y \le \mu_j\ell_j$ and $x \in R^n \setminus M_j^*$

$$G(x,y;jf_j) \le m_4 j \|M_{\mu_j\ell_j} F_j\|_\infty \le \text{const.} j\sqrt{\frac{b_j}{\ell_j}} \le \text{const.}2^{-j},$$

where $M_j^* = \cup_{\xi \in M_j} B(\xi, b_j + \mu_j \ell_j)$. Combining this with (20), we obtain

$$\sup_{x \in R^n \setminus M_j^*,\, c_j \le y \le \mu_j \ell_j} v(x,y) \le \text{const.} 2^{-j} \to 0 \tag{21}$$

as j tends to infinity. Let $E_\alpha = \{\xi; \limsup_{y\to 0,\ (x,y)\in V_\alpha(\xi)} v(x,y) > 0\}$. We infer from (19) and (21) that E_α is included in

$$\{\xi; V_\alpha(\xi) \text{ meets } M_j^* \times [c_j, \mu_j \ell_j] \text{ for infinitely many } j\},$$

and hence

$$E_\alpha \subset \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} M_j^\alpha,$$

where

$$M_j^\alpha = \bigcup_{\xi \in M_j^*} B(\xi, \alpha\mu_j\ell_j) = \bigcup_{k=1}^{n} \bigcup_{\nu=-\infty}^{\infty} \{(x_1, \dots, x_n); |x_k - \nu\ell_j| < b_j + (1+\alpha)\mu_j\ell_j\}.$$

Let $R > 0$. Observing that the number of cubes Q_k of side length ℓ_j that meet $B(0,R)$ is bounded by const.$(R/\ell_j)^n$ for $\ell_j \le R$, we obtain that for $\ell_j \le R$

$$\begin{aligned} |M_j^\alpha \cap B(0,R)| &\le \text{const.} \left(\frac{R}{\ell_j}\right)^n \big(b_j + (1+\alpha)\mu_j\ell_j\big)\ell_j^{n-1} \\ &\le \text{const.} R^n \left(\frac{b_j}{\ell_j} + (1+\alpha)\mu_j\right). \end{aligned}$$

Hence by (14)

$$\sum_{j=1}^{\infty} |M_j^\alpha \cap B(0,R)| < \infty \quad \text{for any } R > 0,$$

which implies that

$$|E_\alpha| \le \left| \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} M_j^\alpha \right| = 0.$$

Accordingly, for all $\alpha > 0$ and a.e. $\xi \in R^n$

$$\lim_{y\to 0,\ (x,y)\in V_\alpha(\xi)} v(x,y) = 0.$$

The theorem is proved.

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