

## DIRICHLET PROBLEM IN AN EXTERIOR DOMAIN WITH POTENTIAL IN A WEIGHTED LEBESGUE CLASS

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ABSTRACT. Let  $1 < p < \infty$  and let  $w$  be a weight in the Muckenhoupt  $A_p$  class. Suppose  $\Omega$  is a smooth exterior domain in  $\mathbb{R}^n$ . Let  $f \in L^{p,w}(\Omega) = \{f : \|f\|_{p,w;\Omega} = (\int_{\Omega} |f|^p w dx)^{1/p} < \infty\}$ . We consider the Dirichlet problem:  $-\Delta u = f$  on  $\Omega$  and  $u = 0$  on  $\partial\Omega \cup \{\infty\}$  with  $f \in L^{p,w}(\Omega)$ . We give sufficient conditions for the Dirichlet problem to have a unique solution  $u$  with estimate  $\sum_{|\alpha|=2} \|D^\alpha u\|_{p,w;\Omega} \leq c\|f\|_{p,w;\Omega}$ .

### 1. INTRODUCTION

Let  $\Omega$  be an exterior domain whose complement consists of finitely many  $C^{1,1}$  bounded domains (cf. [4, p.94]). Without loss of generality we may assume that a ball  $\{x : |x| \leq r_0\}$  lies outside  $\Omega$ . Suppose  $1 < p < \infty$  and  $f \in L^{p,w}(\Omega) = \{f : \|f\|_{p,w;\Omega} < \infty\}$ , where  $\|f\|_{p,w;\Omega} = (\int_{\Omega} |f|^p w dx)^{1/p}$ . Take  $r_1$  so large that  $\mathbb{R}^n \setminus \Omega \subset \{x : |x| < r_1\}$  and let  $\Omega_1 = \{x \in \Omega : |x| < r_1\}$ . Define the weighted Beppo Levi space  $BL^{2,p,w}(\Omega) = \{u : \|u\|_{BL^{2,p,w}(\Omega)} < \infty\}$ , where

$$\|u\|_{BL^{2,p,w}(\Omega)} = \|u\|_{p,w;\Omega_1} + \sum_{|\alpha|=1} \|D^\alpha u\|_{p,w;\Omega_1} + \sum_{|\alpha|=2} \|D^\alpha u\|_{p,w;\Omega}.$$

The weighted Beppo Levi space  $BL^{2,p,w}(\Omega)$  and the weighted Sobolev space

$$W^{2,p,w}(\Omega) = \{u : \|u\|_{W^{2,p,w}(\Omega)} = \sum_{|\alpha|\leq 2} \|D^\alpha u\|_{p,w;\Omega} < \infty\}.$$

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have the same local behavior. Besides the obvious implication  $W^{2,p,w}(\Omega) \subset BL^{2,p,w}(\Omega)$ , we have the opposite  $BL^{2,p,w}(\Omega') \subset W^{2,p,w}(\Omega)$ , if  $\Omega'$  is a bounded subdomain of  $\Omega$ . At  $\infty$ , however, they behave in completely different ways.

Let us consider the Dirichlet problem

$$\begin{aligned} (1) \quad & -\Delta u = f \quad \text{on } \Omega, \\ (2) \quad & u = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Since  $u$  must be “small” at  $\infty$  in a certain sense for  $u \in W^{2,p,w}(\Omega)$ , it follows that (1)-(2) can have no solutions in  $W^{2,p,w}(\Omega)$  if  $f \in L^{p,w}(\Omega)$  is “large” at  $\infty$ . Nevertheless we shall see later in Theorem 3 that (1)-(2) always has a solution in  $BL^{2,p,w}(\Omega)$ . This is the main reason why we introduce the weighted Beppo Levi space. If  $w \equiv 1$ , then each function space reduces to a usual unweighted one and is denoted by a symbol without the superscript  $w$ .

Let us make the meaning of (2) clear. We take the boundary condition in the sense of  $W^{1,q}(\Omega_1)$  for some  $q > 1$ , i.e. (2) holds if and only if there are continuous functions  $u_j$  vanishing near the boundary  $\partial\Omega$  and converging to  $u$  in  $W^{1,q}(\Omega_1)$  for some  $q > 1$ . Since  $w \in A_{p-\varepsilon}$  for some  $\varepsilon > 0$ , it follows that  $w^{-q/(p-q)}$  is locally integrable for some  $q$  with  $1 < q < p$ . Hence the Hölder inequality yields that  $L^{p,w}(\mathbb{R}^n) \subset L_{loc}^q(\Omega_1)$ , and so

$$(*) \quad L^{p,w}(\Omega)|_{\Omega_1} \subset L^q(\Omega_1), \quad BL^{2,p,w}(\Omega)|_{\Omega_1} \subset W^{2,q}(\Omega_1).$$

Thus the above interpretation of (2) is compatible with (1) for  $f \in L^{p,w}(\Omega)$ .

Let  $h_2$  be the Riesz kernel

$$h_2(x) = \begin{cases} \gamma_2^{-1} \log \frac{1}{|x|} & \text{if } n = 2 \\ \gamma_n^{-1} |x|^{2-n} & \text{if } n \geq 3, \end{cases}$$

where  $\gamma_2 = 2\pi$  and  $\gamma_n = 4\pi^{n/2}/\Gamma(\frac{n}{2} - 1)$  if  $n \geq 3$ . In view of the logarithmic growth of  $h_2$  for  $n = 2$ , the two dimensional case is somewhat different from the higher dimensional case. In the sequel we restrict ourselves to the case  $n \geq 3$ . We shall state the case  $n = 2$  in the final section.

Let us consider first the case when  $f$  has compact support. Consider the boundary condition at  $\infty$ :

$$(3) \quad \lim_{|x| \rightarrow \infty} u(x) = 0.$$

Then (1)-(3) has a unique solution.

Maremonti and Solonnikov [6, Theorems 1 and 2] proved the following theorem.

**Theorem A.** *Let  $n \geq 3$  and  $w \equiv 1$ . Suppose  $f \in L^p(\Omega)$  has compact support. Then (1)-(3) has a unique solution  $u$ . Moreover, the estimate  $\|u\|_{BL^{2,p}(\Omega)} \leq c\|f\|_{p;\Omega}$  holds in each one of the following cases:*

- (i)  $1 < p < n/2$ .
- (ii)  $n/2 \leq p < n$  and  $f$  satisfies

$$(4) \quad \int_{\Omega} f(x)h_2(x)dx = 0.$$

- (iii)  $p \geq n$  and  $f$  satisfies (4) and

$$(5) \quad \int_{\Omega} f(x) \frac{\partial h_2}{\partial x_j} dx = 0 \text{ for } j = 1, \dots, n.$$

In [1] we have introduced a subclass  $A_{p,k}$  of the Muckenhoupt  $A_p$  class:

$$A_{p,k} = \left\{ w \in A_p : \int_{\mathbb{R}^n} (1 + |x|)^{(k-n)p/(p-1)} w(x)^{1/(1-p)} dx < \infty \right\}.$$

It is easy to see that

$$\emptyset = A_{p,n} \subset A_{p,n-1} \subset \dots \subset A_{p,1} \subset A_{p,0} = A_p;$$

$$\begin{aligned} 1 \in A_{p,2} &\iff 1 < p < n/2, \\ 1 \in A_{p,1} \setminus A_{p,2} &\iff n/2 \leq p < n, \\ 1 \in A_p \setminus A_{p,1} &\iff p \geq n \end{aligned}$$

(cf. [1, §4]). In view of these facts, we shall show the following generalization of Theorem A.

**Theorem 1.** *Let  $n \geq 3$  and  $w \in A_p$ . Suppose  $f \in L^{p,w}(\Omega)$  has compact support. Then (1)-(3) has a unique solution  $u$ . Moreover, the estimate*

$$(6) \quad \|u\|_{BL^{2,p,w}(\Omega)} \leq c\|f\|_{p,w;\Omega}.$$

*holds in each one of the following cases:*

- (i)  $w \in A_{p,2}$ .
- (ii)  $w \in A_{p,1} \setminus A_{p,2}$  and  $f$  satisfies (4).
- (iii)  $w \in A_p \setminus A_{p,1}$  and  $f$  satisfies (4) and (5).

Let us consider next the case when  $f$  does not necessarily have compact support. In this case we cannot expect a solution of (1)-(3). In order to obtain a solution of (1)-(2) small at  $\infty$  in a sense, we define the subspace  $BL_0^{2,p,w}(\Omega)$  of  $BL^{2,p,w}(\Omega)$  by

$$BL_0^{2,p,w}(\Omega) = \{u \in BL^{2,p,w}(\Omega) : \text{there is } u_j \in BL^{2,p,w}(\Omega) \\ \text{such that } u_j(x) = 0 \text{ for } |x| > j \text{ and } \lim_{j \rightarrow \infty} \|u_j - u\|_{BL^{2,p,w}(\Omega)} = 0\}.$$

A solution  $u \in BL_0^{2,p,w}(\Omega)$  of (1)-(2) may be considered to be small at  $\infty$ . In view of (\*) and the usual trace argument, we see that the first derivatives of  $u$  exist on  $\partial\Omega$ , and they are  $q$ -th integrable with respect to the surface measure  $dS$  for some  $q > 1$  (see the following Lemma 2). In particular, we can consider the integral of the normal derivative  $\partial u / \partial n$  over  $\partial\Omega$ . We shall prove the following.

**Theorem 2.** *Let  $n \geq 3$  and  $w \in A_p$ . Suppose  $f \in L^{p,w}(\Omega)$ .*

- (i) *If  $w \in A_{p,2}$ , then there exists a unique solution  $u \in BL_0^{2,p,w}(\Omega)$  of (1)-(2).*
- (ii) *If  $w \in A_{p,1} \setminus A_{p,2}$ , then there exists a unique solution  $u \in BL_0^{2,p,w}(\Omega)$  of (1)-(2) satisfying the additional condition:*

$$(7) \quad \int_{\partial\Omega} \frac{\partial u}{\partial n} h_2 dS = 0.$$

- (iii) *If  $w \in A_p \setminus A_{p,1}$ , then there exists a unique solution  $u \in BL_0^{2,p,w}(\Omega)$  of (1)-(2) satisfying (7) and*

$$(8) \quad \int_{\partial\Omega} \frac{\partial u}{\partial n} \frac{\partial h_2}{\partial x_j} dS = 0 \text{ for } j = 1, \dots, n.$$

*In each case the solution  $u$  satisfies (6).*

Theorem 2 is a generalization of Maremonti and Solonnikov [6, Theorem 3]. However, there is a significant difference between [6] and ours. A layer potential method and a certain approximation property for  $L^p(\Omega)$  were the main tools in [6]. Since  $w \in A_p$  needs to be neither continuous nor isotropic, the arguments in [6] are not applicable to our weighted case. We shall make use of the same technique as in [1], e.g. modified Riesz potentials, an approximation of polynomials in  $BL^{2,p,w}(\Omega)$  and so on. We shall, in fact, give an explicit representation of the solutions  $u$  of (1)-(2). Although the solutions are not unique, among them there exists a canonical solution which will be written as the difference of a modified Riesz potential and its Poisson integral with respect to  $\Omega$ . We shall show that this canonical solution satisfies (6). Conditions (4), (5), (7) and (8) will imply that the solution considered in each statement must coincide with the canonical one.

## 2. PRELIMINARIES

In this section we collect some basic  $L^p$  estimates which are essentially based on singular integrals and hence applicable to our weighted version in a straightforward fashion. We shall give no proofs since they may be well-known or easy to prove. We denote by  $c$  for a positive constant depending only on  $n, p, w$  and domains whose value may change from one occurrence to the next.

Let us recall first an extension property of  $BL^{2,p,w}(\Omega)$ . Since the weighted Beppo Levi space  $BL^{2,p,w}(\Omega)$  and the weighted Sobolev space  $W^{2,p,w}(\Omega)$  have the same local behavior, it follows from Calderón's extension theorem (see [7, Chapter VI, 4.8]) that a function in  $BL^{2,p,w}(\Omega)$  extends to  $\mathbb{R}^n$ . We have

**Lemma 1.** *If  $u \in BL^{2,p,w}(\Omega)$ , then there exists  $u^* \in BL^{2,p,w}(\mathbb{R}^n)$  such that  $u^* = u$  on  $\Omega$  and  $\|u^*\|_{BL^{2,p,w}(\mathbb{R}^n)} \leq c\|u\|_{BL^{2,p,w}(\Omega)}$ .*

We need also a result for the restriction of elements of  $BL^{2,p,w}(\Omega)$  to  $\partial\Omega$ . Since  $w$  may degenerate on  $\partial\Omega$ , it is, in general, impossible to define an appropriate weighted space over  $\partial\Omega$  associated with  $BL^{2,p,w}(\Omega)$ . We can, however, give a coarse result, which is essentially unweighted. In view of (\*) and [7, Chapter VI, 4.2] we have

**Lemma 2.** (i) *Let  $1 < q < \infty$ . If  $f \in W^{1,q}(\Omega_1)$ , then the trace of  $f$  on  $\partial\Omega$  is defined and*

$$\|f\|_{L^q(\partial\Omega)} \leq c\|f\|_{W^{1,q}(\Omega_1)}.$$

(ii) *Let  $w \in A_p$ . Then there exists  $q > 1$  such that if  $u \in BL^{2,p,w}(\Omega)$ , then the traces of  $u$  and  $\nabla u$  on  $\partial\Omega$  are defined and*

$$\|u\|_{L^q(\partial\Omega)} + \|\nabla u\|_{L^q(\partial\Omega)} \leq c\|u\|_{BL^{2,p,w}(\Omega)}.$$

Let us recall  $L^p$  estimates for solutions of the Dirichlet problem in a bounded  $C^{1,1}$  domain.

**Lemma 3.** (cf. [4, Theorem 9.13 and Lemma 9.17]) *Let  $D$  be a bounded domain in  $\mathbb{R}^n$  with a  $C^{1,1}$  boundary portion  $T \subset \partial D$ . Let  $u \in W^{2,p,w}(D)$  be a solution of  $\Delta u = f$  in  $D$  with  $u = 0$  on  $T$  in the sense of  $W^{1,q}(D)$  for some  $q > 1$ . Then for any domain  $D' \Subset D \cup T$ ,*

$$\|u\|_{W^{2,p,w}(D')} \leq c(\|u\|_{p,w;D} + \|f\|_{p,w;D}).$$

*Moreover, if  $T = \partial D$ , then  $\|u\|_{W^{2,p,w}(D')} \leq c\|f\|_{p,w;D}$ .*

Note that the boundary condition in Lemma 3 is weaker than that in [4, Theorem 9.13]. However, we infer from a careful observation of [4, Lemma 9.12] that the conclusion remains true.

## 3. MODIFIED RIESZ POTENTIALS

Observe that that  $-\Delta h_2$  is the Dirac measure at the origin, so that  $-\Delta(h_2 * f) = f$  if the convolution  $h_2 * f$  exists. Therefore the first attempt to solve (1)-(2) begins with consideration on the Riesz potential  $h_2 * f$ . However, a problem arises: the convolution  $h_2 * f$  does not necessarily exist for all  $f \in L^{p,w}(\Omega)$ . This difficulty can be overcome by means of modified Riesz potentials. Observe that if  $y \neq 0$ , then  $h_2(x - y)$  has a multiple power series expansion in  $x_1, x_2, \dots, x_n$ , convergent in a neighborhood of the origin. We write

$$h_2(x - y) = \sum_{\nu=0}^{\infty} a_{\nu}(x, y),$$

where, for fixed  $\nu$  and  $y \neq 0$ ,  $a_{\nu}(x, y) = \sum_{|\beta|=\nu} \frac{x^{\beta}}{\beta!} D^{\beta} h_2(-y)$  is a homogeneous polynomial in  $x_1$  to  $x_n$  of degree  $\nu$  and continuous in  $x, y$  jointly for  $y \neq 0$  (see [5, Chapter 4]). We set

$$h_{2,k}(x, y) = \sum_{\nu=k}^{\infty} a_{\nu}(x, y)$$

for  $k \geq 0$  and write

$$I_{2,k}(f) = \int_{\mathbb{R}^n} h_{2,k}(\cdot, y) f(y) dy,$$

whenever the right hand side has a meaning. For notational convenience we let  $I_{2,0}(f) = I_2(f)$ . By definition  $I_2(f)$  coincides with the convolution  $h_2 * f$ .

We collect some properties of modified Riesz potentials  $I_{2,k}(f)$ . No proofs will be given. We refer to [1]. Note that if  $f \in L^{p,w}(\mathbb{R}^n)$ ,  $0 \leq k \leq 2$  and  $I_{2,k}(f)$  exists, then  $D^{\alpha} I_{2,k}(f) = (D^{\alpha} h_2) * f$  for  $|\alpha| \geq k$  and

$$(9) \quad \sum_{|\alpha|=2} \|D^{\alpha} I_{2,k}(f)\|_{p,w;\Omega} \leq c \|f\|_{p,w;\Omega}$$

(cf. [2, Theorems I and III] and [1, Lemma 8]). In particular,  $u = I_{2,2}(f)$  satisfies (1). We see that

$$(10) \quad |h_2(x - y) - h_{2,k}(x, y)| \leq c \sum_{\nu=0}^{k-1} |x|^{\nu} |y|^{2-n-\nu};$$

$$|h_{2,k}(x, y)| \leq c |x|^k |y|^{2-n-k} \quad \text{for } 2|x| > |y|$$

(cf. [5, Lemmas 4.1 and 4.2] and [1, Lemma 6]). We infer from Hölder's inequality that  $w \in A_{p,k}$  if and only if

$$(11) \quad \int_{\mathbb{R}^n} (1 + |x|)^{2-n-k} |f(x)| dx \leq c \|f\|_{p,w;\mathbb{R}^n}$$

(cf. [1, Theorem 5]). Hence we obtain from (10) and (11) that  $I_{2,k}(f)$  exists for every  $f \in L^{p,w}(\Omega)$  if and only if  $w \in A_{p,2-k}$ . In particular,  $I_{2,2}(f)$  exists for every  $f \in L^{p,w}(\Omega)$ . Therefore we have

**Lemma 4.** ([1, Theorem 4]) *Let  $w \in A_p$ . If  $u \in BL^{2,p,w}(\mathbb{R}^n)$ , then there are constants  $a$  and  $b_j$  such that  $u = I_{2,2}(-\Delta u) + a + \sum_{j=1}^n b_j x_j$ .*

Let us write

$$BL_0^{2,p,w}(\mathbb{R}^n) = \{u \in BL^{2,p,w}(\mathbb{R}^n) : \text{there is } u_j \in C_0^\infty(\mathbb{R}^n) \text{ such that } \lim_{j \rightarrow \infty} \|u_j - u\|_{BL^{2,p,w}(\mathbb{R}^n)} = 0\}.$$

We have given a characterization of  $BL_0^{2,p,w}(\mathbb{R}^n)$ .

**Lemma 5.** ([1, Corollary]) *Let  $w \in A_p$ .*

- (i)  $BL_0^{2,p,w}(\mathbb{R}^n) = \{I_2(g) : g \in L^{p,w}(\mathbb{R}^n)\} \iff w \in A_{p,2}$ .
- (ii)  $BL_0^{2,p,w}(\mathbb{R}^n) = \{I_{2,1}(g) + a : g \in L^{p,w}(\mathbb{R}^n), a \in \mathbb{R}\} \iff w \in A_{p,1} \setminus A_{p,2}$ .
- (iii)  $BL_0^{2,p,w}(\mathbb{R}^n) = \{I_{2,2}(g) + a + \sum_{j=1}^n b_j x_j : g \in L^{p,w}(\mathbb{R}^n), a, b_j \in \mathbb{R}\} \iff w \in A_p \setminus A_{p,1}$ .

In the proof of Lemma 5 we have used the following approximation property. This may be regarded as an alternative of [6, Lemma 2].

**Lemma 6.** (cf. [1, Lemma 18]) *Let  $w \in A_p$ . Suppose  $\varepsilon > 0$  and  $R > 0$ .*

- (i) *If  $w \in A_{p,1} \setminus A_{p,2}$ , then there is  $g \in L^{p,w}(\mathbb{R}^n)$  with compact support such that  $g(x) = 0$  for  $|x| < R$ ,  $\|g\|_{p,w;\mathbb{R}^n} < \varepsilon$  and  $I_2(g)(0) = 1$ .*
- (ii) *If  $w \in A_p \setminus A_{p,1}$ , then there are  $g_j \in L^{p,w}(\mathbb{R}^n)$ ,  $j = 1, \dots, n$ , with compact support such that  $g_j(x) = 0$  for  $|x| < R$ ,  $\|g_j\|_{p,w;\mathbb{R}^n} < \varepsilon$  and*

$$\frac{\partial}{\partial x_i} I_2(g_j)(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

#### 4. REPRESENTATION OF SOLUTIONS

In this section we shall give an explicit representation of solutions of (1)-(2). We have observed in the last section that  $I_{2,2}(f)$  satisfies (1). In order to obtain a solution of (1)-(2), we subtract from  $I_{2,2}(f)$  a harmonic function in  $\Omega$  having the same boundary values. This harmonic function will be given by the Poisson integral. Let  $G(x, y)$  be the Green function for  $\Omega$ , that is,  $G(x, y)$  satisfies

- (i)  $G(\cdot, y) - h_2(\cdot - y)$  is harmonic on  $\Omega$ ;
- (ii)  $G(\cdot, y)$  vanishes on  $\partial\Omega$ ;
- (iii) if  $n \geq 3$ , then  $G(\cdot, y)$  tends to zero at  $\infty$ ; if  $n = 2$ , then  $G(\cdot, y)$  is bounded at  $\infty$ .

For  $g \in L^1(\partial\Omega)$  we define the Poisson integral by

$$PI(g) = \frac{1}{\gamma_n} \int_{\partial\Omega} \frac{\partial G(x, y)}{\partial n_y} g(y) dS(y),$$

where  $n_y$  is the inward normal unit vector on  $\partial\Omega$  and  $dS(y)$  stands for the surface element. We observe that  $PI(g)$  is harmonic in  $\Omega$ , and that if  $g$  is continuous, then  $PI(g) = g$  on  $\partial\Omega$ . Note that  $PI(g)$  tends to zero at  $\infty$  if  $n \geq 3$ ;  $PI(g)$  is bounded at  $\infty$  if  $n = 2$ . We remark that

$$(12) \quad PI(1) < 1 \quad \text{and} \quad PI(h_2) \equiv h_2 \quad \text{if } n \geq 3;$$

$$PI(1) \equiv 1 \quad \text{and} \quad PI(h_2) < h_2 \quad \text{if } n = 2.$$

By Lemma 2 the trace of  $I_{2,k}(f)$  on  $\partial\Omega$  belongs to  $L^q(\partial\Omega)$ . We write  $PI(I_{2,k}(f))$  for the Poisson integral of the trace  $I_{2,k}(f)$  on  $\partial\Omega$ . One may expect that  $I_{2,2}(f) - PI(I_{2,2}(f))$  is a solution of (1)-(2). This is, in fact, the case.

**Theorem 3.** *Let  $n \geq 3$  and  $w \in A_p$ . Suppose  $f \in L^{p,w}(\Omega)$ . Then (1)-(2) has a solution in  $BL^{2,p,w}(\Omega)$ . Every solution  $u$  of (1)-(2) in  $BL^{2,p,w}(\Omega)$  is represented as*

$$u = I_{2,2}(f) - PI(I_{2,2}(f)) + a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j)),$$

where  $a$  and  $b_j$  are constants. Moreover,  $u = I_{2,2}(f) - PI(I_{2,2}(f))$  satisfies (6).

Let us consider first a special case of Theorem 3 when  $f \equiv 0$ .

**Lemma 7.** *Let  $n \geq 3$ . If  $h = a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j))$ , then  $h$  is a harmonic function in  $BL^{2,p,w}(\Omega)$  satisfying (2), and vice versa. Moreover,  $\|h\|_{BL^{2,p,w}(\Omega)} \leq c(|a| + \sum_{j=1}^n |b_j|)$ .*

*Proof.* It is easy to see that  $h = a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j))$  is a harmonic function vanishing continuously on  $\partial\Omega$ , and hence satisfying (2). Since  $D^\alpha PI(g)(x) = O(|x|^{2-n-|\alpha|})$  as  $|x| \rightarrow \infty$  for  $g \in L^1(\partial\Omega)$ , it follows from (11) that

$$\sum_{|\alpha|=2} \|D^\alpha h\|_{p,w;\Omega \setminus \Omega_1} \leq c(|a| + \sum_{j=1}^n |b_j|).$$

We have from Lemma 3

$$\|h\|_{W^{2,p,w}(\Omega_1)} \leq c(|a| + \sum_{j=1}^n |b_j|),$$

whence  $h \in BL^{2,p,w}(\Omega)$  and the required norm estimate follows.

Conversely, suppose  $h \in BL^{2,p,w}(\Omega)$  is a harmonic function in  $\Omega$  such that  $h = 0$  on  $\partial\Omega$  in the sense of  $W^{1,q}(\Omega_1)$  for some  $q > 1$ . In view of [4, Lemma 9.16] we see that  $h = 0$  on  $\partial\Omega$  in the sense of  $W^{1,q}(\Omega_1)$  for any  $q > 1$ , and hence  $h = 0$  on  $\partial\Omega$  continuously. By Lemma 1 we extend  $h$  to  $\mathbb{R}^n$  so that the

extension  $h^*$  belongs to  $BL^{2,p,w}(\mathbb{R}^n)$ . From Lemma 4 we can find  $a$  and  $b_j$  such that  $h^* = I_{2,2}(-\Delta h^*) + a + \sum_{j=1}^n b_j x_j$ . Since  $\Delta h^*$  is concentrated on  $\mathbb{R}^n \setminus \Omega$ , it follows  $I_{2,2}(-\Delta h^*)$  exists and  $h^* = I_{2,2}(-\Delta h^*) + a + \sum_{j=1}^n b_j x_j$  with different constants  $a$  and  $b_j$ . Observe that  $I_{2,2}(-\Delta h^*)$ ,  $PI(1)$  and  $PI(x_j)$  tend to zero at  $\infty$ , and hence

$$\lim_{|x| \rightarrow \infty, x \in \Omega} h(x) - (a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j))) = 0.$$

Therefore the maximum principle yields  $h = a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j))$ .

*Proof of Theorem 3.* Let  $u_0 = I_{2,2}(f) - PI(I_{2,2}(f))$ . We have seen that  $u_0$  satisfies (1). In order to prove (2) let  $\{f_\varepsilon\}$  be a regularization of  $f$  such that  $f_\varepsilon \rightarrow f$  in  $L^{p,w}(\Omega)$ . Since  $I_{2,2}(f_\varepsilon)$  is continuous, it follows that the continuous function  $u_\varepsilon = I_{2,2}(f_\varepsilon) - PI(I_{2,2}(f_\varepsilon))$  satisfies

$$\begin{aligned} -\Delta u_\varepsilon &= f_\varepsilon & \text{on } \Omega, \\ u_\varepsilon &= 0 & \text{on } \partial\Omega. \end{aligned}$$

Let  $1 < q < n/(n-1)$ . We have from (10)

$$\int_{\Omega_1} |h_{2,2}(x, y)|^q dx \leq c(1 + |y|)^{-nq}.$$

Hence Minkowski's inequality for integrals (see [7, p.271]) and (11) yield

$$(13) \quad \|I_{2,2}(f_\varepsilon)\|_{q;\Omega_1} \leq c\|f_\varepsilon\|_{p,w;\Omega}.$$

Similarly  $\|\frac{\partial}{\partial x_j} I_{2,2}(f_\varepsilon)\|_{q;\Omega_1} \leq c\|f_\varepsilon\|_{p,w;\Omega}$ . By Lemma 2 the trace of  $I_{2,2}(f_\varepsilon)$  on  $\partial\Omega$  satisfies

$$\|I_{2,2}(f_\varepsilon)\|_{L^1(\partial\Omega)} \leq c\|f_\varepsilon\|_{p,w;\Omega}.$$

By the estimate  $\partial G(x, y)/\partial n_y \leq c\delta(x)|x - y|^{-n}$  with  $\delta(x) = \text{dist}(x, \partial\Omega)$  for  $x, y$  near the boundary (cf. [8]), we have  $\sup_{y \in \partial\Omega} \int_{\Omega_1} |\partial G(x, y)/\partial n_y|^q dx < \infty$ . Hence Minkowski's inequality for integrals yields

$$\|PI(I_{2,2}(f_\varepsilon))\|_{q;\Omega_1} \leq c\|I_{2,2}(f_\varepsilon)\|_{L^1(\partial\Omega)} \leq c\|f_\varepsilon\|_{p,w;\Omega}.$$

Therefore we infer from (13) and Lemma 3 that

$$\|u_\varepsilon\|_{W^{2,q}(\Omega_1)} \leq c\|f_\varepsilon\|_{p,w;\Omega}.$$

In particular,  $u_\varepsilon \rightarrow u_0$  in  $W^{1,q}(\Omega_1)$ , so that  $u_0 = 0$  in the sense of  $W^{1,q}(\Omega_1)$ . Thus (2) holds. Consequently,  $u = I_{2,2}(f) - PI(I_{2,2}(f)) + a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j))$  is a solution of (1)-(2) and vice versa by Lemma 7.

Finally let us prove the norm estimate for  $u_0$ . To this end let  $r_2 > r_1$ ,  $\Omega_2 = \{x \in \Omega : |x| < r_2\}$  and take  $\eta \in C_0^\infty(\mathbb{R}^n)$  such that  $0 \leq \eta \leq 1$  on  $\mathbb{R}^n$ ,  $\eta = 1$  on  $\{|x| \leq r_1\}$  and  $\eta = 0$  on  $\{|x| \geq r_2\}$ . Let  $u_1 = \eta u_0$  and  $u_2 = (1 - \eta)u_0$ . An elementary calculation shows that

$$\|\Delta u_1\|_{p,w;\Omega} \leq c(\|f\|_{p,w;\Omega} + \|\nabla u_0\|_{p,w;\Omega_2 \setminus \Omega_1} + \|u_0\|_{p,w;\Omega_2 \setminus \Omega_1}).$$

By (10) we can compare  $D^\alpha I_{2,2}(f)$  ( $|\alpha| \leq 1$ ) and the maximal function  $Mf$  locally, and obtain

$$\sum_{|\alpha| \leq 1} \|D^\alpha I_{2,2}(f)\|_{p,w;\Omega_2} \leq c\|f\|_{p,w;\Omega}.$$

By the estimate of the Green function we have

$$\sup_{x \in \Omega_2 \setminus \Omega_1} |D^\alpha(PI(I_{2,2}(f)))(x)| \leq c\|I_{2,2}(f)\|_{L^1(\partial\Omega)} \leq c\|f\|_{p,w;\Omega}$$

for  $|\alpha| \leq 1$ . Hence

$$(14) \quad \|\Delta u_1\|_{p,w;\Omega} \leq c\|f\|_{p,w;\Omega}.$$

Since  $u_1$  vanishes on  $\{|x| = r_2\}$  continuously and on  $\partial\Omega_1$  in the sense of  $W^{1,q}(\Omega_2)$  for  $1 < q < n/(n-1)$ , it follows from Lemma 3 that

$$\|u_1\|_{W^{2,p,w}(\Omega_2)} \leq c\|f\|_{p,w;\Omega}.$$

By Lemma 1 we extend  $u_2$  to  $\mathbb{R}^n$  and represent it on  $\Omega$  as  $u_2 = I_{2,2}(-\Delta u_2) + a + \sum_{j=1}^n b_j x_j$ . Then by (9) and (14)

$$\sum_{|\alpha|=2} \|D^\alpha u_2\|_{p,w;\Omega} \leq c\|\Delta u_2\|_{p,w;\Omega} = c\|\Delta u - \Delta u_1\|_{p,w;\Omega} \leq c\|f\|_{p,w;\Omega}.$$

Consequently, (6) holds for  $u = u_0$ . The proof is complete.

Since  $u \in BL_0^{2,p,w}(\Omega)$  extends to  $u^* \in BL_0^{2,p,w}(\mathbb{R}^n)$  by Lemma 1, we have from Lemma 5 the following theorem.

**Theorem 4.** *Let  $n \geq 3$  and  $w \in A_p$ . Suppose  $f \in L^{p,w}(\Omega)$ . Then every solution  $u$  of (1)-(2) in  $BL_0^{2,p,w}(\Omega)$  is represented as follows:*

- (i) *If  $w \in A_{p,2}$ , then  $u = I_2(f) - PI(I_2(f))$ .*
- (ii) *If  $w \in A_{p,1} \setminus A_{p,2}$ , then  $u = I_{2,1}(f) - PI(I_{2,1}(f)) + a(1 - PI(1))$ , where  $a$  is a constant.*
- (iii) *If  $w \in A_p \setminus A_{p,1}$ , then  $u = I_{2,2}(f) - PI(I_{2,2}(f)) + a(1 - PI(1)) + \sum_{j=1}^n b_j(x_j - PI(x_j))$ , where  $a$  and  $b_j$  are constants.*

*In each case the canonical solution  $I_{2,k}(f) - PI(I_{2,k}(f))$ , ( $k = 0, 1, 2$  for (i), (ii), (iii), respectively), satisfies (6).*

*Proof.* Only (6) may require a proof. We have observed (6) for (iii) in Theorem 3. Suppose  $w \in A_{p,2}$ . Writing

$$\begin{aligned} I_{2,2}(f) &= I_2(f) - \left( \int_{\Omega} h_2(-y)f(y)dy + \sum_{j=1}^n x_j \int_{\Omega} \frac{\partial h_2}{\partial x_j}(-y)f(y)dy \right) \\ &= I_2(f) - (a' + \sum_{j=1}^n b'_j x_j), \end{aligned}$$

we obtain from (11) that  $|a'| \leq c\|f\|_{p,w;\Omega}$ ,  $|b'_j| \leq c\|f\|_{p,w;\Omega}$ . Hence from Theorem 3 and Lemma 7 we have

$$\begin{aligned} &\|I_2(f) - PI(I_2(f))\|_{BL^{2,p,w}(\Omega)} \\ &\leq \|I_{2,2}(f) - PI(I_{2,2}(f))\|_{BL^{2,p,w}(\Omega)} + \left\| a'(1 - PI(1)) + \sum_{j=1}^n b'_j(x_j - PI(x_j)) \right\|_{BL^{2,p,w}(\Omega)} \\ &\leq c\|f\|_{p,w;\Omega}. \end{aligned}$$

Thus (6) follows for (i). We can prove (6) similarly for (ii).

## 5. PROOF OF THEOREMS 1 AND 2

*Proof of Theorem 1.* Since  $f$  is of compact support, it follows that  $I_2(f)$  exists,  $\lim_{|x| \rightarrow \infty} I_2(f)(x) = 0$  and

$$\begin{aligned} I_2(f) &= I_{2,1}(f) + \int_{\Omega} h_2(-y)f(y)dy \\ &= I_{2,2}(f) + \int_{\Omega} h_2(-y)f(y)dy + \sum_{j=1}^n x_j \int_{\Omega} \frac{\partial h_2}{\partial x_j}(-y)f(y)dy. \end{aligned}$$

Hence Theorem 3 and the maximum principle say that  $u = I_2(f) - PI(I_2(f))$  is a unique solution of (1)-(3) in all cases of (i)-(iii). Conditions (4) and (5)

imply that  $u$  coincides with the canonical solutions  $I_{2,1}(f) - PI(I_{2,1}(f))$  and  $I_{2,2}(f) - PI(I_{2,2}(f))$  in the cases of (ii) and (iii), respectively. Therefore the norm estimate follows from Theorem 4.

*Proof of Theorem 2.* Since (i) has been proved in Theorem 4, we shall prove (ii) and (iii). Let us prove first the unicity. To this end it is sufficient to show that if  $v = 1 - PI(1)$ ,  $v_i = x_i - PI(x_i)$ , then

$$(15) \quad \begin{aligned} \int_{\partial\Omega} \frac{\partial v}{\partial n} h_2 dS &= 1, \\ \int_{\partial\Omega} \frac{\partial v_i}{\partial n} \frac{\partial h_2}{\partial x_j} dS &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

Let  $R > 0$  be sufficiently large. Then the Green formula yields

$$\int_{\partial\Omega} \frac{\partial v}{\partial n} h_2 dS = \int_{|x|=R} \left( \frac{\partial v}{\partial n} h_2 - \frac{\partial h_2}{\partial n} v \right) dS.$$

Since  $PI(1) = O(|x|^{2-n})$  and  $\nabla PI(1) = O(|x|^{1-n})$ , it follows that

$$\int_{\partial\Omega} \frac{\partial v}{\partial n} h_2 dS = - \lim_{R \rightarrow \infty} \int_{|x|=R} \frac{\partial h_2}{\partial n} dS = 1.$$

A similar calculation shows the second assertion.

What remains is to prove that if  $w \in A_{p,1} \setminus A_{p,2}$ , then  $u = I_{2,1}(f) - PI(I_{2,1}(f))$  satisfies (7); and that if  $w \in A_p \setminus A_{p,1}$ , then  $u = I_{2,2}(f) - PI(I_{2,2}(f))$  satisfies (7) and (8). Suppose  $w \in A_{p,1} \setminus A_{p,2}$ . Let  $u = I_{2,1}(f) - PI(I_{2,1}(f))$ . Take  $\varepsilon > 0$ . Split  $u$  into  $u_1 + u_2$ , where  $u_j = I_{2,1}(f_j) - PI(I_{2,1}(f_j))$ ,  $f_1 = \chi_{|x| < R} f$  and  $f_2 = \chi_{|x| > R} f$ . Take  $R > 0$  so large that  $\|f_2\|_{p,w;\Omega} < \varepsilon$  and  $\|D^\alpha I_{2,1}(f_2)\|_{\infty;\Omega_1} < \varepsilon$  for  $|\alpha| \leq 2$ . Then

$$\left| \int_{\partial\Omega} \frac{\partial u_2}{\partial n} h_2 dS \right| < c\varepsilon.$$

Observe that

$$u_1 = I_2(f_1) - PI(I_2(f_1)) - a(1 - PI(1)),$$

where  $a = \int_{\Omega} h_2 f_1 dx$ . By Lemma 6 we find a compactly supported function  $g \in L^{p,w}(\Omega)$  such that

$$\begin{aligned} g(x) &= 0 \text{ for } |x| < r_2 \text{ with } r_2 > r_1, \\ \|g\|_{p,w;\Omega} &< \varepsilon, \\ I_2(g)(0) &= \int h_2 g dx = 1, \\ |I_2(g) - 1| &< \varepsilon \text{ on } \Omega_1, \\ |u_1 - u_3| &< \varepsilon \text{ on } \Omega_1, \end{aligned}$$

where  $u_3 = I_2(f_1 - ag) - PI(I_2(f_1 - ag))$ . Observe that  $u_1 - u_3$  is harmonic on  $\Omega_2 = \{x \in \Omega : |x| < r_2\}$  and vanishes continuously on  $\partial\Omega$ . Hence,

$$\|\nabla(u_1 - u_3)\|_{\infty; \Omega_1} \leq c\|u_1 - u_3\|_{\infty; \Omega_2} \leq c\varepsilon$$

(see [8, Theorem 2.4 and its proof]); in particular

$$\left\| \frac{\partial u_1}{\partial n} - \frac{\partial u_3}{\partial n} \right\|_{\infty; \partial\Omega} \leq c\varepsilon.$$

Since  $u_3$  tends to zero at  $\infty$ , it follows from the Green formula that

$$\int_{\partial\Omega} \frac{\partial u_3}{\partial n} h_2 dS = \int_{\Omega} \Delta u_3 h_2 dx = - \int_{\Omega} (f_1 - ag) h_2 dx = 0.$$

Therefore

$$\left| \int_{\partial\Omega} \frac{\partial u_1}{\partial n} h_2 dS \right| < c\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, (7) follows. In the case of (iii) we can prove (7) and (8) by using the functions  $g_j$  appearing in Lemma 6. Details are left to the reader. The proof is complete.

## 6. TWO DIMENSIONAL CASE

Now let us consider the two dimensional case. Hereafter we let  $n = 2$ . In view of the Phragmén-Lindelöf principle, the boundary condition at  $\infty$  becomes

$$(3') \quad \limsup_{|x| \rightarrow \infty} |u(x)| < \infty.$$

We see from (12) that  $1 - PI(1)$  for  $n \geq 3$  is replaced by  $h_2 - PI(h_2)$ . Note also that if  $f$  has compact support, then

$$I_{2,1}(f) - PI(I_{2,1}(f)) = I_2(f) - PI(I_2(f));$$

$$\lim_{|x| \rightarrow \infty} (I_2(f)(x) - ch_2(x)) = 0 \quad \text{with } c = \int_{\Omega} f dx.$$

Moreover, since  $PI(h_2)$  is harmonic at  $\infty$ , it follows from [3, (2.73) Proposition] that

$$\lim_{R \rightarrow \infty} \int_{|x|=R} \frac{\partial}{\partial n} PI(h_2) dS = 0.$$

Hence we have an alternative of (15):

$$\int_{\partial\Omega} \frac{\partial}{\partial n} (h_2 - PI(h_2)) dS = 1.$$

Using these facts, we obtain the following counterparts of Theorems 1-4. Since  $A_{p,2} = \emptyset$ , we have two cases in each theorem.

**Theorem 1’.** *Let  $n = 2$  and  $w \in A_p$ . Suppose  $f \in L^{p,w}(\Omega)$  has compact support. Then (1)-(2) and (3’) have a unique solution  $u$ . Moreover, (6) holds in each one of the following cases:*

(i)  $w \in A_{p,1} \setminus A_{p,2}$  and  $f$  satisfies

$$(4') \quad \int_{\Omega} f(x) dx = 0.$$

(ii)  $w \in A_p \setminus A_{p,1}$  and  $f$  satisfies (4’) and (5).

**Theorem 2’.** *Let  $n = 2$  and  $w \in A_p$ . Suppose  $f \in L^{p,w}(\Omega)$ .*

(i) *If  $w \in A_{p,1} \setminus A_{p,2}$ , then there exists a unique solution  $u \in BL_0^{2,p,w}(\Omega)$  of (1)-(2) satisfying the additional condition:*

$$(7') \quad \int_{\partial\Omega} \frac{\partial u}{\partial n} dS = 0.$$

(ii) *If  $w \in A_p \setminus A_{p,1}$ , then there exists a unique solution  $u \in BL_0^{2,p,w}(\Omega)$  of (1)-(2) satisfying (7’) and (8).*

*In each case the solution  $u$  satisfies (6).*

**Theorem 3’.** *Let  $n = 2$  and  $w \in A_p$ . Suppose  $f \in L^{p,w}(\Omega)$ . Then (1)-(2) has a solution in  $BL^{2,p,w}(\Omega)$ . Every solution  $u$  of (1)-(2) in  $BL^{2,p,w}(\Omega)$  is represented as*

$$u = I_{2,2}(f) - PI(I_{2,2}(f)) + c(h_2 - PI(h_2)) + \sum_{j=1}^n b_j(x_j - PI(x_j)),$$

*where  $c$  and  $b_j$  are constants. Moreover,  $u = I_{2,2}(f) - PI(I_{2,2}(f))$  satisfies (6).*

**Theorem 4’.** *Let  $n = 2$  and  $w \in A_p$ . Suppose  $f \in L^{p,w}(\Omega)$ . Then every solution  $u$  of (1)-(2) in  $BL_0^{2,p,w}(\Omega)$  is represented as follows:*

(i) *If  $w \in A_{p,1} \setminus A_{p,2}$ , then  $u = I_{2,1}(f) - PI(I_{2,1}(f)) + c(h_2 - PI(h_2))$ , where  $c$  is a constant.*

(ii) *If  $w \in A_p \setminus A_{p,1}$ , then  $u = I_{2,2}(f) - PI(I_{2,2}(f)) + c(h_2 - PI(h_2)) + \sum_{j=1}^n b_j(x_j - PI(x_j))$ , where  $c$  and  $b_j$  are constants.*

*In each case the canonical solution  $I_{2,k}(f) - PI(I_{2,k}(f))$ , ( $k = 1, 2$  for (i), (ii) respectively), satisfies (6).*

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