Generalized Cranston-McConnell inequalities and Martin boundaries of unbounded domains

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Abstract. Let D be a domain in $\mathbf{R}^2$ with the Green function $G(x,y)$ for the Laplace equation. We give a generalization of the Cranston-McConnell inequality concerning the integrability of positive harmonic functions on D. A typical new inequality is

$$\frac{1}{u(x)}\int_D G(x,y)u(y)\prod_{i=1}^n v_i(y)dy \le c_n\int_D\prod_{i=1}^n v_i(y)dy,$$

where u and $v_1,\dots,v_n$ are positive superharmonic functions on D and c_n is a constant depending only on n. The generalized Cranston-McConnell inequality is used for the determination of the Martin boundary of a certain unbounded domain.

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1. Introduction

Let D be a domain in $\mathbf{R}^2$ with the Green function $G(x,y)$ for the Laplace equation. By $|D|$ we denote the area of D. In [7] Cranston and McConnell established the following inequality.

Theorem A. *There exists an absolute constant c such that if D is a domain of finite area and u is a positive harmonic function on D, then for $x \in D$*

$$\frac{1}{u(x)} \int_D G(x,y)u(y)dy \le c|D|.$$

The main aim of this paper is to generalize Theorem A and apply it to determine the Martin boundary of a certain unbounded domain. In what follows, a positive continuous superharmonic function means a $(0,\infty]$-valued continuous (in the extended sense) superharmonic function. Let $\Phi(t_1,\dots,t_n)$ be a nonnegative Borel measurable function on $(0,\infty]^n$. For $\eta > 1$ we define $\Psi(t_1,\dots,t_n) = \Psi_\eta(t_1,\dots,t_n)$ by

$$\Psi(t_1,\dots,t_n) = \Psi_\eta(t_1,\dots,t_n) = \sup_{\eta^{-2}<c_1,\dots,c_n<\eta^2} \Phi(c_1t_1,\dots,c_nt_n).$$

The following theorem is a generalization of Theorem A.

Theorem 1. *For each positive integer n and positive number η there exists a positive constant $c_n = c_n(\eta)$ depending only on n and η such that if u and $v_1,\dots,v_n$ are positive continuous superharmonic functions on D, then for $x \in D$*

$$\frac{1}{u(x)} \int_D G(x,y)u(y)\Phi(v_1(y),\dots,v_n(y))dy \le c_n \int_D \Psi(v_1(y),\dots,v_n(y))dy. \tag{1}$$

If $\Phi(t_1,\dots,t_n)$ is nondecreasing with respect to each t_i, then so is Ψ and (1) holds for a general positive superharmonic functions u, $v_1,\dots,v_n$. In fact, u and $v_1,\dots,v_n$ can be approximated from below by continuous superharmonic functions on every compact subset in D ([9, Theorem 4.20]), and hence the monotone convergence theorem yields (1). In particular, we have

Corollary 1. *For each positive integer n there exists a positive constant c_n depending only on n such that if u and $v_1,\dots,v_n$ are positive superharmonic functions on D, then for $x \in D$*

$$\frac{1}{u(x)} \int_D G(x,y)u(y)\prod_{i=1}^n v_i(y)dy \le c_n \int_D \prod_{i=1}^n v_i(y)dy.$$

We use the above theorem to determine the Martin boundary of a certain unbounded domain. Ioffe and Pinsky [10] investigated positive harmonic functions in a horn-shaped domain $\Omega = \{(x,s) \in \mathbf{R}^N \times \mathbf{R}^1 : |s| < a(|x|)\}$, where $N \ge 2$, and showed the following theorem. Let Ω^*, $\partial_M\Omega$ and K be the Martin compactification, the Martin boundary and the Martin kernel for $(-\Delta,\Omega)$, respectively (cf. [9, Chapter 12]). By H_+ we denote the cone of positive harmonic functions in Ω vanishing continuously on $\partial\Omega$. The set

$$\Gamma = \{\xi \in \partial_M\Omega : K(\cdot,\xi) \in H_+\}$$

is called the Martin boundary at infinity for $(-\Delta,\Omega)$.

Theorem B. *Assume that a is a positive C^2-function on $[0,\infty)$ such that $a' \ge 0, a'' \le 0, a(r)/r$ and the curvature $k(r)$ of the curve $\{(r,a(r)) : r \ge 0\}$ are non-increasing, and $\lim_{r\to\infty} a(r)k(r) = 0$. Then the Martin boundary Γ at infinity for $(-\Delta,\Omega)$ is as follows:*

(i) *If $\displaystyle\int_1^\infty a(r)r^{-2}dr = \infty$, then Γ is a single point.*

(ii) *If $\displaystyle\int_1^\infty a(r)r^{-2}dr < \infty$, then Γ is homeomorphic to the sphere S^{N-1}.*

Thus, (i) implies that H_+ is one-dimensional; and (ii) implies that any element in H_+ is uniquely represented by an integral of minimal elements in H_+ with respect to a finite Borel measure on S^{N-1}. We can extend substantially Theorem B (ii) by making use of Theorem 1 and decomposition methods developed in [14], where the Martin boundary was studied extensively.

Theorem 2. *Let a and b be locally Lipschitz continuous functions on $[1,\infty)$ such that $b < a$. Let E be a Lipschitz domain in S^{N-1} or the whole S^{N-1} and put*

$$\begin{aligned}\Omega &= \{(x,s) \in \mathbf{R}^N \times \mathbf{R}^1 : b(|x|) < s < a(|x|),\ x/|x| \in E,\ |x| > 1\},\\ D &= \{(r,s) \in \mathbf{R}^2 : b(r) < s < a(r),\ 1 < r < \infty\}.\end{aligned}$$

Suppose

$$\int_1^\infty \frac{a(r)-b(r)}{r^2}dr < \infty. \tag{2}$$

Then the Martin compactification Ω^* for $(-\Delta,\Omega)$ is homeomorphic to $(\overline{D}\cup\{\infty\})\times\overline{E}$, the Martin boundary $\partial_M\Omega = \Omega^*\setminus\Omega$ is homeomorphic to $\partial\Omega\cup(\{\infty\}\times\overline{E})$, and any element of $\partial_M\Omega$ is minimal. In particular, the Martin boundary Γ at infinity is homeomorphic to $\overline{E}$.

We observe that Theorem 2 with $b = -a$ and $E = S^{N-1}$ implies Theorem B (ii), since a part of the domain close to the origin does not affect the Martin boundary at infinity (cf. [14, Theorem 7.1]).

The rest of this paper is organized as follows. In Section 2 we establish the so called Basic Estimate, Theorem 3, which is of independent interest and is fundamental in the proof of Theorem 1. The proof is given in Section 3. Theorem 2 is proved in Section 4 as a corollary to a more general result, Theorem 4. As another corollary to Theorem 4, we shall construct a class of domains having a boundary point which corresponds to a continuum of Martin boundary elements (see Theorem 5 in Section 4).

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2. Basic estimates

Throughout this section we let D be a domain in $\mathbf{R}^2$ with the Green function $G(x,y)$. It is well known that there is an absolute constant c_0 such that

$$\sup_{x\in D}\int_D G(x,y)dy \le c_0|D| \tag{3}$$

(cf. [3, Theorem 2.8] and [4, Lemma 1]). We observe that if $u\equiv 1$, then Theorem A is nothing but (3). The basic idea behind Theorem A was how to reduce a general case to (3). This is called the basic estimate. Theorem 1 will follow from a generalized basic estimate, which is valid also for the higher dimensional case.

Theorem 3. *Let $N\ge 2$ and let D be a domain in $\mathbf{R}^N$. Let u be a positive continuous superharmonic function on D. Let $\eta>1$ and put $D_j=\{x\in D:\eta^{j-1}<u(x)<\eta^{j+2}\}$ and $C_j=\{x\in D:\eta^j\le u(x)\le \eta^{j+1}\}$ for an integer j. If f is a nonnegative measurable function on D, then*

(4)
$$\sup_{x\in D}\frac{1}{u(x)}\int_D G(x,y)f(y)dy \le \left(\frac{\eta}{\eta-1}\right)^2\sum_{j=-\infty}^{\infty}\sup_{x\in C_j}\frac{1}{u(x)}\int_{C_j}G_{D_j}(x,y)f(y)dy,$$

$$\sup_{x\in D}\int_D G(x,y)f(y)dy \le \left(\frac{\eta}{\eta-1}\right)^2\sum_{j=-\infty}^{\infty}\sup_{x\in C_j}\int_{C_j}G_{D_j}(x,y)f(y)dy. \tag{5}$$

The original basic estimate was first proved by Cranston-McConnell [7] and then simplified by Chung [6]. Both of them employed a probabilistic method. In the previous paper [2] the first author gave such a basic estimate and proved Theorem A in a purely analytical way. After [2] was submitted the second author found that another analytic proof of the Cranston-McConnell inequality was also given by Bañuelos and Wolff [5]. The proof of Theorem 3 will follow the idea of [5], since it has an advantage that it can deal with superharmonic functions directly.

It is also noteworthy that $\{x\in D: a<u(x)<b\}$ is an **open** set since u is continuous. Therefore, we can consider the Green function G_{D_j} for D_j. For a general superharmonic function u the above set needs not to be open, it is merely finely open. So, it is necessary to work with fine potential theory to handle the general case. Such a treatment will be considered elsewhere. We also note that $\{x\in D: a\le u(x)\le b\}$ is a relatively closed set in the present situation.

Proof of Theorem 3. By the monotone convergence theorem we may assume that f is bounded and has compact support. As a result we observe that for any subset $E\subset D$ the Green potential

$$\int_E G(x,y)f(y)dy$$

is a bounded continuous function which tends to 0 as x approaches a regular boundary point on ∂D, and as $x\to\infty$ if $N\ge 3$ and D is unbounded. The set of irregular boundary points is a polar set and we may say that the potential vanishes q.e. (quasi everywhere) on ∂D. Let $K_j = \{x \in D : u(x) = \eta^j\}$. This is a relatively closed set. Observe that $D \cap \partial D_j \subset K_{j-1} \cup K_{j+2}$ and $D \cap \partial C_j \subset K_j \cup K_{j+1}$. Let

$$\begin{aligned} V_j(x) &= \frac{1}{u(x)} \int_{C_j} G(x,y) f(y) dy \qquad \text{for } x \in D, \\ v_j(x) &= \frac{1}{u(x)} \int_{C_j} G_{D_j}(x,y) f(y) dy \quad \text{for } x \in D_j. \end{aligned}$$

Observe that uV_j and uv_j are bounded positive continuous superharmonic functions on D and D_j, respectively; and they vanish q.e. on ∂D and ∂D_j, respectively. Let $M_j = \sup_{C_j} V_j$. By assumption M_j is finite. Observe that uV_j is harmonic outside C_j and it is continuous on D. By the maximum principle or the Phragmén-Lindelöf principle (cf. [8, Theorem 5.16]) we have

$$uV_j \le M_j u \quad \text{on } D, \tag{6}$$

since the inequality holds trivially on on C_j. Dividing both sides by u, we obtain

$$\sup_D V_j = \sup_{C_j} V_j = M_j. \tag{7}$$

In particular,

$$\sup_{K_{j-1}} V_j \le M_j. \tag{8}$$

Since $u \le \eta^{j+1}$ on C_j, it follows from (6) that $uV_j \le M_j \eta^{j+1}$ on C_j, and so is on D again by the maximum principle. Since $u = \eta^{j+2}$ on K_{j+2}, it follows that

$$\sup_{K_{j+2}} V_j \le \frac{1}{\eta} M_j. \tag{9}$$

Let us compare V_j and v_j on D_j. Since $G(\cdot, y) - G_{D_j}(\cdot, y)$ is harmonic on D_j for $y \in D_j$ and

$$u(x) \left(V_j(x) - v_j(x)\right) = \int_{C_j} [G(x,y) - G_{D_j}(x,y)] f(y) dy,$$

it follows that $u(V_j - v_j)$ is harmonic on D_j. Observing that $u = \eta^i$ on K_i, we obtain from (8) and (9) that

$$u(V_j - v_j) \le \begin{cases} M_j \eta^{j+1} & \text{on } K_{j+2}, \\ M_j \eta^{j-1} & \text{on } K_{j-1}. \end{cases}$$

Hence

$$u(V_j - v_j) - M_j\frac{u}{\eta} \le \begin{cases} 0 & \text{on } K_{j+2}, \\ M_j\eta^{j-1}(1-\dfrac{1}{\eta}) & \text{on } K_{j-1}. \end{cases}$$

Since the left hand side is a bounded subharmonic function on D_j and it is non positive q.e. on $\overline{D_j} \cap \partial D$, it follows from the maximum principle that

$$u(V_j - v_j) - M_j\frac{u}{\eta} \le M_j\eta^{j-1}(1-\frac{1}{\eta}) \text{ on } D_j.$$

Dividing both sides by u, taking the supremum over C_j, and using the inequality $u \ge \eta^j$ on C_j, we obtain

$$\begin{aligned} M_j = \sup_{C_j} V_j &\le \sup_{C_j} v_j + \frac{1}{\eta}M_j + M_j\eta^{j-1}(1-\frac{1}{\eta})\sup_{C_j}\frac{1}{u} \\ &\le \sup_{C_j} v_j + M_j\left(\frac{1}{\eta} + \frac{1}{\eta}(1-\frac{1}{\eta})\right) \\ &= \sup_{C_j} v_j + M_j\frac{2\eta-1}{\eta^2}. \end{aligned}$$

Therefore,

$$M_j \le \left(\frac{\eta}{\eta-1}\right)^2 \sup_{C_j} v_j. \tag{10}$$

Since $\bigcup_j C_j = D$, it follows from (7) and (10) that

$$\sup_D \frac{1}{u}\int_D G(\cdot,y)f(y)dy \le \sup_D \sum_j V_j \le \sum_j \sup_D V_j = \sum_j M_j \le \left(\frac{\eta}{\eta-1}\right)^2 \sum_j \sup_{C_j} v_j. \tag{11}$$

Thus (4) is proved.

For the proof of (5) we retain C_j and K_j and let

$$\begin{aligned} V_j(x) &= \int_{C_j} G(x,y)f(y)dy \qquad \text{for } x \in D, \\ v_j(x) &= \int_{C_j} G_{D_j}(x,y)f(y)dy \quad \text{for } x \in D_j \end{aligned}$$

and $M_j = \sup_{C_j} V_j$. Observe that

$$\begin{aligned} &\sup_D V_j = \sup_{C_j} V_j = M_j, \\ &\sup_{K_{j+2}} V_j \le M_j. \end{aligned}$$

Furthermore, since $V_j \le M_j \eta^{-j} u$ on C_j, it follows that

$$\sup_{K_{j-1}} V_j \le \frac{1}{\eta} M_j.$$

We see that $V_j - v_j$ is bounded and harmonic on D_j and

$$V_j - v_j - \frac{1}{\eta} M_j \le \begin{cases} (1 - \frac{1}{\eta}) M_j & \text{on } K_{j+2}, \\ 0 & \text{on } K_{j-1}. \end{cases}$$

Since $u = \eta^{j+2}$ on K_{j+2}, it follows from the maximum principle that

$$V_j - v_j - \frac{1}{\eta} M_j \le (1 - \frac{1}{\eta}) M_j \frac{u}{\eta^{j+2}} \text{ on } D_j.$$

Taking the supremum over C_j, we obtain

$$\sup_{C_j} V_j \le \sup_{C_j} v_j + \frac{1}{\eta} M_j + (1 - \frac{1}{\eta}) M_j \frac{1}{\eta},$$

because $u \le \eta^{j+1}$ on C_j. This inequality yields (10). In the same way as in (11), we have from (10)

$$\sup_D \int_D G(\cdot, y) f(y) dy \le \sup_D \sum_j V_j \le \sum_j \sup_D V_j = \sum_j M_j \le \left(\frac{\eta}{\eta - 1} \right)^2 \sum_j \sup_{C_j} v_j.$$

Thus (5) follows. The theorem is proved.

3. Proof of Theorem 1

Proof of Theorem 1. First we consider the case $u \equiv 1$. For simplicity let $f(y) = \Phi(v_1(y), \dots, v_n(y))$. Let $C^i_j = \{x \in D : \eta^j \le v_i(x) \le \eta^{j+1}\}$ and $D^i_j = \{x \in D : \eta^{j-1} < v_i(x) < \eta^{j+2}\}$. We apply (5) with $u = v_1$ to obtain

$$\begin{aligned}\int_D G(x,y)f(y)dy &\le \left(\frac{\eta}{\eta-1}\right)^2 \sum_{j_1} \sup_{x \in C^1_{j_1}} \int_{C^1_{j_1}} G_{D^1_{j_1}}(x,y)f(y)dy \\ &\le \left(\frac{\eta}{\eta-1}\right)^2 \sum_{j_1} \sup_{x \in D^1_{j_1}} \int_{C^1_{j_1}} G_{D^1_{j_1}}(x,y)f(y)dy.\end{aligned}$$

Repeating this n times, we obtain

(12)
$$\int_D G(x,y)f(y)dy \le \left(\frac{\eta}{\eta-1}\right)^{2n} \sum_{j_1,\dots,j_n} \sup_{x \in D^1_{j_1}\cap\dots\cap D^n_{j_n}} \int_{C^1_{j_1}\cap\dots\cap C^n_{j_n}} G_{j_1,\dots,j_n}(x,y)f(y)dy,$$

where $G_{j_1,\dots,j_n}$ denotes the Green function $G_{D^1_{j_1}\cap\dots\cap D^n_{j_n}}$ for $D^1_{j_1}\cap\dots\cap D^n_{j_n}$. Let

$$\psi(t_1,\dots,t_n) = \sup_{1 \le c_1,\dots,c_n \le \eta} \Phi(c_1 t_1, \dots, c_n t_n).$$

Observe that

$$\begin{aligned}&f(y) = \Phi(v_1(y),\dots,v_n(y)) \le \psi(\eta^{j_1},\dots,\eta^{j_n}) && \text{for } y \in C^1_{j_1}\cap\dots\cap C^n_{j_n}, \\ &\psi(\eta^{j_1},\dots,\eta^{j_n}) \le \Psi(v_1(y),\dots,v_n(y)) && \text{for } y \in D^1_{j_1}\cap\dots\cap D^n_{j_n}.\end{aligned}$$

Hence by (3)

$$\begin{aligned}\int_{C^1_{j_1}\cap\dots\cap C^n_{j_n}} G_{j_1,\dots,j_n}(x,y)f(y)dy &\le \psi(\eta^{j_1},\dots,\eta^{j_n}) \int_{C^1_{j_1}\cap\dots\cap C^n_{j_n}} G_{j_1,\dots,j_n}(x,y)dy \\ &\le c_0 \psi(\eta^{j_1},\dots,\eta^{j_n}) |D^1_{j_1}\cap\dots\cap D^n_{j_n}| \\ &\le c_0 \int_{D^1_{j_1}\cap\dots\cap D^n_{j_n}} \Psi(v_1(y),\dots,v_n(y))dy\end{aligned}$$

for $x \in D^1_{j_1}\cap\dots\cap D^n_{j_n}$. This, together with (12), yields

$$\int_D G(x,y)f(y)dy \le c_0 \left(\frac{\eta}{\eta-1}\right)^{2n} \sum_{j_1,\dots,j_n} \int_{D^1_{j_1}\cap\dots\cap D^n_{j_n}} \Psi(v_1(y),\dots,v_n(y))dy.$$

Since $\{D^1_{j_1}\cap\dots\cap D^n_{j_n}\}_{j_1,\dots,j_n}$ covers D at most 3^n times, we obtain (1) for the case $u \equiv 1$ with $c_n = 3^n c_0[\eta/(\eta-1)]^{2n}$.

Let us consider the general case. Let f be as above. We apply (4) with fu replacing f to obtain

$$\begin{aligned}\frac{1}{u(x)}\int_D G(x,y)f(y)u(y)dy &\le \left(\frac{\eta}{\eta-1}\right)^2 \sum_j \eta^{-j} \sup_{x\in C_j}\int_{C_j} G_{D_j}(x,y)f(y)u(y)dy \\ &\le \left(\frac{\eta}{\eta-1}\right)^2 \sum_j \eta^{-j}\eta^{j+1} \sup_{x\in D_j}\int_{D_j} G_{D_j}(x,y)f(y)dy \\ &= \frac{\eta^3}{(\eta-1)^2}\sum_j \sup_{x\in D_j}\int_{D_j} G_{D_j}(x,y)f(y)dy.\end{aligned}$$

By letting $\eta = 3$ we obtain

$$\frac{1}{u(x)}\int_D G(x,y)f(y)u(y)dy \le \frac{3^3}{4}\sum_j \sup_{x\in D_j}\int_{D_j} G_{D_j}(x,y)f(y)dy.$$

Estimating the supremum in the right hand side by using the first case and observing $\{D_j\}$ covers D at most 3 times, we obtain (1) with $c_n = \frac{1}{4}3^{n+4}c_0[\eta/(\eta-1)]^{2n}$. The theorem is proved.

4. Martin boundary

In this section we prove Theorem 2 and shall deal with several differential operators. In particular, the N-dimensional and the two dimensional Laplacians are considered. To distinguish them we write Δ_N and Δ_2, respectively. We shall need many positive constants. So, for simplicity, by the symbol c we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $c_1, c_2, \ldots,$ to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $c \ge 1$ such that $c^{-1}f_1 \le f_2 \le cf_1$. The constant c will be called the constant of comparison.

Theorem 2 comes from the following general result.

Theorem 4. *Let D be a domain in $\{(r,s)\in \mathbf{R}^2 : r > 0\}$ such that every boundary point of D is regular with respect to the Dirichlet problem. Let E be a Lipschitz domain in S^{N-1} or the whole S^{N-1} and put*

$$\Omega = \{(x,s)\in \mathbf{R}^N\times \mathbf{R}^1 : (|x|,s)\in D, x/|x|\in E\}.$$

Suppose

$$\iint_D \frac{drds}{r^2} < \infty. \tag{13}$$

Then the Martin compactification Ω^* *for* $(-\Delta_{N+1}, \Omega)$ *is homeomorphic to* $D^* \times \overline{E}$, *where* D^* *is the Martin compactification of* D *for* $(-\Delta_2, D)$*. In particular,* $\partial_M \Omega$ *is homeomorphic to* $(D \times \partial E) \cup (\partial_M D \times \overline{E})$*. Moreover, if any element of the Martin boundary* $\partial_M D = D^* \setminus D$ *is minimal, then so is any element of the Martin boundary* $\partial_M \Omega = \Omega^* \setminus \Omega$.

Before giving a proof of Theorem 4, we complete the proof of Theorem 2 and give another corollary of Theorem 4.

Proof of Theorem 2. Let $D = \{(r, s) : b(r) < s < a(r), 1 < r < \infty\}$. Considering D as a subdomain of the complex plane, we see, by the Carathéodory theorem (cf. [16]), that there exists a homeomorphism from $\overline{D} \cup \{\infty\}$ onto the closed unit disc which is conformal in D. Hence the Martin compactification D^* for $(-\Delta_2, D)$ is homeomorphic to $\overline{D} \cup \{\infty\}$ and any element of the Martin boundary $\partial_M D$ is minimal. Since (2) implies (13), Theorem 2 is straightforward from Theorem 4.

Let us proceed to another consequence of Theorem 4, which yields a class of domains having a boundary point which corresponds to a continuum of Martin boundary elements. In [11, Example 3] Martin treated a domain in $\mathbf{R}^3$ which is bounded by two spheres, one internally tangent to the other at the origin 0, and a plane containing the common diameter. He showed that the Martin boundary at 0 for this domain is homeomorphic to $[-\pi/2, \pi/2]$. See also [1]. Subsequently, Maz'ja [12] threw a light on this phenomenon and announced a class of domains having similar properties. The following theorem, which follows from Theorem 4 as Theorem 2 does, implies [11, Example 3] and [12, Theorem 2].

Theorem 5. *Let* $\delta > 0$ *and let* a *and* b *be locally Lipschitz continuous functions on* $(0, \delta]$ *such that* $b < a$ *and* $\lim_{r\to 0} a(r) = \lim_{r\to 0} b(r) = 0$*. Let* E *be a Lipschitz domain in* S^{N-1} *or the whole* S^{N-1} *and put*

$$\Omega = \{(x, s) \in \mathbf{R}^N \times \mathbf{R}^1 : b(|x|) < s < a(|x|),\ x/|x| \in E,\ 0 < |x| < \delta\},$$
$$D = \{(r, s) \in \mathbf{R}^2 : b(r) < s < a(r),\ 0 < r < \delta\}.$$

Suppose

$$\int_0^\delta \frac{a(r) - b(r)}{r^2} dr < \infty.$$

Then the Martin compactification Ω^* *for* $(-\Delta, \Omega)$ *is homeomorphic to* $\overline{D} \times \overline{E}$*, the Martin boundary* $\partial_M \Omega$ *is homeomorphic to* $(\partial\Omega \setminus \{0\}) \cup (\{0\} \times \overline{E})$*, and any element of* $\partial_M \Omega$ *is minimal. In particular, the Martin boundary at the origin is homeomorphic to* $\overline{E}$.

Now let us proceed to the proof of Theorem 4. We write the N-dimensional Laplacian Δ_N in polar coordinates,

$$\Delta_N = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{\Lambda}{r^2},$$

where Λ is the Laplace-Beltrami operator on the sphere S^{N-1}. By the change of an unknown function u to $v = r^{(N-1)/2}u$, the Laplace equation in Ω becomes the equation

$$Pv = \left(-\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2} - \frac{\Lambda}{r^2} + \frac{(N-1)(N-3)}{4r^2} \right) v = 0 \quad \text{in } D \times E.$$

Observe that the Green function for $(-\Delta_{N+1}, \Omega)$ with argument at (x, s) and pole at $(\widetilde{x}, \widetilde{s})$ is equal to

$$\left(\frac{|x|}{|\widetilde{x}|} \right)^{\frac{1-N}{2}} \mathcal{G}(|x|, s, \frac{x}{|x|}; |\widetilde{x}|, \widetilde{s}, \frac{\widetilde{x}}{|\widetilde{x}|}),$$

where $\mathcal{G}$ is the Green function for $(P, D \times E)$. By this form of the Green function it is easy to see that $(-\Delta_{N+1}, \Omega)$ and $(P, D \times E)$ have homeomorphic Martin compactifications. Thus Theorem 4 follows from the next theorem.

Theorem 6. *Let D and E be as in Theorem 4 and let P be as above. If (13) holds, then the Martin compactification for $(P, D \times E)$ is homeomorphic to $D^* \times \overline{E}$. Moreover, if any element of the Martin boundary $\partial_M D$ is minimal, then so is any element of the Martin boundary for $(P, D \times E)$.*

We shall show Theorem 6 by making use of the decomposition method for constructing the Martin boundary which is developed in [14], and is a generalized version of the separation of variables method used in [12] and [13]. Hereafter we assume that (13) holds.

From now on we consider functions on the plane domain D and change the variables (r, s) to (x_1, x_2). Let $L^2(E)$ be the space of square integrable functions over E. Let $\lambda_0 < \lambda_1 \leq \ldots$ be the eigenvalues repeated according to multiplicity of the Dirichlet realization of $-\Lambda + (N-1)(N-3)/4$ on $L^2(E)$. By L_j we denote the operator defined by $L_j v = (-\Delta_2 + V_j)v$ with $V_j(x) = \lambda_j x_1^{-2}$. Note that for $j = 0, 1, \ldots,$

$$L_j(x_1^{1/2}) = (\lambda_j + \frac{1}{4})x_1^{-3/2} \geq 0 \quad \text{on } \mathbf{R}^2_+ = \{(x_1, x_2) : x_1 > 0\};$$

in particular there is a positive supersolution to $L_j u = 0$ on $\mathbf{R}^2_+$. Since $|\mathbf{R}^2_+ \setminus \overline{D}| > 0$, we see from [14, Theorem 1.8] that there exists the Green function $H_j(x, y)$ for (L_j, D). By $G(x, y)$ we denote the Green function for $(-\Delta_2, D)$.

Let x_0 be a fixed point in D and let $D_\delta = \{x \in D : G(x, x_0) < \delta\}$ for $\delta > 0$. By $L_{j,\delta}$ we denote the operator defined by $L_{j,\delta} v = (-\Delta_2 + V_{j,\delta})v$ with $V_{j,\delta} = \chi_{D_\delta} V_j$. Since L_j and $L_{j,\delta}$ differ only on a compact subset of D, we have the following lemma from [15, Theorem 2.9] (see also [13, Theorem 2.11]).

Lemma 1. *There is a positive constant $c = c_{j,\delta}$ such that*

$$c^{-1} H_j(x, y) \leq H_{j,\delta}(x, y) \leq c H_j(x, y) \quad \textit{for } x, y \in D.$$

Let us apply Theorem 1 with $\eta = 2$, $n = 2$, $\Phi(t_1, t_2) = |\lambda_j|\chi_{(0,\delta)}(t_1)t_2^{-2}$, $v_1(z) = G(z, x_0)$ and $v_2(z) = z_1$. Then we have

$$\frac{1}{u(x)}\int_{D_\delta} G(x,z)\frac{|\lambda_j|}{z_1^2}u(z)dz \le c_2\int_{D_{4\delta}}\frac{|\lambda_j|}{z_1^2}dz \tag{14}$$

for every positive continuous superharmonic function u on D.

From this estimate we have

Lemma 2. *There is a positive constant $c = c_j$ such that*

$$c^{-1}G(x,y) \le H_j(x,y) \le cG(x,y) \quad \textit{for } x, y \in D.$$

Proof. Assumption (13) gives $\delta = \delta_j$ such that the right hand side of (14) is less than $1/4$. Letting $u = G(\cdot, y)$, we obtain

$$\int_D G(x,z)|V_{j,\delta}(z)|G(z,y)dz \le \frac{1}{4}G(x,y) \quad \text{for } x, y \in D. \tag{15}$$

We define $G_i(x,y)$ inductively by $G_0(x,y) = G(x,y)$ and

$$G_i(x,y) = \int_D G(x,z)V_{j,\delta}(z)G_{i-1}(z,y)dz \quad \text{for } i \ge 1.$$

In view of (15)

$$G_i(x,y) \le 4^{-i}G(x,y), \tag{16}$$

and hence $H(x,y) = \sum_{i=0}^{\infty}(-1)^i G_i(x,y)$ is convergent and

$$\frac{2}{3}G(x,y) \le H(x,y) \le \frac{4}{3}G(x,y).$$

Moreover, (16) and the dominated convergence theorem yield

$$\begin{aligned}\int_D G(x,z)V_{j,\delta}(z)H(z,y)dz &= \int_D G(x,z)V_{j,\delta}(z)\sum_{i=0}^{\infty}(-1)^i G_i(z,y)dz \\ &= \sum_{i=0}^{\infty}(-1)^i\int_D G(x,z)V_{j,\delta}(z)G_i(z,y)dz \\ &= \sum_{i=0}^{\infty}(-1)^i G_{i+1}(x,y) = G(x,y) - H(x,y).\end{aligned}$$

Thus $H(x,y)$ satisfies the resolvent equation

$$H(x,y) = G(x,y) - \int_D G(x,z)V_{j,\delta}(z)H(z,y)dz \quad \text{for } x,y \in D.$$

We observe from the resolvent equation that $H(x,y)$ actually coincides with the Green function $H_{j,\delta}(x,y)$. Thus

$$\frac{2}{3}G(x,y) \le H_{j,\delta}(x,y) \le \frac{4}{3}G(x,y),$$

and hence we obtain the required estimate by Lemma 1.

Let $K(x,y) = G(x,y)/G(x_0,y)$. By definition the limit $\lim_{y\to\xi} K(x,y)$ exists for each $x \in D$ when $y \in D$ tends to $\xi \in \partial_M D$ in the topology of D^*. This limit is called the Martin kernel $K(x,\xi)$ at ξ (with reference point at x_0). This is a positive harmonic function such that $K(x_0,\xi) = 1$.

Lemma 3. *Let $x \in D$ and $\xi \in \partial_M D$. Suppose $y \in D$ tends to ξ in the topology of D^*. Then the limit $\lim_{y\to\xi} H_j(x,y)/G(x_0,y)$ exists and is positive.*

Proof. In view of Lemma 2, the limit is positive if it exists. Let us show the existence of the limit. Let ε be an arbitrary positive number. We use (14) again and find $\delta_\varepsilon > 0$ so small that

$$\int_{D_{\delta_\varepsilon}} G(x,z)\frac{|\lambda_j|}{z_1^2}u(z)dz \le \varepsilon u(x)$$

for $x \in D$ and for $u \ge 0$ continuous and superharmonic on D. Let us apply this estimate to $u = K(\cdot,y)$ for $y \in D$. We have

$$\int_{D_{\delta_\varepsilon}} G(x,z)\frac{|\lambda_j|}{z_1^2}K(z,y)dz \le \varepsilon K(x,y) \quad \text{for } x,y \in D. \tag{17}$$

Since $H_j(x,y)$ is the Green function for (L_j, D), we have the resolvent equation

$$H_j(x,y) = G(x,y) - \int_D H_j(x,z)\frac{\lambda_j}{z_1^2}G(z,y)dz.$$

Divide both sides by $G(x_0,y)$ to obtain

$$\frac{H_j(x,y)}{G(x_0,y)} = K(x,y) - \int_{D\setminus D_{\delta_\varepsilon}} H_j(x,z)\frac{\lambda_j}{z_1^2}K(z,y)dz - \int_{D_{\delta_\varepsilon}} H_j(x,z)\frac{\lambda_j}{z_1^2}K(z,y)dz.$$

Let $y \to \xi$ in the topology of D^*. Since $K(\cdot,y) \to K(\cdot,\xi)$ uniformly on every compact subset of D, it follows that the first two terms in the right hand side are convergent. By (17) and Lemma 2 the last term is estimated as

$$\begin{aligned}\limsup_{y\to\xi}\left|\int_{D_{\delta_\varepsilon}} H_j(x,z)\frac{\lambda_j}{z_1^2}K(z,y)dz\right| &\le c\limsup_{y\to\xi}\int_{D_{\delta_\varepsilon}} G(x,z)\frac{|\lambda_j|}{z_1^2}K(z,y)dz \\ &\le c\varepsilon\limsup_{y\to\xi} K(x,y) = c\varepsilon K(x,\xi).\end{aligned}$$

Since ε is arbitrary, it follows that $\lim_{y\to\xi} H_j(x,y)/G(x_0,y)$ exists. The lemma is proved.

By D^{*j} we denote the Martin compactification for (L_j, D). Let us consider a convergence property for D^{*j}.

Lemma 4. *Let $x \in D$ and $\eta \in D^{*j} \setminus D$. Suppose $y \in D$ tends to η in the topology of D^{*j}. Then the limit $\lim_{y\to\eta} G(x,y)/H_j(x_0,y)$ exists and is positive.*

Proof. The proof is similar to that of Lemma 3. But for the completeness we give a proof. We have the resolvent equation

$$G(x,y) = H_j(x,y) + \int_D G(x,z)\frac{\lambda_j}{z_1^2}H_j(z,y)dz.$$

Let $\varepsilon > 0$ be an arbitrary number. We choose δ_ε so that (17) holds. Divide both sides by $H_j(x_0,y)$ to obtain

$$\frac{G(x,y)}{H_j(x_0,y)} = \frac{H_j(x,y)}{H_j(x_0,y)} + \int_{D\setminus D_{\delta_\varepsilon}} G(x,z)\frac{\lambda_j}{z_1^2}\frac{H_j(z,y)}{H_j(x_0,y)}dz + \int_{D_{\delta_\varepsilon}} G(x,z)\frac{\lambda_j}{z_1^2}\frac{H_j(z,y)}{H_j(x_0,y)}dz.$$

Let $y \to \eta$ in the topology of D^{*j}. We observe that first two terms in the right hand side are convergent. By (17) and Lemma 2 the last term is estimated as

$$\begin{aligned}\limsup_{y\to\eta}\left|\int_{D_{\delta_\varepsilon}} G(x,z)\frac{\lambda_j}{z_1^2}\frac{H_j(z,y)}{H_j(x_0,y)}dz\right| &\le c\limsup_{y\to\eta}\int_{D_{\delta_\varepsilon}} G(x,z)\frac{|\lambda_j|}{z_1^2}K(z,y)dz\\ &\le c\varepsilon\limsup_{y\to\eta} K(x,y) \le c\varepsilon\limsup_{y\to\eta}\frac{H_j(x,y)}{H_j(x_0,y)}.\end{aligned}$$

Since y converges in the topology of D^{*j}, it follows that the last upper limit is convergent. Since ε is arbitrary, it follows that $\lim_{y\to\eta} G(x,y)/H_j(x_0,y)$ exists. The lemma is proved.

Lemma 5. *The Martin compactifications D^* and D^{*j} are homeomorphic. Moreover, if any element of $\partial_M D$ is minimal, then so is any element of $D^{*j} \setminus D$.*

Proof. Suppose $\{y_k\} \subset D$ is a sequence convergent in the topology of D^*. Writing

$$\frac{H_j(x,y_k)}{H_j(x_0,y_k)} = \frac{H_j(x,y_k)/G(x_0,y_k)}{H_j(x_0,y_k)/G(x_0,y_k)},$$

we observe from Lemma 3 that the limit of $H_j(x,y_k)/H_j(x_0,y_k)$ exists. Similarly, if $\{y_k\} \subset D$ is a sequence convergent in the topology of D^{*j}, then the limit of $G(x,y_k)/G(x_0,y_k)$ exists. From these observations we conclude the Martin compactifications D^* and D^{*j} are homeomorphic. The last assertion follows from Lemma 2 and [15, Theorem 2.3].

Proof of Theorem 6. Suppose $y \in D$ tends to $\xi \in D^{*0} \setminus D$ in the topology of D^{*0}. Then, by Lemma 5, y converges also in the topology of D^*. Writing

$$\frac{H_j(x,y)}{H_0(x_0,y)} = \frac{H_j(x,y)}{G(x_0,y)} \cdot \frac{G(x_0,y)}{H_0(x_0,y)},$$

we obtain from Lemmas 3 and 4 that the limits

$$\lim_{y \to \xi} \frac{H_j(x,y)}{H_0(x_0,y)}, \quad j = 1, 2, \dots$$

exist and are positive for each $x \in D$. Hence [14, Theorem 3.5] implies that the Martin compactification for $(P, D \times E)$ is homeomorphic to $D^{*0} \times \overline{E}$, and hence to $D^* \times \overline{E}$ by Lemma 5. The last assertion also follows from [14, Theorem 3.5] and Lemma 5. The theorem is proved.

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