Generalized Cranston-McConnell inequalities for discontinuous superharmonic functions

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Abstract. Let Ω be an open set in $\mathbf{R}^2$ with Green function $G(x,y)$ for the Laplace equation. We give a generalization of the Cranston-McConnell inequality concerning the integrability of positive harmonic functions on Ω.

1. Introduction

Throughout this paper we let Ω be an open set in $\mathbf{R}^2$ with Green function $G(x,y)$. In [5] Cranston and McConnell established the following inequality.

Theorem A. *Let Ω be an open set of finite area $|\Omega|$. Then there exists an absolute constant c such that if u is a positive harmonic function on Ω, then*

$$\sup_\Omega \frac{1}{u}\int_\Omega G(\cdot,y)u(y)dy \le c|\Omega|.$$

We study a generalization of Theorem A. Let $\Phi(t_1,\dots,t_n)$ be a nonnegative Borel measurable function on $(0,\infty]^n$. For $\eta>1$ we define $\Psi(t_1,\dots,t_n)=\Psi_\eta(t_1,\dots,t_n)$ by

$$\Psi(t_1,\dots,t_n)=\Psi_\eta(t_1,\dots,t_n)=\sup_{\eta^{-2}<c_1,\dots,c_n<\eta^2}\Phi(c_1t_1,\dots,c_nt_n).$$

The main aim of this paper is to show the following theorem.

Theorem. *For each positive integer n and $\eta>1$ there exists a positive constant $c_n=c_n(\eta)$ depending only on n and η such that if u and $v_1,\dots,v_n$ are positive superharmonic functions on Ω, then*

$$\sup_\Omega \frac{1}{u}\int_\Omega G(\cdot,y)u(y)\Phi(v_1(y),\dots,v_n(y))dy \le c_n\int_\Omega \Psi(v_1(y),\dots,v_n(y))dy. \tag{1}$$

In the previous paper [1] we proved (1) for the case whether $v_1,\dots,v_n$ are continuous or $\Phi(t_1,\dots,t_n)$ is nondecreasing with respect to each t_i. For the general case a difficulty arises since the sets $\{x\in\Omega : a<u(x)<b\}$ and $\{x\in\Omega : a<v_i(x)<b\}$

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need not to be open. They are finely open sets. We shall overcome this difficulty by making use of the so-called fine potential theory.

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2. Preliminaries

The coarsest topology that makes all superharmonic functions continuous is called the fine topology. An open (resp. closed) set in this topology is called a finely open (resp. closed) set. Thus, if u is a superharmonic function on Ω, then the set $\{x \in \Omega : a < u(x) < b\}$ is a finely open set. For basic properties of fine topology and other fine notions, we refer to Fuglede [6]. It is known that the restriction of an ordinary superharmonic function to a finely open set is finely superharmonic there. For $E \subset \Omega$ we denote by $\complement E$ the complement of E in Ω. Let U be a finely open subset of Ω. We define the Green function for U by

$$G_U(\cdot, y) = G(\cdot, y) - \widehat{R}^{\complement U}_{G(\cdot,y)},$$

where $\widehat{R}^{\complement U}_{G(\cdot,y)}$ denotes the regularized reduced function. The Green function G_U is Borel measurable and vanishes quasi everywhere (q.e.) on $\complement U$. Here "q.e." means that the property holds outside a polar set. For a measure μ we denote by $G_U\mu$ the Green potential $\int G_U(\cdot, y)d\mu(y)$. We observe that $G_U\mu = 0$ q.e. on $\complement U$ and that if V is a finely open set with $\mu(V) = 0$ and $G_U\mu$ is finite on V, then $G_U\mu$ is finely harmonic on V ([7, Theorem 2.7]). We shall frequently use the fine boundary minimum principle.

Lemma A. *([6, Theorem 9.1]) Let u be a bounded finely superharmonic function in a finely open set $U \subset \Omega$. Suppose*

$$\operatorname*{fine\,lim\,inf}_{x\to y,\ x\in U} u(x) \ge 0,$$

for q.e. $y \in \partial_f U$, the fine boundary of U. Then $u \ge 0$ on U.

In fact, the boundedness of u in the above lemma can be relaxed. But for our purpose the present form is sufficient. The following lemma is well-known if U is an ordinary open set (cf. [2, Theorem 2.8] and [3, Lemma 1]).

Lemma 1. *Suppose Ω is bounded. Let U be a finely open subset of Ω and let m_U be the Lebesgue measure restricted over U. Then*

$$\sup_U G_U m_U \le c_0|U|,$$

where $c_0 > 0$ is an absolute constant.

Proof. Take $x \in U$. We show that there is $B = B(x, r)$, the open disk with center at x and radius r, such that

$$|U| \le |B| \le 4|U|, \tag{2}$$

$$\omega_B(x, U \cap \partial B) \le \frac{1}{2}, \tag{3}$$

where ω_B is the harmonic measure for B. Let $R > 0$ be such that $|U| = \pi R^2$. Then by the polar coordinate

$$\pi R^2 = |U| = \int_0^\infty \sigma(U \cap \partial B(x,r))dr \geq \int_R^{2R} \sigma(U \cap \partial B(x,r))dr,$$

where by $\sigma(E)$ we denote the one dimensional Hausdorff measure of E. Hence there is $r \in [R, 2R]$ such that $\sigma(U \cap \partial B(x,R)) \leq \pi R$. Let $B = B(x,r)$ for this r. Then

$$\omega_B(x, U \cap \partial B) = \frac{\sigma(U \cap \partial B)}{\sigma(\partial B)} \leq \frac{\pi R}{2\pi r} \leq \frac{1}{2}.$$

Thus (3) holds. It is easy to see (2) holds.

Let $M = \sup_U G_U m_U$. Since $U \subset \Omega$ and Ω is bounded, we have $M < \infty$. We claim

$$G_U m_U \leq G_{U\cap B} m_{U \cap B} + M\omega_B(\cdot, U \cap \partial B) \quad \text{on } U \cap B. \tag{4}$$

Let $u = G_U m_U - G_{U\cap B} m_{U\cap B}$. By [7, (3) on p.197] we see that

$$G_U m_{U\cap B} - G_{U\cap B} m_{U\cap B} = \widehat{R}^{\complement(U\cap B)}_{Gm_{U\cap B}} - \widehat{R}^{\complement U}_{Gm_{U\cap B}}.$$

The reduced functions $\widehat{R}^{\complement(U\cap B)}_{Gm_{U\cap B}}$ and $\widehat{R}^{\complement U}_{Gm_{U\cap B}}$ are finely harmonic in $U \cap B$ and in U respectively, according to [6, Corollary on p.86]. By [7, Theorem 2.7] $G_U m_{U\setminus B}$ is finely harmonic on $U \cap B$. Thus u is finely harmonic on $U \cap B$. Let χ be the indicator function of $U \cap \partial B$. Since U is measurable with respect to the harmonic measure $\omega_B(\cdot,\cdot)$, we have $\omega_B(\cdot, U \cap \partial B) = H^B_\chi$, where H^B_χ denotes the Dirichlet solution (cf. [6, Section 14]). In view of [6, Theorem 14.7], we have

$$\operatorname*{fine\,lim}_{\substack{x\to y\\ x\in B}} \omega_B(x, U\cap B) = \operatorname*{fine\,lim}_{\substack{x\to y\\ x\in B}} H^B_\chi(x) = 1 \qquad \text{for } y \in U \cap \partial B.$$

Since $u \leq G_U m_U = 0$ q.e. on $\complement U$, it follows from the minimum principle (Lemma A) that $u \leq M\omega_B(\cdot, U\cap \partial B)$ on $U \cap B$, so that (4) follows. Evaluating (4) at x, we obtain from (3) that

$$G_U m_U(x) \leq G_{U\cap B} m_{U\cap B}(x) + \frac{1}{2}M \leq G_B m_B(x) + \frac{1}{2}M.$$

By an elementary calculation we have $G_B m_B(x) = c_1|B|$ with absolute constant $c_1 > 0$. Hence, taking the supremum for $x \in U$, we obtain

$$M \leq c_1|B| + \frac{1}{2}M,$$

and so by (2)

$$M \leq 2c_1|B| \leq 8c_1|U|.$$

The lemma follows.

3. Proof of Theorem

We observe that if $u \equiv 1$, then Theorem A readily follows from the inequality in Lemma 1. The basic idea behind Theorem A was how to reduce a general case to this inequality. This was done by the so-called basic estimate.

Lemma 2. *Suppose Ω is bounded. Let u be a positive superharmonic function on Ω and let D be a finely open subset of Ω. For an integer j we let $D_j = \{x \in D : \eta^{j-1} < u(x) < \eta^{j+2}\}$ and $C_j = \{x \in D : \eta^j \le u(x) \le \eta^{j+1}\}$. If f is a nonnegative measurable function on D, then*

$$\sup_{C_j} \int_{C_j} G_D(\cdot, y) f(y) dy \le \left(\frac{\eta}{\eta - 1}\right)^2 \sup_{C_j} \int_{C_j} G_{D_j}(\cdot, y) f(y) dy.$$

Proof. By the monotone convergence theorem we may assume that f is bounded and has compact support in D. It is easy to see that the Green potential Gf is bounded on Ω. Let $K_j = \{x \in D : u(x) = \eta^j\}$ and let

$$V_j = \int_{C_j} G_D(\cdot, y) f(y) dy, \quad v_j = \int_{C_j} G_{D_j}(\cdot, y) f(y) dy$$

and $M_j = \sup_{C_j} V_j$. From the boundedness of Gf we have $M_j < \infty$. Since V_j is finely harmonic on $D \setminus C_j$ ([7, Theorem 2.7]), it follows from the minimum principle (Lemma A) that

$$\sup_D V_j = \sup_{C_j} V_j = M_j. \tag{5}$$

By definition $u \ge \eta^j$ on C_j and hence $V_j \le M_j u/\eta^j$ on C_j. Again by the minimum principle the same inequality holds on D. In particular $V_j \le M_j \eta^{-1}$ on K_{j-1}. Observe that

$$D \cap \partial_f D_j \subset K_{j-1} \cup K_{j+2}$$

and

$$V_j - v_j - \frac{M_j}{\eta} \le V_j - \frac{M_j}{\eta} \le \begin{cases} (1 - \frac{1}{\eta}) M_j & \text{on } K_{j+2}, \\ 0 & \text{on } K_{j-1}. \end{cases} \tag{6}$$

Let $w_j = \int_{C_j} G(\cdot, y) f(y) dy$. In view of [7, (3) on p.197] we have

$$V_j - v_j = \widehat{R}_{w_j}^{\complement D} - \widehat{R}_{w_j}^{\complement D_j}.$$

By [6, Corollary on p.86], the reduced functions $\widehat{R}_{w_j}^{\complement D}$ and $\widehat{R}_{w_j}^{\complement D_j}$ are finely harmonic in D and in D_j, respectively. Hence $V_j - v_j$ is finely harmonic in D_j. Since $u = \eta^{j+2}$ on K_{j+2}, it follows from (6) and the minimum principle (Lemma A) that

$$V_j - v_j - \frac{M_j}{\eta} \le (1 - \frac{1}{\eta}) M_j \frac{u}{\eta^{j+2}} \text{ on } D_j.$$

Taking the supremum over C_j, we obtain

$$M_j = \sup_{C_j} V_j \le \sup_{C_j} v_j + \frac{M_j}{\eta} + (1-\frac{1}{\eta})M_j\frac{1}{\eta} = \sup_{C_j} v_j + \frac{2\eta-1}{\eta^2}M_j,$$

because $u \le \eta^{j+1}$ on C_j. Hence

$$M_j \le \left(\frac{\eta}{\eta-1}\right)^2 \sup_{C_j} v_j.$$

Thus the lemma follows.

In an analogous way we shall show the following lemma. For a continuous superharmonic function u and an ordinary open set D the lemma was proved essentially by [4].

Lemma 3. *Let Ω, u, D, C_j, D_j and f be as in Lemma 2. Then*

$$\sup_{C_j} \frac{1}{u}\int_{C_j} G_D(\cdot,y)f(y)dy \le \left(\frac{\eta}{\eta-1}\right)^2 \sup_{C_j}\frac{1}{u}\int_{C_j} G_{D_j}(\cdot,y)f(y)dy.$$

Proof. Let f, V_j, v_j, M_j and K_j be as in the proof of Lemma 2. Let $\mathcal{M}_j = \sup_{C_j} V_j/u$. Since $u \ge \eta^j$ on C_j, it follows that $\mathcal{M}_j \le \eta^{-j}M_j < \infty$. By definition $V_j \le \mathcal{M}_j u \le \mathcal{M}_j\eta^{j+1}$ on C_j. Since V_j is finely harmonic on $D \setminus C_j$, it follows from the minimum principle (Lemma A) that

$$V_j \le \mathcal{M}_j u \quad \text{and} \quad V_j \le \mathcal{M}_j\eta^{j+1} \qquad \text{on } D. \tag{7}$$

Dividing both sides of the first inequality by u, we obtain $\sup_D V_j/u \le \mathcal{M}_j$ and so $\sup_D V_j/u = \mathcal{M}_j$. In particular, $V_j \le \mathcal{M}_j\eta^{j-1}$ on K_{j-1}. It follows from the second inequality of (7) that $V_j \le \mathcal{M}_j\eta^{-1}u$ on K_{j+2}. Hence

$$V_j - v_j - \frac{\mathcal{M}_j}{\eta}u \le V_j - \frac{\mathcal{M}_j}{\eta}u \le \begin{cases} \mathcal{M}_j\eta^{j-1}(1-\frac{1}{\eta}) & \text{on } K_{j-1}, \\ 0 & \text{on } K_{j+2} \end{cases}$$

and $V_j - v_j - \mathcal{M}_j\eta^{-1}u$ is finely subharmonic in D_j. Therefore the minimum principle (Lemma A) yields

$$V_j - v_j - \frac{\mathcal{M}_j}{\eta}u \le \mathcal{M}_j\eta^{j-1}(1-\frac{1}{\eta}) \qquad \text{on } D_j.$$

Dividing by u and taking the supremum over C_j, we obtain

$$\mathcal{M}_j \le \sup_{C_j}\frac{v_j}{u} + \frac{\mathcal{M}_j}{\eta} + \mathcal{M}_j\eta^{j-1}(1-\frac{1}{\eta})\sup_{C_j}\frac{1}{u} \le \sup_{C_j}\frac{v_j}{u} + \frac{2\eta-1}{\eta^2}\mathcal{M}_j,$$

which yields the required inequality.

From Lemmas 2 and 3 we obtain a version of the basic estimates of [1, Theorem 3].

Corollary 4. *Let Ω, u, D, C_j, D_j and f be as in Lemma 2. Then*

$$\sup_D \int_D G_D(\cdot,y)f(y)dy \le \left(\frac{\eta}{\eta-1}\right)^2 \sum_j \sup_{C_j} \int_{C_j} G_{D_j}(\cdot,y)f(y)dy,$$

$$\sup_D \frac{1}{u}\int_D G_D(\cdot,y)f(y)dy \le \left(\frac{\eta}{\eta-1}\right)^2 \sum_j \sup_{C_j} \frac{1}{u}\int_{C_j} G_{D_j}(\cdot,y)f(y)dy.$$

Proof. Let V_j be as in the proof of Lemma 2. Since $D = \bigcup_j C_j$, it follows that $\int_D G_D(\cdot,y)f(y)dy \le \sum_j V_j$, so that

$$\sup_D \int_D G_D(\cdot,y)f(y)dy \le \sup_D \sum_j V_j \le \sum_j \sup_D V_j = \sum_j \sup_{C_j} V_j,$$

where the last equality follows from (5). Similarly,

$$\sup_D \frac{1}{u}\int_D G_D(\cdot,y)f(y)dy \le \sup_D \frac{1}{u}\sum_j V_j \le \sum_j \sup_D \frac{1}{u} V_j = \sum_j \sup_{C_j} \frac{1}{u} V_j.$$

Hence the required inequality follows from Lemmas 2 and 3.

The following lemma is a preliminary version of Theorem.

Lemma 5. *Suppose Ω is bounded and let $v_1,\dots,v_n$ be positive superharmonic functions on Ω. If D is a finely open subset of Ω, then*

$$\sup_D \int_D G_D(\cdot,y)\Phi(v_1(y),\dots,v_n(y))dy \le c_n' \int_D \Psi(v_1(y),\dots,v_n(y))dy.$$

Proof. For simplicity we let $f(y)=\Phi(v_1(y),\dots,v_n(y))$. Let $C_j^i = \{x\in D : \eta^j \le v_i(x) \le \eta^{j+1}\}$ and $D_j^i = \{x \in D : \eta^{j-1} < v_i(x) < \eta^{j+2}\}$. Observe that D_j^i are finely open sets. We apply Corollary 4 with $u = v_1$ to obtain

$$\sup_D \int_D G(\cdot,y)f(y)dy \le \left(\frac{\eta}{\eta-1}\right)^2 \sum_{j_1} \sup_{D_{j_1}^1} \int_{C_{j_1}^1} G_{D_{j_1}^1}(\cdot,y)f(y)dy.$$

Next, we apply Corollary 4 with $D = D_{j_1}^1$ and $u = v_2$ to obtain

$$\sup_{D_{j_1}^1} \int_{C_{j_1}^1} G_{D_{j_1}^1}(\cdot,y)f(y)dy \le \left(\frac{\eta}{\eta-1}\right)^2 \sum_{j_2} \sup_{D_{j_2}^2} \int_{C_{j_1}^1\cap C_{j_2}^2} G_{D_{j_1}^1\cap D_{j_2}^2}(\cdot,y)f(y)dy.$$

Repeating this, we arrive at

$$\text{(8)} \quad \sup_D \int_D G(\cdot,y)f(y)dy$$
$$\le \left(\frac{\eta}{\eta-1}\right)^{2n} \sum_{j_1,\dots,j_n} \sup_{D_{j_1}^1\cap\cdots\cap D_{j_n}^n} \int_{C_{j_1}^1\cap\cdots\cap C_{j_n}^n} G_{j_1,\dots,j_n}(\cdot,y)f(y)dy,$$

where $G_{j_1,\dots,j_n}$ denotes the Green function $G_{D^1_{j_1}\cap\cdots\cap D^n_{j_n}}$ for $D^1_{j_1}\cap\cdots\cap D^n_{j_n}$. Let

$$\psi(t_1,\dots,t_n)=\sup_{1\le c_1,\dots,c_n\le\eta}\Phi(c_1t_1,\dots,c_nt_n).$$

Observe that

$$\begin{aligned} &f(y)=\Phi(v_1(y),\dots,v_n(y))\le\psi(\eta^{j_1},\dots,\eta^{j_n}) && \text{for } y\in C^1_{j_1}\cap\cdots\cap C^n_{j_n},\\ &\psi(\eta^{j_1},\dots,\eta^{j_n})\le\Psi(v_1(y),\dots,v_n(y)) && \text{for } y\in D^1_{j_1}\cap\cdots\cap D^n_{j_n}. \end{aligned}$$

We have from Lemma 1

$$\int_{D^1_{j_1}\cap\cdots\cap D^n_{j_n}} G_{j_1,\dots,j_n}(\cdot,y)dy\le c_0|D^1_{j_1}\cap\cdots\cap D^n_{j_n}|.$$

Hence

$$\begin{aligned}\int_{C^1_{j_1}\cap\cdots\cap C^n_{j_n}} G_{j_1,\dots,j_n}(\cdot,y)f(y)dy &\le \psi(\eta^{j_1},\dots,\eta^{j_n})\int_{C^1_{j_1}\cap\cdots\cap C^n_{j_n}} G_{j_1,\dots,j_n}(\cdot,y)dy\\ &\le c_0\psi(\eta^{j_1},\dots,\eta^{j_n})|D^1_{j_1}\cap\cdots\cap D^n_{j_n}|\\ &\le c_0\int_{D^1_{j_1}\cap\cdots\cap D^n_{j_n}}\Psi(v_1(y),\dots,v_n(y))dy.\end{aligned}$$

This, together with (8), yields

$$\sup_D\int_D G(\cdot,y)f(y)dy\le c_0\left(\frac{\eta}{\eta-1}\right)^{2n}\sum_{j_1,\dots,j_n}\int_{D^1_{j_1}\cap\cdots\cap D^n_{j_n}}\Psi(v_1(y),\dots,v_n(y))dy.$$

Since $\{D^1_{j_1}\cap\cdots\cap D^n_{j_n}\}_{j_1,\dots,j_n}$ covers D at most 3^n times, we obtain the required inequality with $c'_n=3^nc_0[\eta/(\eta-1)]^{2n}$.

We know the following monotone property of the Green function: if $\{\Omega_i\}$ is a monotone increasing sequence of open sets such that $\bigcup_i\Omega_i=\Omega$, then $G_{\Omega_i}\uparrow G$ on $\Omega\times\Omega$ (e.g. [8, Theorem 5.15]). This, together with the monotone convergence theorem, reduces Theorem to the case when Ω is bounded.

Proof of Theorem. In view of the above remark we may assume that Ω is bounded. Let $f(y)=\Phi(v_1(y),\dots,v_n(y))$ and let $C_j=\{x\in\Omega:\eta^j\le u(x)\le\eta^{j+1}\}$ and $D_j=\{x\in\Omega:\eta^{j-1}<u(x)<\eta^{j+2}\}$. In view of the second inequality of Corollary 4 we have

$$\begin{aligned}\sup_\Omega\frac{1}{u}\int_\Omega G(\cdot,y)f(y)u(y)dy &\le\left(\frac{\eta}{\eta-1}\right)^2\sum_j\eta^{-j}\sup_{C_j}\int_{C_j}G_{D_j}(\cdot,y)f(y)u(y)dy\\ &\le\left(\frac{\eta}{\eta-1}\right)^2\sum_j\eta^{-j}\eta^{j+1}\sup_{D_j}\int_{D_j}G_{D_j}(\cdot,y)f(y)dy\\ &=\frac{\eta^3}{(\eta-1)^2}\sum_j\sup_{D_j}\int_{D_j}G_{D_j}(\cdot,y)f(y)dy.\end{aligned}$$

By Lemma 5

$$\sup_{D_j} \int_{D_j} G_{D_j}(\cdot,y)f(y)dy \le c_n' \int_{D_j} \Psi(v_1(y),\dots,v_n(y))dy.$$

Hence by letting $\eta = 3$ we obtain

$$\begin{aligned}\sup_{\Omega} \frac{1}{u}\int_{\Omega} G(\cdot,y)f(y)u(y)dy &\le \frac{27}{4}c_n' \sum_j \int_{D_j} \Psi(v_1(y),\dots,v_n(y))dy \\ &\le \frac{81}{4}c_n' \int_{\Omega} \Psi(v_1(y),\dots,v_n(y))dy,\end{aligned}$$

since $\{D_j\}$ covers Ω at most 3 times. Thus (1) holds with $c_n = \frac{1}{4}3^{n+4}c_0[\eta/(\eta-1)]^{2n}$. The theorem is proved.

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