DIRICHLET PROBLEM IN AN EXTERIOR DOMAIN WITH
POTENTIAL IN A WEIGHTED LEBESGUE CLASS

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(Received February 1, 1999)

Abstract. Let \( 1 < p < \infty \) and let \( w \) be a weight in the Muckenhoupt \( A_p \) class. Suppose \( \Omega \) is a smooth exterior domain in \( \mathbb{R}^n \). Let \( f \in L^{p,w}(\Omega) = \{ f : \|f\|_{p,w;\Omega} = \left( \int_{\Omega} |f|^p w dx \right)^{1/p} < \infty \} \). We consider the Dirichlet problem: \(-\Delta u = f \) on \( \Omega \) and \( u = 0 \) on \( \partial \Omega \cup \{ \infty \} \) with \( f \in L^{p,w}(\Omega) \). We give sufficient conditions for the Dirichlet problem to have a unique solution \( u \) with estimate \( P |\alpha| = 2 \| D^\alpha u \|_{p,w;\Omega} \leq c \|f\|_{p,w;\Omega} \).

1. Introduction

Let \( \Omega \) be an exterior domain whose complement consists of finitely many \( C^{1,1} \) bounded domains (cf. [4, p.94]). Without loss of generality we may assume that a ball \( \{ x : |x| \leq r_0 \} \) lies outside \( \Omega \). Suppose \( 1 < p < \infty \) and \( f \in L^{p,w}(\Omega) = \{ f : \|f\|_{p,w;\Omega} < \infty \} \), where \( \|f\|_{p,w;\Omega} = \left( \int_{\Omega} |f|^p w dx \right)^{1/p} \). Take \( r_1 \) so large that \( \mathbb{R}^n \setminus \Omega \subset \{ x : |x| < r_1 \} \) and let \( \Omega_1 = \{ x \in \Omega : |x| < r_1 \} \). Define the weighted Beppo Levi space \( BL^{2,p,w}(\Omega) = \{ u : \|u\|_{BL^{2,p,w}(\Omega)} < \infty \} \), where

\[
\|u\|_{BL^{2,p,w}(\Omega)} = \|u\|_{p,w;\Omega_1} + \sum_{|\alpha|=1} \|D^\alpha u\|_{p,w;\Omega_1} + \sum_{|\alpha|=2} \|D^\alpha u\|_{p,w;\Omega}.
\]

The weighted Beppo Levi space \( BL^{2,p,w}(\Omega) \) and the weighted Sobolev space

\[
W^{2,p,w}(\Omega) = \{ u : \|u\|_{W^{2,p,w}(\Omega)} = \sum_{|\alpha|\leq 2} \|D^\alpha u\|_{p,w;\Omega} < \infty \}.
\]

1991 Mathematics Subject Classification. Primary 31B10, 31B35, 46E39.
Key words and phrases. Dirichlet problem, exterior domain, Muckenhoupt class, Beppo Levi function, modified Riesz potential, integral representation.

This work was supported in part by Grant-in-Aid for Scientific Research (B) (No. 09440062), Japanese Ministry of Education, Science and Culture.
have the same local behavior. Besides the obvious implication $W^{2,p,w}(\Omega) \subset BL^{2,p,w}(\Omega)$, we have the opposite $BL^{2,p,w}(\Omega') \subset W^{2,p,w}(\Omega)$, if $\Omega'$ is a bounded subdomain of $\Omega$. At $\infty$, however, they behave in completely different ways.

Let us consider the Dirichlet problem

\begin{align}
-\Delta u &= f \quad \text{on } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align}

Since $u$ must be “small” at $\infty$ in a certain sense for $u \in W^{2,p,w}(\Omega)$, it follows that (1)-(2) can have no solutions in $W^{2,p,w}(\Omega)$ if $f \in L^{p,w}(\Omega)$ is “large” at $\infty$. Nevertheless we shall see later in Theorem 3 that (1)-(2) always has a solution in $BL^{2,p,w}(\Omega)$. This is the main reason why we introduce the weighted Beppo Levi space. If $w \equiv 1$, then each function space reduces to a usual unweighted one and is denoted by a symbol without the superscript $w$.

Let us make the meaning of (2) clear. We take the boundary condition in the sense of $W^{1,q}(\Omega_1)$ for some $q > 1$, i.e. (2) holds if and only if there are continuous functions $u_j$ vanishing near the boundary $\partial \Omega$ and converging to $u$ in $W^{1,q}(\Omega_1)$ for some $q > 1$. Since $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$, it follows that $w^{-q/(p-q)}$ is locally integrable for some $q$ with $1 < q < p$. Hence the Hölder inequality yields that $L^{p,w}(\mathbb{R}^n) \subset L^{q}_{\text{loc}}(\Omega_1)$, and so

$$ L^{p,w}(\Omega)|_{\Omega_1} \subset L^{q}(\Omega_1), \quad BL^{2,p,w}(\Omega)|_{\Omega_1} \subset W^{2,q}(\Omega_1). $$

Thus the above interpretation of (2) is compatible with (1) for $f \in L^{p,w}(\Omega)$.

Let $h_2$ be the Riesz kernel

$$ h_2(x) = \begin{cases} 
\gamma_2^{-1}\log \frac{1}{|x|} & \text{if } n = 2, \\
\gamma_n^{-1}|x|^{2-n} & \text{if } n \geq 3,
\end{cases} $$

where $\gamma_2 = 2\pi$ and $\gamma_n = 4\pi^{n/2}/\Gamma\left(\frac{n}{2}\right) - 1$ if $n \geq 3$. In view of the logarithmic growth of $h_2$ for $n = 2$, the two dimensional case is somewhat different from the higher dimensional case. In the sequel we restrict ourselves to the case $n \geq 3$. We shall state the case $n = 2$ in the final section.

Let us consider first the case when $f$ has compact support. Consider the boundary condition at $\infty$:

\begin{equation}
\lim_{|x| \to \infty} u(x) = 0.
\end{equation}

Then (1)-(3) has a unique solution.

Maremonti and Solonnikov [6, Theorems 1 and 2] proved the following theorem.
Theorem A. Let \( n \geq 3 \) and \( w \equiv 1 \). Suppose \( f \in L^p(\Omega) \) has compact support. Then (1)-(3) has a unique solution \( u \). Moreover, the estimate \( \|u\|_{BL^2,p(\Omega)} \leq c\|f\|_{p;\Omega} \) holds in each one of the following cases:

(i) \( 1 < p < n/2 \).
(ii) \( n/2 \leq p < n \) and \( f \) satisfies

\[
\int_{\Omega} f(x)h_2(x)dx = 0.
\]

(iii) \( p \geq n \) and \( f \) satisfies (4) and

\[
\int_{\Omega} f(x)\frac{\partial h_2}{\partial x_j}dx = 0 \text{ for } j = 1, \ldots, n.
\]

In [1] we have introduced a subclass \( A_{p,k} \) of the Muckenhoupt \( A_p \) class:

\[
A_{p,k} = \left\{ w \in A_p : \int_{\mathbb{R}^n} (1 + |x|)^{(k-n)p/(p-1)}w(x)^{1/(1-p)}dx < \infty \right\}.
\]

It is easy to see that

\[
\emptyset = A_{p,n} \subset A_{p,n-1} \subset \cdots \subset A_{p,1} \subset A_{p,0} = A_p;
\]

\[
1 \in A_{p,2} \iff 1 < p < n/2,
\]
\[
1 \in A_{p,1} \setminus A_{p,2} \iff n/2 \leq p < n,
\]
\[
1 \in A_p \setminus A_{p,1} \iff p \geq n
\]

(cf. [1, §4]). In view of these facts, we shall show the following generalization of Theorem A.

Theorem 1. Let \( n \geq 3 \) and \( w \in A_p \). Suppose \( f \in L^{p,w}(\Omega) \) has compact support. Then (1)-(3) has a unique solution \( u \). Moreover, the estimate

\[
\|u\|_{BL^2,p,w(\Omega)} \leq c\|f\|_{p,w;\Omega}.
\]

holds in each one of the following cases:

(i) \( w \in A_{p,2} \).
(ii) \( w \in A_{p,1} \setminus A_{p,2} \) and \( f \) satisfies (4).
(iii) \( w \in A_p \setminus A_{p,1} \) and \( f \) satisfies (4) and (5).
Let us consider next the case when $f$ does not necessarily have compact support. In this case we cannot expect a solution of (1)-(3). In order to obtain a solution of (1)-(2) small at $\infty$ in a sense, we define the subspace $BL^{2,p,w}_{0}(\Omega)$ of $BL^{2,p,w}(\Omega)$ by

$$BL^{2,p,w}_{0}(\Omega) = \{ u \in BL^{2,p,w}(\Omega) : \text{there is } u_j \in BL^{2,p,w}(\Omega) \text{ such that } u_j(x) = 0 \text{ for } |x| > j \text{ and } \lim_{j \to \infty} \| u_j - u \|_{BL^{2,p,w}(\Omega)} = 0 \}.$$  

A solution $u \in BL^{2,p,w}_{0}(\Omega)$ of (1)-(2) may be considered to be small at $\infty$. In view of (*) and the usual trace argument, we see that the first derivatives of $u$ exist on $\partial \Omega$, and they are $q$-th integrable with respect to the surface measure $dS$ for some $q > 1$ (see the following Lemma 2). In particular, we can consider the integral of the normal derivative $\partial u / \partial n$ over $\partial \Omega$. We shall prove the following.

**Theorem 2.** Let $n \geq 3$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$.

(i) If $w \in A_{p,2}$, then there exists a unique solution $u \in BL^{2,p,w}_{0}(\Omega)$ of (1)-(2).

(ii) If $w \in A_{p,1} \setminus A_{p,2}$, then there exists a unique solution $u \in BL^{2,p,w}_{0}(\Omega)$ of (1)-(2) satisfying the additional condition:

$$\int_{\partial \Omega} \frac{\partial u}{\partial n} h^2 dS = 0.$$  

(iii) If $w \in A_p \setminus A_{p,1}$, then there exists a unique solution $u \in BL^{2,p,w}_{0}(\Omega)$ of (1)-(2) satisfying (7) and

$$\int_{\partial \Omega} \frac{\partial u}{\partial n} \frac{\partial h^2}{\partial x_j} dS = 0 \text{ for } j = 1, \ldots, n.$$  

In each case the solution $u$ satisfies (6).

Theorem 2 is a generalization of Maremonti and Solonnikov [6, Theorem 3]. However, there is a significant difference between [6] and ours. A layer potential method and a certain approximation property for $L^p(\Omega)$ were the main tools in [6]. Since $w \in A_p$ needs to be neither continuous nor isotropic, the arguments in [6] are not applicable to our weighted case. We shall make use of the same technique as in [1], e.g. modified Riesz potentials, an approximation of polynomials in $BL^{2,p,w}(\Omega)$ and so on. We shall, in fact, give an explicit representation of the solutions $u$ of (1)-(2). Although the solutions are not unique, among them there exists a canonical solution which will be written as the difference of a modified Riesz potential and its Poisson integral with respect to $\Omega$. We shall show that this canonical solution satisfies (6). Conditions (4), (5), (7) and (8) will imply that the solution considered in each statement must coincide with the canonical one.
2. Preliminaries

In this section we collect some basic $L^p$ estimates which are essentially based on singular integrals and hence applicable to our weighted version in a straightforward fashion. We shall give no proofs since they may be well-known or easy to prove. We denote by $c$ for a positive constant depending only on $n, p, w$ and domains whose value may change from one occurrence to the next.

Let us recall first an extension property of $BL^{2,p,w} (\Omega)$. Since the weighted Beppo Levi space $BL^{2,p,w} (\Omega)$ and the weighted Sobolev space $W^{2,p,w} (\Omega)$ have the same local behavior, it follows from Calderón’s extension theorem (see [7, Chapter VI, 4.8]) that a function in $BL^{2,p,w} (\Omega)$ extends to $\mathbb{R}^n$. We have

**Lemma 1.** If $u \in BL^{2,p,w} (\Omega)$, then there exists $u^* \in BL^{2,p,w} (\mathbb{R}^n)$ such that $u^* = u$ on $\Omega$ and $\|u^*\|_{BL^{2,p,w} (\mathbb{R}^n)} \leq c \|u\|_{BL^{2,p,w} (\Omega)}$.

We need also a result for the restriction of elements of $BL^{2,p,w} (\Omega)$ to $\partial \Omega$. Since $w$ may degenerate on $\partial \Omega$, it is, in general, impossible to define an appropriate weighted space over $\partial \Omega$ associated with $BL^{2,p,w} (\Omega)$. We can, however, give a coarse result, which is essentially unweighted. In view of (*) and [7, Chapter VI, 4.2] we have

**Lemma 2.** (i) Let $1 < q < \infty$. If $f \in W^{1,q} (\Omega_1)$, then the trace of $f$ on $\partial \Omega$ is defined and

$$\|f\|_{L^q (\partial \Omega)} \leq c \|f\|_{W^{1,q} (\Omega_1)}.$$  

(ii) Let $w \in A_p$. Then there exists $q > 1$ such that if $u \in BL^{2,p,w} (\Omega)$, then the traces of $u$ and $\nabla u$ on $\partial \Omega$ are defined and

$$\|u\|_{L^q (\partial \Omega)} + \|\nabla u\|_{L^q (\partial \Omega)} \leq c \|u\|_{BL^{2,p,w} (\Omega)}.$$  

Let us recall $L^p$ estimates for solutions of the Dirichlet problem in a bounded $C^{1,1}$ domain.

**Lemma 3.** (cf. [4, Theorem 9.13 and Lemma 9.17]) Let $D$ be a bounded domain in $\mathbb{R}^n$ with a $C^{1,1}$ boundary portion $T \subset \partial D$. Let $u \in W^{2,p,w} (D)$ be a solution of $\Delta u = f$ in $D$ with $u = 0$ on $T$ in the sense of $W^{1,q} (D)$ for some $q > 1$. Then for any domain $D' \Subset D \cup T$,

$$\|u\|_{W^{2,p,w} (D')} \leq c (\|u\|_{p,w;D} + \|f\|_{p,w;D}).$$  

Moreover, if $T = \partial D$, then $\|u\|_{W^{2,p,w} (D')} \leq c \|f\|_{p,w;D}$.

Note that the boundary condition in Lemma 3 is weaker than that in [4, Theorem 9.13]. However, we infer from a careful observation of [4, Lemma 9.12] that the conclusion remains true.
3. Modified Riesz potentials

Observe that that $-\Delta h_2$ is the Dirac measure at the origin, so that $-\Delta (h_2 * f) = f$ if the convolution $h_2 * f$ exists. Therefore the first attempt to solve (1)-(2) begins with consideration on the Riesz potential $h_2 * f$. However, a problem arises: the convolution $h_2 * f$ does not necessarily exist for all $f \in L^{p,w}(\Omega)$. This difficulty can be overcome by means of modified Riesz potentials. Observe that if $y \neq 0$, then $h_2(x - y)$ has a multiple power series expansion in $x_1, x_2, \ldots, x_n$, convergent in a neighborhood of the origin. We write

$$h_2(x - y) = \sum_{\nu=0}^{\infty} a_{\nu}(x, y),$$

where, for fixed $\nu$ and $y \neq 0, a_{\nu}(x, y) = \sum_{|\beta|=\nu} x^\beta \beta! D^\beta h_2(-y)$ is a homogeneous polynomial in $x_1$ to $x_n$ of degree $\nu$ and continuous in $x, y$ jointly for $y \neq 0$ (see [5, Chapter 4]). We set

$$h_{2,k}(x, y) = \sum_{\nu=k}^{\infty} a_{\nu}(x, y)$$

for $k \geq 0$ and write

$$I_{2,k}(f) = \int_{\mathbb{R}^n} h_{2,k}(\cdot, y) f(y) dy,$$

whenever the right hand side has a meaning. For notational convenience we let $I_{2,0}(f) = I_2(f)$. By definition $I_2(f)$ coincides with the convolution $h_2 * f$.

We collect some properties of modified Riesz potentials $I_{2,k}(f)$. No proofs will be given. We refer to [1]. Note that if $f \in L^{p,w}(\mathbb{R}^n), 0 \leq k \leq 2$ and $I_{2,k}(f)$ exists, then $D^\alpha I_{2,k}(f) = (D^\alpha h_2) * f$ for $|\alpha| \geq k$ and

$$(9) \quad \sum_{|\alpha|=2} \|D^\alpha I_{2,k}(f)\|_{p,w;\Omega} \leq c \|f\|_{p,w;\Omega}$$

(cf. [2, Theorems I and III] and [1, Lemma 8]). In particular, $u = I_{2,2}(f)$ satisfies (1). We see that

$$(10) \quad |h_2(x - y) - h_{2,k}(x, y)| \leq c \sum_{\nu=0}^{k-1} |x|^\nu |y|^{2n-\nu};$$

$$(11) \quad |h_{2,k}(x, y)| \leq c |x|^k |y|^{2n-k} \quad \text{for } 2|x| > |y|$$

(cf. [5, Lemmas 4.1 and 4.2] and [1, Lemma 6]). We infer from Hölder’s inequality that $w \in A_{p,k}$ if and only if

$$(11) \quad \int_{\mathbb{R}^n} (1 + |x|)^{2n-k} |f(x)| dx \leq c \|f\|_{p,w;\mathbb{R}^n}$$

(cf. [1, Theorem 5]). Hence we obtain from (10) and (11) that $I_{2,k}(f)$ exists for every $f \in L^{p,w}(\Omega)$ if and only if $w \in A_{p,2-k}$. In particular, $I_{2,2}(f)$ exists for every $f \in L^{p,w}(\Omega)$. Therefore we have
Lemma 4. ([1, Theorem 4]) Let \( w \in A_p \). If \( u \in BL^{2,p,w}(\mathbb{R}^n) \), then there are constants \( a \) and \( b_j \) such that \( u = I_{2,2}(-\Delta u) + a + \sum_{j=1}^{n} b_j x_j \).

Let us write

\[
BL^{2,p,w}_0(\mathbb{R}^n) = \{ u \in BL^{2,p,w}(\mathbb{R}^n) : \text{there is } u_j \in C_0^\infty(\mathbb{R}^n) \text{ such that } \lim_{j \to \infty} \| u_j - u \|_{BL^{2,p,w}(\mathbb{R}^n)} = 0 \}.
\]

We have given a characterization of \( BL^{2,p,w}_0(\mathbb{R}^n) \).

Lemma 5. ([1, Corollary]) Let \( w \in A_p \).

(i) \( BL^{2,p,w}_0(\mathbb{R}^n) = \{ I_{2}(g) : g \in L^{p,w}(\mathbb{R}^n) \} \iff w \in A_{p,2} \).

(ii) \( BL^{2,p,w}(\mathbb{R}^n) = \{ I_{2,1}(g) + a : g \in L^{p,w}(\mathbb{R}^n), a \in \mathbb{R} \} \iff w \in A_{p,1} \setminus A_{p,2} \).

(iii) \( BL^{2,p,w}(\mathbb{R}^n) = \{ I_{2,2}(g) + a + \sum_{j=1}^{n} b_j x_j : g \in L^{p,w}(\mathbb{R}^n), a, b_j \in \mathbb{R} \} \iff w \in A_p \setminus A_{p,1} \).

In the proof of Lemma 5 we have used the following approximation property. This may be regarded as an alternative of [6, Lemma 2].

Lemma 6. (cf. [1, Lemma 18]) Let \( w \in A_p \). Suppose \( \varepsilon > 0 \) and \( R > 0 \).

(i) If \( w \in A_{p,1} \setminus A_{p,2} \), then there is \( g \in L^{p,w}(\mathbb{R}^n) \) with compact support such that \( g(x) = 0 \) for \( |x| < R \), \( ||g||_{p,w;\mathbb{R}^n} < \varepsilon \) and \( I_2(g)(0) = 1 \).

(ii) If \( w \in A_p \setminus A_{p,1} \), then there are \( g_j \in L^{p,w}(\mathbb{R}^n) \), \( j = 1, \ldots, n \), with compact support such that \( g_j(x) = 0 \) for \( |x| < R \), \( ||g_j||_{p,w;\mathbb{R}^n} < \varepsilon \) and

\[
\frac{\partial}{\partial x_i} I_{2}(g_j)(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

4. Representation of solutions

In this section we shall give an explicit representation of solutions of (1)-(2). We have observed in the last section that \( I_{2,2}(f) \) satisfies (1). In order to obtain a solution of (1)-(2), we subtract from \( I_{2,2}(f) \) a harmonic function in \( \Omega \) having the same boundary values. This harmonic function will be given by the Poisson integral. Let \( G(x,y) \) be the Green function for \( \Omega \), that is, \( G(x,y) \) satisfies

(i) \( G(\cdot, y) - h_2(\cdot - y) \) is harmonic on \( \Omega \);

(ii) \( G(\cdot, y) \) vanishes on \( \partial \Omega \);

(iii) if \( n \geq 3 \), then \( G(\cdot, y) \) tends to zero at \( \infty \); if \( n = 2 \), then \( G(\cdot, y) \) is bounded at \( \infty \).

For \( g \in L^1(\partial \Omega) \) we define the Poisson integral by

\[
PI(g) = \frac{1}{\gamma_n} \int_{\partial \Omega} \frac{\partial G(x,y)}{\partial n_y} g(y)dS(y),
\]
where $n_y$ is the inward normal unit vector on $\partial \Omega$ and $dS(y)$ stands for the surface element. We observe that $PI(g)$ is harmonic in $\Omega$, and that if $g$ is continuous, then $PI(g) = g$ on $\partial \Omega$. Note that $PI(g)$ tends to zero at $\infty$ if $n \geq 3$; $PI(g)$ is bounded at $\infty$ if $n = 2$. We remark that

$$PI(1) < 1 \quad \text{and} \quad PI(h_2) \equiv h_2 \quad \text{if } n \geq 3;$$

$$PI(1) \equiv 1 \quad \text{and} \quad PI(h_2) < h_2 \quad \text{if } n = 2.$$  

By Lemma 2 the trace of $I_{2,k}(f)$ on $\partial \Omega$ belongs to $L^q(\partial \Omega)$. We write $PI(I_{2,k}(f))$ for the Poisson integral of the trace $I_{2,k}(f)$ on $\partial \Omega$. One may expect that $I_{2,2}(f) - PI(I_{2,2}(f))$ is a solution of (1)-(2). This is, in fact, the case.

**Theorem 3.** Let $n \geq 3$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$. Then (1)-(2) has a solution in $BL^{2,p,w}(\Omega)$. Every solution $u$ of (1)-(2) in $BL^{2,p,w}(\Omega)$ is represented as

$$u = I_{2,2}(f) - PI(I_{2,2}(f)) + a(1 - PI(1)) + \sum_{j=1}^{n} b_j(x_j - PI(x_j)),$$

where $a$ and $b_j$ are constants. Moreover, $u = I_{2,2}(f) - PI(I_{2,2}(f))$ satisfies (6).

Let us consider first a special case of Theorem 3 when $f \equiv 0$.

**Lemma 7.** Let $n \geq 3$. If $h = a(1 - PI(1)) + \sum_{j=1}^{n} b_j(x_j - PI(x_j))$, then $h$ is a harmonic function in $BL^{2,p,w}(\Omega)$ satisfying (2), and vice versa. Moreover, $\|h\|_{BL^{2,p,w}(\Omega)} \leq c(|a| + \sum_{j=1}^{n} |b_j|)$.

**Proof.** It is easy to see that $h = a(1 - PI(1)) + \sum_{j=1}^{n} b_j(x_j - PI(x_j))$ is a harmonic function vanishing continuously on $\partial \Omega$, and hence satisfying (2). Since $D^\alpha PI(g)(x) = O(|x|^{2-n-|\alpha|})$ as $|x| \to \infty$ for $g \in L^1(\partial \Omega)$, it follows from (11) that

$$\sum_{|\alpha|=2} \|D^\alpha h\|_{p,w;\Omega \setminus \Omega_1} \leq c(|a| + \sum_{j=1}^{n} |b_j|).$$

We have from Lemma 3

$$\|h\|_{W^{2,p,w}(\Omega_1)} \leq c(|a| + \sum_{j=1}^{n} |b_j|),$$

whence $h \in BL^{2,p,w}(\Omega)$ and the required norm estimate follows.

Conversely, suppose $h \in BL^{2,p,w}(\Omega)$ is a harmonic function in $\Omega$ such that $h = 0$ on $\partial \Omega$ in the sense of $W^{1,q}(\Omega_1)$ for some $q > 1$. In view of [4, Lemma 9.16] we see that $h = 0$ on $\partial \Omega$ in the sense of $W^{1,q}(\Omega_1)$ for any $q > 1$, and hence $h = 0$ on $\partial \Omega$ continuously. By Lemma 1 we extend $h$ to $\mathbb{R}^n$ so that the
extension \( h^* \) belongs to \( BL^{2,p,w}(\mathbb{R}^n) \). From Lemma 4 we can find \( a \) and \( b_j \) such that \( h^* = I_{2,2}(-\Delta h^*) + a + \sum_{j=1}^{n} b_j x_j \). Since \( \Delta h^* \) is concentrated on \( \mathbb{R}^n \setminus \Omega \), it follows \( I_{2}(\Delta h^*) \) exists and \( h^* = I_{2}(-\Delta h^*) + a + \sum_{j=1}^{n} b_j x_j \) with different constants \( a \) and \( b_j \). Observe that \( I_{2}(-\Delta h^*) \), \( PI(1) \) and \( PI(x_j) \) tend to zero at \( \infty \), and hence

\[
\lim_{|x| \to \infty, x \in \Omega} h(x) - (a(1 - PI(1)) + \sum_{j=1}^{n} b_j (x_j - PI(x_j))) = 0.
\]

Therefore the maximum principle yields \( h = a(1 - PI(1)) + \sum_{j=1}^{n} b_j (x_j - PI(x_j)) \).

**Proof of Theorem 3.** Let \( u_0 = I_{2,2}(f) - PI(I_{2,2}(f)) \). We have seen that \( u_0 \) satisfies (1). In order to prove (2) let \( \{f_\varepsilon\} \) be a regularization of \( f \) such that \( f_\varepsilon \to f \) in \( L^{p,w}(\Omega) \). Since \( I_{2,2}(f_\varepsilon) \) is continuous, it follows that the continuous function \( u_\varepsilon = I_{2,2}(f_\varepsilon) - PI(I_{2,2}(f_\varepsilon)) \) satisfies

\[
-\Delta u_\varepsilon = f_\varepsilon \quad \text{on } \Omega,
\]

\[
u_\varepsilon = 0 \quad \text{on } \partial \Omega.
\]

Let \( 1 < q < n/(n-1) \). We have from (10)

\[
\int_{\Omega_1} |h_{2,2}(x, y)|^q dx \leq c(1 + |y|)^{-nq}.
\]

Hence Minkowski’s inequality for integrals (see [7, p.271]) and (11) yield

\[
\|I_{2,2}(f_\varepsilon)\|_{q, \Omega_1} \leq c\|f_\varepsilon\|_{p,w; \Omega}.
\]

Similarly \( \| \partial_{x_j} I_{2,2}(f_\varepsilon) \|_{q, \Omega_1} \leq c\|f_\varepsilon\|_{p,w; \Omega} \). By Lemma 2 the trace of \( I_{2,2}(f_\varepsilon) \) on \( \partial \Omega \) satisfies

\[
\|I_{2,2}(f_\varepsilon)\|_{L^1(\partial \Omega)} \leq c\|f_\varepsilon\|_{p,w; \Omega}.
\]

By the estimate \( \partial G(x, y)/\partial n_y \leq c\delta(x)|x - y|^{-n} \) with \( \delta(x) = \text{dist}(x, \partial \Omega) \) for \( x, y \) near the boundary (cf. [8]), we have \( \sup_{y \in \partial \Omega} \int_{\Omega_1} |\partial G(x, y)/\partial n_y|^q dx < \infty \). Hence Minkowski’s inequality for integrals yields

\[
\|PI(I_{2,2}(f_\varepsilon))\|_{q, \Omega_1} \leq c\|I_{2,2}(f_\varepsilon)\|_{L^1(\partial \Omega)} \leq c\|f_\varepsilon\|_{p,w; \Omega}.
\]

Therefore we infer from (13) and Lemma 3 that

\[
\|u_\varepsilon\|_{W^{2,q}(\Omega_1)} \leq c\|f_\varepsilon\|_{p,w; \Omega}.
\]
In particular, \( u_\varepsilon \to u_0 \) in \( W^{1,q}(\Omega_1) \), so that \( u_0 = 0 \) in the sense of \( W^{1,q}(\Omega_1) \). Thus (2) holds. Consequently, \( u = I_{2,2}(f - PI(I_{2,2}(f)) + a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j)) \) is a solution of (1)-(2) and vice versa by Lemma 7.

Finally let us prove the norm estimate for \( u_0 \). To this end let \( r_2 > r_1, \Omega_2 = \{ x \in \Omega : |x| < r_2 \} \) and take \( \eta \in C_0^\infty(\mathbb{R}^n) \) such that \( 0 \leq \eta \leq 1 \) on \( \{ |x| \leq r_1 \} \) and \( \eta = 0 \) on \( \{ |x| \geq r_2 \} \). Let \( u_1 = \eta u_0 \) and \( u_2 = (1 - \eta)u_0 \). An elementary calculation shows that

\[
\|\Delta u_1\|_{p,w;\Omega} \leq c(\|f\|_{p,w;\Omega} + \|\nabla u_0\|_{p,w;\Omega_2\setminus\Omega_1} + \|u_0\|_{p,w;\Omega_2\setminus\Omega_1}).
\]

By (10) we can compare \( D^\alpha I_{2,2}(f) \) (\(|\alpha| \leq 1\)) and the maximal function \( Mf \) locally, and obtain

\[
\sum_{|\alpha|\leq 1} \|D^\alpha I_{2,2}(f)\|_{p,w;\Omega_2} \leq c\|f\|_{p,w;\Omega}.
\]

By the estimate of the Green function we have

\[
\sup_{x \in \Omega_2 \setminus \Omega_1} |D^\alpha(PI(I_{2,2}(f)))(x)| \leq c\|I_{2,2}(f)\|_{L^1(\partial\Omega)} \leq c\|f\|_{p,w;\Omega}
\]

for \(|\alpha| \leq 1\). Hence

(14)

\[
\|\Delta u_1\|_{p,w;\Omega} \leq c\|f\|_{p,w;\Omega}.
\]

Since \( u_1 \) vanishes on \( \{ |x| = r_2 \} \) continuously and on \( \partial\Omega_1 \) in the sense of \( W^{1,q}(\Omega_2) \) for \( 1 < q < n/(n-1) \), it follows from Lemma 3 that

\[
\|u_1\|_{W^{2,p,w}(\Omega_2)} \leq c\|f\|_{p,w;\Omega}.
\]

By Lemma 1 we extend \( u_2 \) to \( \mathbb{R}^n \) and represent it on \( \Omega \) as \( u_2 = I_{2,2}(-\Delta u_2) + a + \sum_{j=1}^n b_jx_j \). Then by (9) and (14)

\[
\sum_{|\alpha|=2} \|D^\alpha u_2\|_{p,w;\Omega} \leq c\|\Delta u_2\|_{p,w;\Omega} = c\|\Delta u - \Delta u_1\|_{p,w;\Omega} \leq c\|f\|_{p,w;\Omega}.
\]

Consequently, (6) holds for \( u = u_0 \). The proof is complete.

Since \( u \in BL^{2,p,w}_0(\Omega) \) extends to \( u^* \in BL^{2,p,w}(\mathbb{R}^n) \) by Lemma 1, we have from Lemma 5 the following theorem.
Theorem 4. Let \( n \geq 3 \) and \( w \in A_p \). Suppose \( f \in L^{p,w}(\Omega) \). Then every solution \( u \) of (1)-(2) in \( BL^2_{0,p,w}(\Omega) \) is represented as follows:

(i) If \( w \in A_p, \), then \( u = I_2(f) - PI(I_2(f)) \).

(ii) If \( w \in A_{p,1} \setminus A_{p,2} \), then \( u = I_2,1(f) - PI(I_2,1(f)) + a(1 - PI(1)), \) where \( a \) is a constant.

(iii) If \( w \in A_{p,2} \setminus A_p \), then \( u = I_2,2(f) - PI(I_2,2(f)) + a(1 - PI(1)) + \sum_{j=1}^{n} b_j(x_j - PI(x_j)), \) where \( a \) and \( b_j \) are constants.

In each case the canonical solution \( I_2,k(f) - PI(I_2,k(f)) \), \( (k = 0, 1, 2) \) for (i), (ii), (iii), respectively, satisfies (6).

Proof. Only (6) may require a proof. We have observed (6) for (iii) in Theorem 3. Suppose \( w \in A_{p,2} \). Writing

\[
I_2,2(f) = I_2(f) = \left( \int_{\Omega} h_2(-y)f(y)dy + \sum_{j=1}^{n} x_j \int_{\Omega} \frac{\partial h_2}{\partial x_j}(-y)f(y)dy \right)
\]

we obtain from (11) that \( |a'| \leq c\|f\|_{p,w;\Omega} \), \( |b'_j| \leq c\|f\|_{p,w;\Omega} \). Hence from Theorem 3 and Lemma 7 we have

\[
\|I_2(f) - PI(I_2(f))\|_{BL^2,p,w(\Omega)} \\
\leq \|I_2,2(f) - PI(I_2,2(f))\|_{BL^2,p,w(\Omega)} + \|a'(1 - PI(1)) + \sum_{j=1}^{n} b'_j(x_j - PI(x_j))\|_{BL^2,p,w(\Omega)} \\
\leq c\|f\|_{p,w;\Omega}.
\]

Thus (6) follows for (i). We can prove (6) similarly for (ii).

5. Proof of Theorems 1 and 2

Proof of Theorem 1. Since \( f \) is of compact support, it follows that \( I_2(f) \) exists, \( \lim_{|x| \to \infty} I_2(f)(x) = 0 \) and

\[
I_2(f) = I_2,1(f) + \int_{\Omega} h_2(-y)f(y)dy = I_2,2(f) + \int_{\Omega} h_2(-y)f(y)dy + \sum_{j=1}^{n} x_j \int_{\Omega} \frac{\partial h_2}{\partial x_j}(-y)f(y)dy.
\]

Hence Theorem 3 and the maximum principle say that \( u = I_2(f) - PI(I_2(f)) \) is a unique solution of (1)-(3) in all cases of (i)-(iii). Conditions (4) and (5)
imply that \( u \) coincides with the canonical solutions \( I_{2,1}(f) - PI(I_{2,1}(f)) \) and \( I_{2,2}(f) - PI(I_{2,2}(f)) \) in the cases of (ii) and (iii), respectively. Therefore the norm estimate follows from Theorem 4.

\[ \int_{\partial \Omega} \frac{\partial v}{\partial n} h_2 ds = 1, \]
\[ \int_{\partial \Omega} \frac{\partial v_i}{\partial n} \frac{\partial h_2}{\partial x_j} ds = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \]

Let \( R > 0 \) be sufficiently large. Then the Green formula yields
\[ \int_{\partial \Omega} \frac{\partial v}{\partial n} h_2 ds = \int_{|x| = R} \left( \frac{\partial v}{\partial n} h_2 - \frac{\partial h_2}{\partial n} v \right) ds. \]
Since \( PI(1) = O(|x|^{-n}) \) and \( \nabla PI(1) = O(|x|^{1-n}) \), it follows that
\[ \int_{\partial \Omega} \frac{\partial v}{\partial n} h_2 ds = - \lim_{R \to \infty} \int_{|x| = R} \frac{\partial h_2}{\partial n} ds = 1. \]
A similar calculation shows the second assertion.

What remains is to prove that if \( w \in A_{p,1} \setminus A_{p,2} \), then \( u = I_{2,1}(f) - PI(I_{2,1}(f)) \) satisfies (7); and that if \( w \in A_p \setminus A_{p,1} \), then \( u = I_{2,2}(f) - PI(I_{2,2}(f)) \) satisfies (7) and (8). Suppose \( w \in A_{p,1} \setminus A_{p,2} \). Let \( u = I_{2,1}(f) - PI(I_{2,1}(f)) \). Take \( \varepsilon > 0 \). Split \( u \) into \( u_1 + u_2 \), where \( u_j = I_{2,1}(f_j) - PI(I_{2,1}(f_j)) \), \( f_1 = \chi_{|x| < R}f \) and \( f_2 = \chi_{|x| > R}f \). Take \( R > 0 \) so large that \( \|f_2\|_{p,w;\Omega} < \varepsilon \) and \( \|D^\alpha I_{2,1}(f_2)\|_{\infty;\Omega_1} < \varepsilon \) for \( |\alpha| \leq 2 \). Then
\[ \left| \int_{\partial \Omega} \frac{\partial u_2}{\partial n} h_2 ds \right| < c\varepsilon. \]
Observe that
\[ u_1 = I_2(f_1) - PI(I_2(f_1)) - a(1 - PI(1)), \]
where \( a = \int_{\Omega} h_2 f_1 dx \). By Lemma 6 we find a compactly supported function \( g \in L^{p,w}(\Omega) \) such that
\[ g(x) = 0 \text{ for } |x| < r_2 \text{ with } r_2 > r_1, \]
\[ \|g\|_{p,w;\Omega} < \varepsilon, \]
\[ I_2(g)(0) = \int h_2 g dx = 1, \]
\[ |I_2(g) - 1| < \varepsilon \text{ on } \Omega_1, \]
\[ |u_1 - u_3| < \varepsilon \text{ on } \Omega_1, \]
where \( u_3 = I_2(f_1 - ag) - PI(I_2(f_1 - ag)) \). Observe that \( u_1 - u_3 \) is harmonic on \( \Omega_2 = \{ x \in \Omega : |x| < r_2 \} \) and vanishes continuously on \( \partial \Omega \). Hence,
\[
\| \nabla (u_1 - u_3) \|_{\infty; \Omega_1} \leq c \| u_1 - u_3 \|_{\infty; \Omega_2} \leq c \varepsilon
\]
(see [8, Theorem 2.4 and its proof]); in particular
\[
\left\| \frac{\partial u_1}{\partial n} - \frac{\partial u_3}{\partial n} \right\|_{\infty; \partial \Omega} \leq c \varepsilon.
\]
Since \( u_3 \) tends to zero at \( \infty \), it follows from the Green formula that
\[
\int_{\partial \Omega} \frac{\partial u_3}{\partial n} h_2 dS = \int_\Omega \Delta u_3 h_2 dx = - \int_\Omega (f_1 - ag) h_2 dx = 0.
\]
Therefore
\[
\left| \int_{\partial \Omega} \frac{\partial u_1}{\partial n} h_2 dS \right| < c \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, (7) follows. In the case of (iii) we can prove (7) and (8) by using the functions \( g_j \) appearing in Lemma 6. Details are left to the reader. The proof is complete.

6. TWO DIMENSIONAL CASE

Now let us consider the two dimensional case. Hereafter we let \( n = 2 \). In view of the Phragmén-Lindelöf principle, the boundary condition at \( \infty \) becomes
\[
(3') \quad \limsup_{|x| \to \infty} |u(x)| < \infty.
\]
We see from (12) that \( 1 - PI(1) \) for \( n \geq 3 \) is replaced by \( h_2 - PI(h_2) \). Note also that if \( f \) has compact support, then
\[
I_{2,1}(f) - PI(I_{2,1}(f)) = I_2(f) - PI(I_2(f));
\]
\[
\lim_{|x| \to \infty} (I_2(f)(x) - ch_2(x)) = 0 \quad \text{with} \quad c = \int_\Omega f dx.
\]
Moreover, since \( PI(h_2) \) is harmonic at \( \infty \), it follows from [3, (2.73) Proposition] that
\[
\lim_{R \to \infty} \int_{|x|=R} \frac{\partial}{\partial n} PI(h_2) dS = 0.
\]
Hence we have an alternative of (15):
\[
\int_{\partial \Omega} \frac{\partial}{\partial n} (h_2 - PI(h_2)) dS = 1.
\]
Using these facts, we obtain the following counterparts of Theorems 1-4. Since \( A_{p,2} = \emptyset \), we have two cases in each theorem.
Theorem 1’. Let $n = 2$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$ has compact support. Then (1)-(2) and (3’) have a unique solution $u$. Moreover, (6) holds in each one of the following cases:

(i) $w \in A_{p,1} \setminus A_{p,2}$ and $f$ satisfies

$$\int_{\Omega} f(x) dx = 0. \quad (4')$$

(ii) $w \in A_p \setminus A_{p,1}$ and $f$ satisfies (4’) and (5).

Theorem 2’. Let $n = 2$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$.

(i) If $w \in A_{p,1} \setminus A_{p,2}$, then there exists a unique solution $u \in BL^{2,p,w}_0(\Omega)$ of (1)-(2) satisfying the additional condition:

$$\int_{ \partial \Omega} \frac{\partial u}{\partial n} dS = 0. \quad (7')$$

(ii) If $w \in A_p \setminus A_{p,1}$, then there exists a unique solution $u \in BL^{2,p,w}_0(\Omega)$ of (1)-(2) satisfying (7’) and (8).

In each case the solution $u$ satisfies (6).

Theorem 3’. Let $n = 2$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$. Then (1)-(2) has a solution in $BL^{2,p,w}(\Omega)$. Every solution $u$ of (1)-(2) in $BL^{2,p,w}(\Omega)$ is represented as

$$u = I_{2,2}(f) - PI(I_{2,2}(f)) + c(h_2 - PI(h_2)) + \sum_{j=1}^n b_j(x_j - PI(x_j)),$$

where $c$ and $b_j$ are constants. Moreover, $u = I_{2,2}(f) - PI(I_{2,2}(f))$ satisfies (6).

Theorem 4’. Let $n = 2$ and $w \in A_p$. Suppose $f \in L^{p,w}(\Omega)$. Then every solution $u$ of (1)-(2) in $BL^{2,p,w}_0(\Omega)$ is represented as follows:

(i) If $w \in A_{p,1} \setminus A_{p,2}$, then $u = I_{2,1}(f) - PI(I_{2,1}(f)) + c(h_2 - PI(h_2))$, where $c$ is a constant.

(ii) If $w \in A_p \setminus A_{p,1}$, then $u = I_{2,2}(f) - PI(I_{2,2}(f)) + c(h_2 - PI(h_2)) + \sum_{j=1}^n b_j(x_j - PI(x_j))$, where $c$ and $b_j$ are constants.

In each case the canonical solution $I_{2,k}(f) - PI(I_{2,k}(f))$, ($k = 1, 2$ for (i), (ii) respectively), satisfies (6).

References


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