EXTENDED HARNACK INEQUALITIES WITH EXCEPTIONAL SETS AND A BOUNDARY HARNACK PRINCIPLE

HIROAKI AIKAWA

Abstract. The Harnack inequality is one of the most fundamental inequalities for positive harmonic functions and, more generally, for positive solutions to elliptic equations and parabolic equations. This article gives a different view point of generalization. We generalize Harnack chains rather than equations. More precisely, we allow a small exceptional set; and yet we obtain a similar Harnack inequality. The size of an exceptional set is measured by capacity. The results are new even for classical harmonic functions. Our extended Harnack inequality includes information for the boundary behavior of positive harmonic functions. It yields a boundary Harnack principle for a very nasty domain whose boundary is given locally by the graph of a function with modulus of continuity worse than Hölder continuity.

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1. Introduction

1.1. An extended Harnack inequality with exceptional set. The Harnack inequality is one of the most fundamental inequalities for positive harmonic functions and, more generally, for positive solutions to elliptic equations and parabolic equations (De Giorgi-Nash-Moser theory). This article gives a different view point of generalization. We generalize Harnack chains rather than equations. More precisely, we allow a small exceptional set in the Harnack chain and yet we obtain a similar Harnack inequality.

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For simplicity we restrict ourselves to harmonic functions in this paper. Generalizations to positive solutions to uniformly elliptic equations are straightforward since we use only fundamental properties of harmonic functions such as the maximum principle and the linearity.

Let us begin with the Harnack inequality for positive harmonic functions in the Euclidean space \( \mathbb{R}^n \) with \( n \geq 2 \). We write \( B(x, r) \) for the open ball of center at \( x \) and radius \( r \). Let \( h \) be a positive harmonic function on \( B(x_0, r) \). If \( 0 < \alpha < 1 \), then

\[
(1.1) \quad \frac{1 - \alpha}{(1 + \alpha)^{n-1}} \leq \frac{h(x)}{h(x_0)} \leq \frac{1 + \alpha}{(1 - \alpha)^{n-1}} \quad \text{for } x \in B(x_0, \alpha r).
\]

Suppose two open balls \( B(x_0, r_0) \) and \( B(x_1, r_1) \) intersect. If \( h \) is a positive harmonic function on \( B(x_0, \alpha^{-1} r_0) \cup B(x_1, \alpha^{-1} r_1) \), then (1.1) gives

\[
\frac{1 - \alpha}{(1 + \alpha)^{n-1}} \leq \frac{h(z)}{h(x_0)} \quad \text{and} \quad \frac{h(z)}{h(x_1)} \leq \frac{1 + \alpha}{(1 - \alpha)^{n-1}} \quad \text{for } z \in B(x_0, r_0) \cap B(x_1, r_1),
\]

so that

\[
\frac{(1 - \alpha)^n}{(1 + \alpha)^n} \leq \frac{h(x_1)}{h(x_0)} \leq \frac{(1 + \alpha)^n}{(1 - \alpha)^n}.
\]

By the repeated application of the above inequality, we obtain the following theorem.

**Theorem A.** Let \( 0 < \alpha < 1 \) and let \( \{B(x_j, r_j)\}_{j=0}^J \) be a chain of open balls such that \( B(x_{j-1}, r_{j-1}) \cap B(x_j, r_j) \neq \emptyset \) for \( j = 1, \ldots, J \). If \( h \) is a positive harmonic function on \( B(x_0, \alpha^{-1} r_0) \cup \cdots \cup B(x_J, \alpha^{-1} r_J) \), then

\[
A^{-J} \leq \frac{h(x_J)}{h(x_0)} \leq A^J,
\]

where \( A = (1 + \alpha)^n/(1 - \alpha)^n \).

The above chain \( \{B(x_j, r_j)\}_{j=0}^J \) is referred to as a Harnack chain of length \( J \) with factor \( \alpha^{-1} \). Let \( D \) be a proper subdomain in \( \mathbb{R}^n \) and let \( \delta_D(x) = \text{dist}(x, \partial D) \). Let \( x, y \in D \). We say that a Harnack chain \( \{B(x_j, r_j)\}_{j=0}^J \) with factor \( \alpha^{-1} \) connects \( x \) and \( y \) in \( D \) if \( x \in B(x_0, r_0), y \in B(x_J, r_J) \) and \( \cup_{j=0}^J B(x_j, \alpha^{-1} r_j) \subset D \), or equivalently, \( 0 < r_j \leq \alpha \delta_D(x_j) \) for \( j = 0, \ldots, J \). The shortest length of Harnack chains connecting \( x \) and \( y \) in \( D \) is estimated by the quasihyperbolic metric \( k_D(x, y) \) defined by

\[
k_D(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds}{\delta_D(y(s))},
\]

where the infimum is taken over all rectifiable curves \( \gamma \) connecting \( x \) and \( y \) in \( D \); \( \gamma \) is parameterized as \( y(s), 0 \leq s \leq \ell(\gamma) \), by arc length \( s \) with \( \ell(\gamma) \) being the length of \( \gamma \). See Proposition 2.1 below, which provides a more specific Harnack chain. Hence Theorem A yields the following theorem.

**Theorem B.** Let \( D \) be a proper subdomain in \( \mathbb{R}^n \). Suppose \( h \) is a positive harmonic function in \( D \). Then

\[
(1.2) \quad \exp(-Ak_D(x, y)) \leq \frac{h(y)}{h(x)} \leq \exp(Ak_D(x, y)) \quad \text{for } x, y \in D,
\]

where \( A > 0 \) depends only on the dimension \( n \).

This theorem can be generalized to a positive solution to a uniformly elliptic (even nonlinear) partial differential equation. There have been extensively many papers in this direction. Our motivation is different. We are interested in the shape of a domain on which \( h \) is positive and
harmonic. More precisely, we would like to show that a small exceptional set can be allowed in the domain.

Let us begin with simple observations. Let $E$ be a set of finitely many points in $D$. Suppose $h$ is positive and harmonic in $D \setminus E$. Then (1.2) holds with different $A$, for we can take another Harnack chain connecting $x$ and $y$ in $D \setminus E$. More generally, $E$ may be a large compact set as long as $D \setminus E$ has a sufficiently wide passage from $x$ to $y$. However, the constant of comparison in (1.2) depends on the geometrical configuration of $D, E, x$ and $y$.

Another straightforward generalization of Theorem B is given by the removable singularity theorem. We call a set $E$ polar if there is a superharmonic function $u$ on a neighborhood of $E$ such that $u = \infty$ on $E$. Let $E$ be a closed polar set and suppose that $h$ is bounded positive harmonic function in $D \setminus E$. Then the removable singularity theorem shows that $h$ has a harmonic extension in $D$ and hence (1.2) holds with $A$ retained. Note that $k_{D,E}(x,y)$ and $k_D(x,y)$ are not comparable as a polar set $E$ may destroy the geometric nature and we have to make a long detour to avoid $E$.

In this paper, we shall show that an exceptional set $E$ can be bigger than a polar set, provided $E$ is apart from $x$ and $y$. In order to measure the size of exceptional sets, we introduce capacity with respect to an open set $U$. For $E \subset U$ we define

$$\text{Cap}_U(E) = \inf \int_U |\nabla u(x)|^2 \, dx : u(x) \geq 1 \text{ on } E, \ u \in C_0^\infty(U) \}.$$ 

Observe that $\text{Cap}_U(E)$ coincides with the Green capacity of $E$ with respect to $U$. See Section 4 below. Let $\{Q_k\}_k$ be the Whitney decomposition of $D$, i.e., $\{Q_k\}_k$ are closed cubes with sides parallel to the coordinate axes such that $\{\text{int } Q_k\}_k$ are mutually disjoint, $D = \bigcup_k Q_k$, and $\text{diam}(Q_k) \leq \text{dist}(Q_k, \partial D) \leq 4 \text{ diam}(Q_k)$.

See e.g. [Ste70, Chapter VI].

**Theorem 1.1.** Let $D$ be a domain with Whitney decomposition $\{Q_k\}$ and let $0 < \alpha_0 < 1$. Then there exists a positive constant $\varepsilon_0 < 1$ depending only on the dimension $n$ and $\alpha_0$ with the following property: Let $E$ be a closed set with

$$\frac{\text{Cap}_{Q_k^*(E \cap Q_k)}}{\text{Cap}_{Q_k^*(Q_k)}} < \varepsilon_0 \quad \text{for each Whitney cube } Q_k,$$

where $Q_k^*$ is the expanded open cube of $Q_k$ with factor $3/2$ and the same center. Let $x, y \in D$. If $h$ is a positive harmonic function in $(D \setminus E) \cup B(x, \alpha_0 \delta_D(x)) \cup B(y, \alpha_0 \delta_D(y))$, then

$$\exp(-A k_D(x,y)) \leq \frac{h(y)}{h(x)} \leq \exp(A k_D(x,y)),$$

where $A > 1$ depends only on the dimension $n$, $\alpha_0$ and $\varepsilon_0$. See Figure 1.

![Figure 1. An extended Harnack inequality with exceptional sets.](image-url)
1.2. **Application to a boundary Harnack principle.** As an application of the extended Harnack inequality, we shall prove a boundary Harnack principle for a domain whose boundary is locally given by the graph of a continuous function in $\mathbb{R}^{n-1}$. Let $\psi(t)$ be a nondecreasing continuous function for $t \geq 0$ with $\psi(0) = 0$. We say that a function $\varphi$ in $\mathbb{R}^{n-1}$ is $\psi$-Hölder continuous if $|\varphi(x') - \varphi(y')| \leq A \psi(|x' - y'|)$ for $x', y' \in \mathbb{R}^{n-1}$. We say that a bounded domain in $\mathbb{R}^n$ is a $\psi$-Hölder domain if its boundary is locally given by the graph of a $\psi$-Hölder continuous function in $\mathbb{R}^{n-1}$. If $0 < \alpha \leq 1$, then a $t^\alpha$-Hölder domain is simply called an $\alpha$-Hölder domain. A 1-Hölder domain is called a Lipschitz domain.

For a domain $D$, we consider a pair $(V, K)$ of a bounded open set $V \subset \mathbb{R}^n$ and a compact set $K \subset \mathbb{R}^n$ such that

$$K \subset V, \quad K \cap D \neq \emptyset \text{ and } K \cap \partial D \neq \emptyset. \quad (1.5)$$

**Definition 1.2.** We say that $D$ enjoys the **global boundary Harnack principle** if for each pair $(V, K)$ with (1.5) the following property holds: If $u$ and $v$ are positive superharmonic functions on $D$ such that

(i) $u$ and $v$ are bounded, positive and harmonic in $V \cap D$,
(ii) $u$ and $v$ vanish on $V \cap \partial D$ outside a polar set,

then

$$u(x)/u(y) \leq A \quad v(x)/v(y) \leq A \quad \text{for } x, y \in K \cap D,$$

where $A$ depends only on $D$, $V$, and $K$.

Ancona [Anc78], Dahlberg [Dah77] and Wu [Wu78] independently proved the global boundary Harnack principle for a Lipschitz domain. Note that Ancona and Wu actually obtained the local boundary Harnack principle, which is stronger than the global boundary Harnack principle. See [Aik08] for these accounts. Bass-Burdzy [BB91] proved probabilistically the global boundary Harnack principle for an $\alpha$-Hölder domain for $1/2 < \alpha \leq 1$, and then Bañuelos-Bass-Burdzy [BBB91] extended the range of $\alpha$ to $0 < \alpha \leq 1$. The main tool of [BBB91] is [BB92, Lemma 2.4], which inspires our present work. In [Aik09] we gave an analytic proof of the global boundary Harnack principle for an $\alpha$-Hölder domain. (Note that the proof of [Aik09] included a mistake and the proof actually worked only for $1/2 < \alpha \leq 1$.) By Theorem 1.1 (more precisely its variant Theorem 5.5) we have a global boundary Harnack principle for more general domains including all $\alpha$-Hölder domains with $0 < \alpha \leq 1$.

**Theorem 1.3.** Let $\psi(t)$ be a nondecreasing continuous function for $t \geq 0$ with $\psi(0) = 0$. Suppose that $\psi$ satisfies the Dini condition:

$$\int_0^1 \frac{\psi(t)}{t} \, dt < \infty \quad (1.6)$$

and

$$\limsup_{t \to 0} \frac{\psi(Mt)}{\psi(t)} < M \quad \text{for some } M > 1. \quad (1.7)$$

Then every $\psi$-Hölder domain enjoys the global boundary Harnack principle.

**Remark 1.4.** The Dini condition (1.6) is essential, while (1.7) is somewhat technical. These conditions are satisfied by many general functions $\psi$. 
(i) If $0 < \alpha < 1$, then $\psi(t) = t^\alpha$ satisfies (1.6) and (1.7). Actually, Lipschitz continuity $\psi(t) = t$ does not satisfy (1.7). However, $t$ is majorized by $t^\alpha$ for $0 < t \leq 1$ with $0 < \alpha < 1$. So, every $\alpha$-Hölder domain with $0 < \alpha \leq 1$ satisfies the global boundary Harnack principle. Hence an analytic proof of [BBB91] is retrieved.

(ii) For $\alpha > 0$ let $\psi_\alpha(t) = (-\log t)^{-\alpha}$ for $0 < t < 1/e^{\alpha+1}$ and extend it by constant for $t \geq 1/e^{\alpha+1}$. If $\alpha > 1$, then $\psi(t) = \psi_\alpha(t)$ satisfies (1.6) and (1.7), so that every $\psi_\alpha$-Hölder domain satisfies the global boundary Harnack principle. Obviously, $\psi_\alpha$-continuity is worse than Hölder continuity. See [Aik10], [Shi09], [Shi10], and [Ito12] for $\psi_\alpha$ in different contexts.

1.3. Plan of the paper. The plan of the paper is as follows. In Section 2 we shall construct a special Harnack chain in association with the quasihyperbolic metric. In Section 3 we formulate an extended Harnack inequality for this special Harnack chain. We observe that this extended Harnack inequality is reduced to an estimate of harmonic measure (Lemma 3.2). Section 4 is devoted to several relationships among harmonic measure, reduced function and capacity. In Section 5 we prove Lemma 3.2 by induction. We also give a stronger version of extended Harnack inequality (Theorem 5.5).

The validity of the boundary Harnack principle on a domain $D$ heavily depends on a lower estimate of the Green function for $D$. In Section 6 we establish a very precise lower estimate of the Green function for a $\psi$-Hölder domain (Lemma 6.3). The proof is based on Theorem 5.5. Capacitary width is a useful tool for the proof of the boundary Harnack principle. In Section 7 we define a modified capacitary width, which is comparable to the classical capacitary width. By the lower estimate of the Green function, we measure the modified capacitary width (and hence the classical capacitary width) of the set where the Green function is small (Lemma 7.6 and Corollary 7.7). In Section 8 we give some equivalent formulation of the boundary Harnack principle. These preparations culminate in the proof of the boundary Harnack principle.

We freely use basic properties in potential theory. See [AG01] for these accounts. We use the following notation. Let $S(x, r)$ be the sphere with center at $x$ and radius $r$. By the symbol $A$ we denote an absolute positive constant whose value is unimportant and may change from one occurrence to the next. If necessary, we use $A_0, A_1, \ldots$, to specify constants. If two positive quantities $f$ and $g$ satisfies $A^{-1} \leq f/g \leq A$ with some constant $A \geq 1$, then we say $f$ and $g$ are comparable and write $f \approx g$. The constant $A$ is referred to as the constant of comparison. We must be careful for the dependency of the constant of comparison. Throughout the paper, constants may depend on the dimension $n$, although the dependency on $n$ is not explicitly mentioned.

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2. Quasihyperbolic metric and a special Harnack chain

The first step of the proof of Theorem 1.1 is to construct a special Harnack chain in association with the quasihyperbolic metric. Throughout this section we let $D$ be a proper subdomain of $\mathbb{R}^n$.

Proposition 2.1. Let $x, y \in D$. Then there exists a Harnack chain $\{B(x_j, r_j)\}_{j=0}^J = \{B_j\}_{j=0}^J$ connecting $x$ and $y$ in $D$ with the following properties:

(i) $J \leq A_0 k_D(x, y)$. 


Let us begin with elementary properties of quasihyperbolic metric. The quasihyperbolic metric \( k_D(x, y) \) is decreasing with respect to \( D \), i.e., if \( D \subset D' \subset \mathbb{R}^n \), then
\[
k_D(x, y) \geq k_{D'}(x, y) \quad \text{for } x, y \in D.
\]
It is known that there exists a quasihyperbolic geodesic \( \tilde{\gamma} \) connecting \( x \) and \( y \) in \( D \), i.e.
\[
k_D(x, y) = \int_{\tilde{\gamma}} \frac{ds}{\delta_D(\tilde{\gamma}(s))}.
\]
If \( D \) is a ball, then the quasihyperbolic metric is explicitly calculated.

**Lemma 2.2.** Let \( r > 0 \), \( 0 < \alpha < 1 \) and \( x \in \mathbb{R}^n \). Then
\[
k_{B(x, r)}(x, y) = \log \frac{1}{1 - \alpha} \quad \text{for } y \in S(x, \alpha r).
\]

**Proof.** By dilation, translation and rotation we may assume that \( r = 1 \), \( x = 0 \) and \( y = (\alpha, 0, \ldots, 0) \).

It is easy to see that the line segment \( \overline{0y} \) is the quasihyperbolic geodesic connecting \( 0 \) and \( y \) in \( B(0, 1) \) and
\[
k_{B(0,1)}(0, y) = \int_{\overline{0y}} \frac{ds}{\delta_{B(0,1)}((s,0,\ldots,0))} = \int_0^\alpha \frac{ds}{1 - s} = \log \frac{1}{1 - \alpha}.
\]

For a general domain \( D \) we have a lower estimate of \( k_D(x, y) \). Let \( \gamma \) be a curve in \( D \). For \( x, y \in \gamma \), we let \( \gamma(x, y) \) be the subcurve of \( \gamma \) connecting \( x \) and \( y \).

**Lemma 2.3.** Let \( x \in D \) and \( 0 < \alpha < 1 \). Then
\[
k_D(x, y) \geq \log(1 + \alpha) \quad \text{for } y \in S(x, \alpha \delta_D(x)).
\]

**Proof.** Let \( y \in S(x, \alpha \delta_D(x)) \) and let \( \tilde{\gamma} \) be the quasihyperbolic geodesic connecting \( x \) and \( y \). Then, by definition,
\[
k_D(x, y) = \int_{\tilde{\gamma}} \frac{ds}{\delta_D(\tilde{\gamma}(s))} \geq \int_0^{\ell(\tilde{\gamma})} \frac{ds}{\delta_D(x) + s} = \log(\delta_D(x) + \ell(\tilde{\gamma})) - \log \delta_D(x)
\]
\[
\geq \log(\delta_D(x) + \alpha \delta_D(x)) - \log \delta_D(x) = \log(1 + \alpha),
\]
where we have used \( \delta_D(\tilde{\gamma}(s)) \leq \delta_D(x) + s \) in the first inequality.

**Proof of Proposition 2.1.** Let \( x, y \in D \). Let us construct a Harnack chain \( \{B(x_j, r_j)\}_{j=0}^J = \{B_j\}_{j=0}^J \) connecting \( x \) and \( y \) in \( D \) and satisfying the required properties with \( \tau = 9/8 \) and \( A_0 > 1 \) to be determined later. Let
\[
0 < \alpha < \frac{1}{75} \quad \text{and} \quad \beta = \frac{3\alpha}{2}.
\]
We shall make \( \alpha \) sufficiently small in the the proof of Theorem 1.1.
Let $x_0 = x$. If $y \in B(x_0, \beta \delta_D(x_0))$, then we let $r_0 = \beta \delta_D(x_0)$, and observe that the chain $\{B(x_0, r_0)\}$ of a simple ball satisfies the required properties. So, we assume that $y \notin B(x_0, \beta \delta_D(x_0))$. Let $\tilde{y}$ be the quasihyperbolic geodesic connecting $x = x_0$ and $y$ in $D$. Let $x_1$ be the last hit of $\tilde{y}$ to $S(x_0, \beta \delta_D(x_0))$. (Note that $\tilde{y}$ must hit $S(x_0, \beta \delta_D(x_0))$ since $y \notin B(x_0, \beta \delta_D(x_0))$.) If $y \in B(x_1, \beta \delta_D(x_1))$, then we stop with $r_0 = \alpha \delta_D(x_0)$, $r_1 = \beta \delta_D(x_1)$ and $\{B(x_j, r_j)\}_{j=0}^1$. Otherwise, we inductively choose points $x_j$ for $j \geq 2$ as the last hit of $\tilde{y}(x_{j-1}, y)$ to $S(x_{j-1}, \beta \delta_D(x_{j-1}))$, where $\tilde{y}(x_{j-1}, y)$ is the subarc of $\tilde{y}$ connecting $x_{j-1}$ and $y$. If $y \in B(x_j, \beta \delta_D(x_j))$, then we stop with $r_0 = \alpha \delta_D(x_0)$, $\ldots$, $r_{j-1} = \alpha \delta_D(x_{j-1})$, $r_j = \beta \delta_D(x_j)$. Otherwise, we continue the same procedure. By compactness we end up with $y \in B(x_j, \beta \delta_D(x_j))$ for some $J < \infty$ and $x_j \in \tilde{y}(x_{j-1}, y) \cap S(x_{j-1}, \beta \delta_D(x_{j-1}))$ for $j = 1, \ldots, J$. Let $r_0 = \alpha \delta_D(x_0)$, $\ldots$, $r_{j-1} = \alpha \delta_D(x_{j-1})$ and let $r_j = \beta \delta_D(x_j)$. By construction $x \in B(x_0, r_0)$ and $y \in B(x_j, r_j)$. Let us observe that $\{B(x_j, r_j)\}_{j=0}^J$ is a Harnack chain with the required properties.

By the triangle inequality $(1 - \beta)\delta_D(x_{j-1}) \leq \delta_D(x_j) \leq (1 + \beta)\delta_D(x_{j-1})$ for $1 \leq j \leq J$, so that

$$(1 - \beta)r_{j-1} \leq r_j \leq (1 + \beta)r_{j-1} \quad \text{for } 1 \leq j \leq J - 1$$

and $\frac{3}{2}(1 - \beta)r_{j-1} \leq r_j \leq \frac{3}{2}(1 + \beta)r_{j-1}$, Hence we have the first inequalities of (2.1). By definition $|x_{j-1} - x_j| = \beta \delta_D(x_{j-1}) = \frac{3}{2}r_{j-1}$, so that

$$\frac{r_j - r_{j-1}}{|x_{j-1} - x_j|} = \frac{r_j - r_{j-1}}{\frac{3}{2}r_{j-1}} = \frac{2 - \beta}{3/2} \geq \frac{2 - 1/50}{3/2} > 1.$$ 

This gives the second property of (2.1) with $A_0 > 1$ determined by $\alpha$. By construction we have

$$B(x_j, (\beta - \alpha)\delta_D(x_{j-1})) = B(x_j, \frac{\alpha}{2}\delta_D(x_{j-1})) \subset B_j \setminus B_{j-1}.$$ 

This yields the third property of (2.1) with $A_0 > 1$ determined by $\alpha$. Thus we have (ii).

By Lemma 2.3 we have

$$k_D(x_{j-1}, x_j) \geq \log(1 + \beta) \quad \text{for } 1 \leq j \leq J.$$ 

Since $x_0, \ldots, x_j$ lie on the quasihyperbolic geodesic $\tilde{y}$ connecting $x$ and $y$, it follows that

$$(2.2) \quad |i - j| \log(1 + \beta) \leq k_D(x_i, x_{i+1}) + \cdots + k_D(x_{j-1}, x_j) = k_D(x_i, x_j) \leq k_D(x, y)$$

for $0 \leq i < j \leq J$. Letting $i = 0$ and $j = J$, we obtain $J \leq A_0k_D(x, y)$ with $A_0 \geq \log(1 + \beta)^{-1}$. Thus (i) follows.

Finally let us prove (iii), in other words,

$$(2.3) \quad B^*_i \cap B^*_j \neq \emptyset \implies |i - j| \leq 1.$$ 

Recall $B^*_j = B(x_j, \tau \alpha \delta_D(x_j))$ for $0 \leq j \leq J - 1$ and $B^*_j = B(x_j, \tau \beta \delta_D(x_j))$. Suppose first $0 \leq i < j = J$. Take $z \in B^*_i \cap B^*_j$. It follows from (2.2) and Lemma 2.2 that

$$|i - j| \log(1 + \beta) \leq k_D(x_i, x_j) \leq k_D(x_i, z) + k_D(z, x_j) \leq k_{B(x_i, \delta_D(x_i))}(x_i, z) + k_{B(x_j, \delta_D(x_j))}(z, x_j) \leq \log \frac{1}{1 - \tau \alpha} + \log \frac{1}{1 - \tau \beta}.$$ 

Hence elementary inequalities

$$- \log(1 - t) \leq \frac{t}{1 - t} \quad \text{and} \quad \log(1 + t) \geq \frac{t}{1 + t} \quad \text{for } 0 \leq t < 1,$$

$$\frac{1 + p}{1 - q} \leq 1 + p + 2q \quad \text{for } 0 < p, q < \frac{1}{3}.$$
yield

\[ |i - J| \leq -\frac{\log(1 - \tau \alpha)}{\log(1 + \beta)} + \frac{\log(1 - \tau \beta)}{\log(1 + \beta)} \]

\[ \leq \frac{\tau \alpha}{1 - \tau \alpha} \cdot \frac{1 + \beta}{\beta} + \frac{\tau \beta}{1 - \tau \beta} \cdot \frac{1 + \beta}{\beta} \]

\[ \leq \frac{3}{4} (1 + \beta + 2 \tau \alpha) + \frac{9}{8} (1 + \beta + 2 \tau \beta) = \frac{15}{8} + \frac{177}{32} \beta < 2, \]

as

\[ \tau = \frac{9}{8} \quad \text{and} \quad 0 < \beta = \frac{3\alpha}{2} < \frac{1}{50} < \frac{4}{177}. \]

Thus (2.3) follows in this case.

Suppose next \( 0 \leq i < j \leq J - 1 \). Take \( z \in B_i \cap B_j \). In the same way as above,

\[ |i - j| \log(1 + \beta) \leq k_{B(x_j, \alpha \delta_B(x_j))}(x_i, z) + k_{B(x_j, \alpha \delta_B(x_j))}(z, x_j) \leq 2 \log \frac{1}{1 - \tau \alpha}, \]

so that \( |i - j| < 2 \). Thus (2.3) follows in this case, too. The proof is complete. \( \square \)

3. Proof of Theorem 1.1 — Reduction to harmonic measure for a special Harnack chain —

The second step of the proof of Theorem 1.1 is to give an extended Harnack inequality for a special Harnack chain given in Proposition 2.1.

**Theorem 3.1.** Let \( \{ B_j \}_{j=0}^J \) be a Harnack chain satisfying (ii) and (iii) of Proposition 2.1. Then there exist positive constants \( \varepsilon_0 < 1 \) and \( A_1 > 1 \) depending only on \( n, A_0, \) and \( \tau \) with the following property: Let \( E \) be a closed set in \( \mathbb{R}^n \) with \( E \cap (B_0 \cup B_J) = \emptyset \) and

\[ \text{Cap}_{B_j}(E \cap B_j) \leq \varepsilon_0 \text{Cap}_{B_j}(B_j) \]

for \( j = 1, \ldots, J - 1 \). If \( h \) is a positive harmonic function in \( B_0 \cup \cdots \cup B_J \setminus E \), then

\[ \frac{h(x_j)}{h(x_0)} \leq \exp(A_1 J). \]

**Proof of Theorem 1.1.** Let \( x, y \in D \) and let \( \{ B_j \}_{j=0}^J \) be the Harnack chain given in Proposition 2.1. Letting \( \alpha > 0 \) be small in the proof of Proposition 2.1, we may assume that \( B_0 \subseteq B(x, \alpha_0 \delta_B(x)) \) and \( B_J \subseteq B(y, \alpha_0 \delta_B(y)) \). Let \( E' = E' \setminus (B_0 \cup B_J) \). Then \( E' \) is a closed set with \( E' \cap (B_0 \cup B_J) = \emptyset \). Let us examine \( E' \) satisfies (3.1). For simplicity consider a generic ball \( B = B(z, \alpha \delta_B(z)) \) with \( z \in D \) and \( 0 < \alpha < 1 \). Observe that the number of Whitney cubes \( Q_k \) intersecting \( B \) is bounded by a constant depending only on \( \alpha \) and the dimension \( n \); if \( B \cap Q_k \neq \emptyset \), then \( \text{diam}(B) = 2 \alpha \delta_B(z) \approx \text{diam}(Q_k) \approx \text{dist}(Q_k, \partial D) \), and

\[ \text{Cap}_{B_k}(E \cap B \cap Q_k) \approx \text{Cap}_{Q_k}(E \cap B \cap Q_k) \quad \text{for} \quad E \subseteq D, \]

where \( B' = B(z, \tau \alpha \delta_B(z)) \) with \( 1 < \tau < 1/\alpha \) and the constant of comparison depends only on \( \alpha \) and \( \tau \). See Lemma 4.8 below. Let us now apply the above observations to \( B = B_j \). We have from (1.3)

\[ \frac{\text{Cap}_{B_j}(E \cap B_j)}{\text{Cap}_{B_j}(B_j)} \leq \sum_{Q_k \cap B_j \neq \emptyset} \frac{\text{Cap}_{B_j}(E \cap B_j \cap Q_k)}{\text{Cap}_{B_j}(B_j)} \leq A \sum_{Q_k \cap B_j \neq \emptyset} \frac{\text{Cap}_{Q_k}(E \cap B_j \cap Q_k)}{\text{Cap}_{Q_k}(Q_k)} \leq A \varepsilon_0. \]
3.1. Let $A \subset \partial U$. Then

$$h(y) \leq \exp(A_1 J).$$

This is the right inequality of (1.4). Replacing the roles of $x$ and $y$, we obtain the left inequality of (1.4). The theorem follows. \hfill \square

Theorem 3.1, in turn, is reduced to an estimate of harmonic measure. Let $U$ be an open set and $E \subset \partial U$. By $\omega(E, U)$ we denote the harmonic measure of $E$ in $U$, i.e., the Dirichlet solution of the boundary function $\chi_E$ on $U$. The Dirichlet solution is taken in the Perron-Wiener-Brelot sense; $\omega(E, U)$ is harmonic in $U$ and takes boundary values $\chi_E$ on $\partial U$ outside a polar set. We write $\omega(E, U) = \chi_E$ quasi-everywhere (q.e.) on $\partial U$ for this property. The value of the harmonic measure $\omega(E, U)$ at $x$ is designated by $\omega^x(E, U)$. If $E$ is a compact subset of $U$, then we write $\omega(E, U)$ for $\omega(\partial U, U \setminus E)$. This can be regraded as a regularized reduced function. See Section 4.

Lemma 3.2. Let $\{B_j\}_{j=0}^J$ be a Harnack chain satisfying (ii) and (iii) of Proposition 2.1. Then there exist positive constants $\varepsilon_0 < 1$ and $A_2 > 1$ depending only on $n, A_0$, and $\tau$ with the following property: if a closed set $E$ satisfies $E \cap B_0 = \emptyset$ and (3.1) for $j = 1, \ldots, J - 1$, then

$$\omega^x(b_j, \Omega_j \cup B_j) \geq \exp(-A_2 j) \quad \text{for } j = 1, \ldots, J,$$

where $\Omega_j = (B_0 \cup \cdots \cup B_j) \setminus E$.

The proof of Lemma 3.2 requires some properties among harmonic measure, reduced function and capacity in Section 4. The proof will be postponed in Section 5. Let us complete the proof of Theorem 3.1 by using Lemma 3.2.

Proof of Theorem 3.1. Since $E \cap B_J = \emptyset$, it follows from the ordinary Harnack inequality that $h \approx h(x_J)$ on $b_J$, and hence

$$h \geq Ah(x_J)\omega(b_J, \Omega_j) = Ah(x_J)\omega(b_J, \Omega_j \cup B_j) \quad \text{on } \Omega_j \setminus b_J$$

by the maximum principle. Evaluate the inequality at $x_0$. We have from Lemma 3.2 with $j = J$ that

$$h(x_0) \geq Ah(x_J)\omega^x(b_J, \Omega_j \cup B_j) \geq Ah(x_J)\exp(-A_2 J),$$

so that

$$\frac{h(x_J)}{h(x_0)} \leq A^{-1}\exp(A_2 J) \leq \exp(A'_2 J),$$

where $A'_2$ is a constant bigger than $A_2$. Hence (3.2) follows. The proof is complete. \hfill \square

4. Harmonic measure, reduced function and capacity

The proof of Lemma 3.2 is based on some relationships among harmonic measure, reduced function and capacity. Let $U$ be an open set. For $E \subset U$ and a nonnegative function $u$ on $E$, we define the reduced function $U R^E_u$ by

$$U R^E_u(x) = \inf\{v(x) : v \geq 0 \text{ is superharmonic in } U \text{ and } v \geq u \text{ on } E\} \quad \text{for } x \in U.$$
The lower semicontinuous regularization of $^{U\mathbb{R}_{+}}E$ is called the \textit{regularized reduced function} or \textit{balayage} and is denoted by $^{U\mathbb{R}_{+}E}$. It is known that $^{U\mathbb{R}_{+}E}$ is a nonnegative superharmonic function, $^{U\mathbb{R}_{+}E} \leq ^{U\mathbb{R}_{+}u}E$ on $U$ and the equality sign holds q.e. on $U$. If $u$ is a nonnegative superharmonic function on $U$, then $^{U\mathbb{R}_{+}u}E \leq u$ on $U$. We suppress $U$ and write $^{u\mathbb{R}_{+}E}$ if the open set $U$ can be understood from the context. Observe that if $E$ is a compact subset of $U$, then

$^{U\mathbb{R}_{+}E} = \omega(\partial E, U \setminus E)$ on $U \setminus E$.

Thus $^{U\mathbb{R}_{+}E}$ may be regarded as the extension of the harmonic measure $\omega(\partial E, U \setminus E)$ to $U$. For the convenience sake, we also write $\omega(E, U)$ for this \textit{extended harmonic measure}. Note that $\omega(E, U)$ is superharmonic in $U$ and harmonic in $U \setminus E$; $0 \leq \omega(E, U) \leq 1$ on $U$, $\omega(E, U) = 0$ q.e. on $\partial U \setminus E$ and $\omega(E, U) = 1$ q.e. on $E$.

Let $G_{U}$ be the Green function for $U$, i.e., $G_{U}(x, y)$ is a function of $(x, y) \in U \times U$ satisfying the following three conditions for each $y \in D$:

(i) $G_{U}(:, y)$ is harmonic in $U \setminus \{y\}$;
(ii) $-\Delta G_{U}(:, y)$ is the Dirac measure at $y$;
(iii) $G_{U}(\cdot, y) = 0$ q.e. on $\partial U$.

For a (Radon) measure $\mu$ on $U$ we let $G_{U}\mu = \int_{U} G(\cdot, y) d\mu(y)$ be the Green potential of $\mu$.

Let $E$ be a compact subset of $U$. Recall the definition of the capacity of $E$ in $U$:

$$\text{Cap}_{U}(E) = \inf \{ \int_{U} |\nabla u(x)|^{2}dx : u \geq 1 \text{ on } E, \ u \in C_{0}^{\infty}(U) \}.$$  

Suppose $u \in C_{0}^{\infty}(U)$. Then $u = G_{U}\mu$ with $\mu = -\Delta u$ (the Riesz decomposition). (Note our normalization (ii) for the Green function.) Integration by parts shows

$$\int_{U} |\nabla u(x)|^{2}dx = -\int_{U} u \Delta u dx = \int G_{U} d\mu.$$

Thus the capacity of $E$ in $U$ is given as the infimum of the Green energy $\int G_{U} d\mu$ for $u = G_{U}\mu \in C_{0}^{\infty}(U)$ with $u \geq 1$ on $E$. In general, there is no minimizer in $C_{0}^{\infty}(U)$ attaining the infimum. It is well known that the infimum is attained by $^{U\mathbb{R}_{+}E}$, which is superharmonic in $U$, harmonic in $U \setminus E$ and equal to $1$ q.e. on $E$. Hence $^{U\mathbb{R}_{+}E} = G_{U}\mu_{E}$ with $\mu_{E} = -\Delta (^{U\mathbb{R}_{+}E})$. Since $\mu_{E}$ is supported on $E$ and a polar set is of $\mu_{E}$ measure zero, we have

$$\text{Cap}_{U}(E) = \int_{U} ^{U\mathbb{R}_{+}E}d\mu_{E} = \int_{U} G_{\mu_{E}} d\mu_{E} = ||\mu_{E}||.$$

We call $\mu_{E}$ and $^{U\mathbb{R}_{+}E} = G_{U}\mu_{E}$ the \textit{capacitary distribution} of $E$ and the \textit{capacitary potential} of $E$, respectively. Further, by the minimax theorem in [Fug65] we have the following characterization:

$$\text{Cap}_{U}(E) = \sup \{ ||\mu|| : G_{U}\mu \leq 1 \text{ on } U, \text{supp} \mu \subset E \}$$

$$= \inf \{ ||\mu|| : G_{U}\mu \geq 1 \text{ on } E \}.$$  

This characterization extends to every Borel subset $E$ of $U$.

Using these observations, we can easily estimate the capacity of a ball.

**Lemma 4.1.** \textit{Let $B$ be an open ball with expanded ball $B^{*}$. Then}

$$\text{Cap}_{B^{*}}(B) = A \text{ diam}(B)^{n-2},$$

\textit{where $A > 0$ depends only on the dimension $n$ and the factor of expansion $B^{*}$}.
Proof. Let $B = B(x, r)$ and $B^* = B(x, \tau r)$ with $\tau > 1$. By translation and dilation we have
\[
\Cap_{B^*}(B) = \Cap_{B(x,\tau r)}(B(x, r)) = r^{n-2} \Cap_{B(0, r)}(B(0, 1)),
\]
which gives the lemma with $A = 2^{2-n} \Cap_{B(0, r)}(B(0, 1))$. □

The following lemma gives a relationship among harmonic measures, capacities and regularized reduced functions. For simplicity let us state the lemma for $n \geq 3$. The lemma has somewhat different form for the $n = 2$ case because of the lack of the Green function for $\mathbb{R}^2$.

Lemma 4.2. Let $n \geq 3$. Let $B$ be an open ball with expanded ball $B^*$ and let $B_0$ be an open ball with center at $x_0$. Assume that
\[
(4.1) \quad \text{diam}(B) \approx \text{diam}(B^*) \approx \text{diam}(B_0).
\]
Then there exists a positive constant $A$ depending only on the constant of comparison in (4.1) with the following property: If $E \subset B \setminus B_0$, then
\[
(4.2) \quad \omega^{x_0}(E, \mathbb{R}^n) = \frac{\mathbb{R}^n_{\overline{B}}(x_0)}{\mathbb{R}^n_{\overline{B}^*}(x_0)} \leq A \frac{\Cap_{B^*}(E)}{\Cap_{B^*}(B)}.
\]
See Figure 2.

![Figure 2. Harmonic measure and capacity.](image)

Proof. In this proof $A$ and constants of comparison depend only on the constant of comparison of (4.1). For simplicity let us write $G$ for the Green function $G_{\mathbb{R}^n}$. Observe from (4.1) that
\[
(4.3) \quad G_B(x, y) \leq G(x, y) \leq A G_B(x, y) \quad \text{for } x, y \in \overline{B}
\]
and
\[
(4.4) \quad G(x_0, y) \leq A \text{diam}(B)^{2-n} \quad \text{for } y \in B \setminus B_0.
\]
Observe that
\[
\Cap_{B^*}(E) = \|\mu_E\| \quad \text{with } \mathbb{R}^n_{\overline{B}} = G_{B^*} \mu_E \text{ and } \text{supp } \mu_E \subset \overline{E} \subset B \setminus B_0.
\]
Since $G \mu_E \geq G_B \mu_E$ on $B$ by (4.3), it follows that $G \mu_E \geq 1$ on $E$. Since $G \mu_E$ is a nonnegative superharmonic function in $\mathbb{R}^n$, we have
\[
\mathbb{R}^n_{\overline{B}} \leq G \mu_E \quad \text{in } \mathbb{R}^n.
\]
By (4.4) and Lemma 4.1 we obtain
\[
\mathbb{R}^n_{\overline{B}}(x_0) \leq G \mu_E(x_0) \leq A \text{diam}(B)^{2-n} \|\mu_E\| \leq A \frac{\Cap_{B^*}(E)}{\Cap_{B^*}(B)}.
\]
Hence (4.2) follows. The lemma is proved. □
**Corollary 4.3.** Let \( n \geq 3 \). Let \( B \) be an open ball with expanded ball \( B^* \). If \( E \subset B \), then
\[
\omega(E, \mathbb{R}^n) = \mathbb{R}^n E \leq A \frac{\text{Cap}_B(E)}{\text{Cap}_B(B)} \quad \text{on } \mathbb{R}^n \setminus B^*,
\]
where \( A \) depends only on the ratio \( \text{diam}(B^*)/\text{diam}(B) \).

**Proof.** For each \( x_0 \in \partial B^* \) we take a ball \( B_0 \) with center at \( x_0 \) such that \( \text{diam}(B_0) \approx \text{diam}(B) \) and \( B_0 \cap B = \emptyset \), where the constant of comparison depends only on the ratio \( \text{diam}(B^*)/\text{diam}(B) \). Then Lemma 4.2 shows that
\[
\mathbb{R}^n E(x_0) \leq A \frac{\text{Cap}_B(E)}{\text{Cap}_B(B)}.
\]
The arbitrary nature of \( x_0 \in \partial B^* \) and the maximum principle yield (4.5). The corollary is proved. \( \square \)

Since \( \mathbb{R}^2 \) has no Green function, we remove a closed ball \( K \) from \( \mathbb{R}^2 \) so that the complement \( \mathbb{R}^2 \setminus K \) has the Green function. We have estimates of harmonic measure with respect to \( \mathbb{R}^2 \setminus K \). The estimates holds not only for \( n = 2 \) but also for \( n \geq 3 \). In case \( n \geq 3 \), however, the simpler estimates (Lemma 4.2 and Corollary 4.3) are sufficient for future purposes.

**Lemma 4.4.** Let \( B \) be an open ball with expanded ball \( B^* \), \( B_0 \) an open ball with center at \( x_0 \) and \( K \) a closed ball with \( K \cap (B^* \cup B_0) = \emptyset \). Assume that
\[
\text{diam}(B) \approx \text{diam}(B^*) \approx \text{diam}(B_0) \approx \text{diam}(K) \approx \text{dist}(K, B^*) \approx \text{dist}(K, B_0).
\]
Then there exists a positive constant \( A \) depending only on the constant of comparison in (4.6) with the following property: If \( E \subset B \setminus B_0 \), then
\[
\omega^\omega(E, \mathbb{R}^n \setminus K) = \mathbb{R}^n \setminus K E(x_0) \leq A \frac{\text{Cap}_B(E)}{\text{Cap}_B(B)}.
\]

**Proof.** In view of (4.6) we have
\[
G_B(x, y) \leq G_{\mathbb{R}^n \setminus K}(x, y) \leq AG_B(x, y) \quad \text{for } x, y \in \overline{B},
\]
\[
G_{\mathbb{R}^n \setminus K}(x_0, y) \leq A \text{diam}(B)^{2-n} \quad \text{for } y \in B \setminus B_0,
\]
where the constants of comparison depends only on that of (4.6) and the dimension \( n \). Replace (4.3) and (4.4) by these comparisons. Then the same proof as for Lemma 4.2 works. \( \square \)

We readily have a counterpart of Corollary 4.3.

**Corollary 4.5.** Let \( B \) be an open ball with expanded ball \( B^* \) and let \( K \) be a closed ball such that
\[
\text{diam}(B) \approx \text{diam}(B^*) \approx \text{diam}(K) \approx \text{dist}(K, B^*).
\]
If \( E \subset B \), then
\[
\omega(E, \mathbb{R}^n \setminus K) = \mathbb{R}^n \setminus K E(x_0) \leq A \frac{\text{Cap}_B(E)}{\text{Cap}_B(B)} \quad \text{on } \mathbb{R}^n \setminus B^*,
\]
where \( A \) depends only on the constant of comparison in (4.7).

We also have an opposite inequality in some sense. See [Aik98, Lemma 2].

**Lemma 4.6.** Let \( B \) an open ball with expanded ball \( B^* \) such that \( \text{diam}(B) \approx \text{diam}(B^*) \). If \( E \subset B \), then
\[
\mathbb{R}^n E(x) \geq A \frac{\text{Cap}_B(E)}{\text{Cap}_B(B)} \quad \text{on } B,
\]
where \( A \) depends only on the constant of comparison of \( \text{diam}(B) \approx \text{diam}(B^*) \).
Proof. Let $\mu_E$ be the capacitary distribution of $E$ with respect to $B^*$, i.e., $b^*\hat{R}_1^E = G_{B^*}\mu_E$. Then $\mu_E$ is concentrated on $E$ and $\operatorname{Cap}_{B^*}(E) = \|\mu_E\|$. Since $G_{B^*}(x, y) \geq A \operatorname{diam}(B)^{2-n}$ for $x, y \in \overline{B}$, it follows from Lemma 4.1 that
\[
b^*\hat{R}_1^E(x) \geq A \operatorname{diam}(B)^{2-n}\|\mu_E\| \geq A \frac{\operatorname{Cap}_{B^*}(E)}{\operatorname{Cap}_{B^*}(B)} \quad \text{for } x \in B.
\]
The lemma follows. \hfill \Box

The regularized reduced function $\hat{U}\hat{R}_n^E$ increases and the capacity $\operatorname{Cap}_U(E)$ decreases, as $U$ increases. We have a converse in a certain sense.

Lemma 4.7. Let $B, B^*$ and $\tilde{B}$ be open balls with $B \subset B^* \subset \tilde{B}$ and
\begin{equation}
\operatorname{diam}(B) \approx \operatorname{diam}(B^*) \approx \operatorname{diam}(\tilde{B}) \approx \operatorname{dist}(B, \mathbb{R}^n \setminus B^*) \approx \operatorname{dist}(B^*, \mathbb{R}^n \setminus \tilde{B}).
\end{equation}
If $E \subset B$, then
\begin{align}
\hat{b}^*\hat{R}_1^E \leq \hat{b}^*\hat{R}_1^E \leq A \hat{b}^*\hat{R}_1^E & \quad \text{on } B, \\
\operatorname{Cap}_{\tilde{B}}(E) \leq \operatorname{Cap}_{B^*}(E) \leq A \operatorname{Cap}_{B^*}(E),
\end{align}
where $A$ depends only on the constant of comparison of (4.8).

Proof. By definition we have $b^*\hat{R}_1^E \leq \hat{b}^*\hat{R}_1^E \leq b^*\hat{R}_1^E$ on $B^*$ and $\operatorname{Cap}_{\tilde{B}}(E) \leq \operatorname{Cap}_{B^*}(E)$. Let $\mu_E$ be the capacitary distribution of $E$ with respect to $B^*$, i.e., $b^*\hat{R}_1^E = G_{B^*}\mu_E$, $\operatorname{Cap}_{B^*}(E) = \|\mu_E\|$ and $\mu_E$ is concentrated on $E$. By the geometrical hypothesis
\[
G_{B^*}(x, y) \leq G_{\tilde{B}}(x, y) \leq AG_{B^*}(x, y) \quad \text{for } x, y \in \overline{B}.
\]
Hence
\begin{equation}
b^*\hat{R}_1^E = G_{B^*}\mu_E \leq G_{\tilde{B}}\mu_E \leq AG_{B^*}\mu_E = A b^*\hat{R}_1^E \quad \text{on } \overline{B}.
\end{equation}
This means that $G_{\tilde{B}}\mu_E$ is a nonnegative superharmonic function in $\tilde{B}$ such that $G_{\tilde{B}}\mu_E \geq 1$ q.e. on $E$. By the definition of the regularized reduced function $\hat{b}^*\hat{R}_1^E$, we have $\hat{b}^*\hat{R}_1^E \leq G_{\tilde{B}}\mu_E$ on $\tilde{B}$. This, together with (4.11), gives the second inequality of (4.9).

Let $\nu_E$ be the capacitary distribution of $E$ with respect to $\tilde{B}$, i.e., $\hat{b}^*\hat{R}_1^E = G_{\tilde{B}}\nu_E$, $\operatorname{Cap}_{\tilde{B}}(E) = \|\nu_E\|$ and $\nu_E$ is concentrated on $E$. Integrate (4.11) with respect to $\nu_E$ and apply the Fubini theorem. We have
\[
\operatorname{Cap}_{B^*}(E) = \|\mu_E\| = \int G_{\tilde{B}}\nu_E d\mu_E \leq A \int G_{B^*}\mu_E d\nu_E \leq A \|\nu_E\| = A \operatorname{Cap}_{B^*}(E).
\]
Hence the second inequality of (4.10) follows. \hfill \Box

In the same way, we have a similar comparison for cubes.

Lemma 4.8. Let $Q$ be an open cube with expanded cube $Q^*$ and let $B$ be an open ball with expanded ball $B^*$. Suppose $Q \cap B \neq \emptyset$ and
\[
diam(Q) \approx diam(Q^*) \approx diam(B) \approx diam(B^*) \approx \operatorname{dist}(Q, \mathbb{R}^n \setminus Q^*) \approx \operatorname{dist}(B, \mathbb{R}^n \setminus B^*).
\]
Then
\[
\operatorname{Cap}_Q(E) \approx \operatorname{Cap}_{B^*}(E) \quad \text{for } E \subset Q \cap B,
\]
where the constant of comparison depends only on that of (4.12).
5. Proof of Lemma 3.2

Lemma 3.2 will be proved by induction on \( j \). Throughout this section, we let \( \{B_j\}_{j=0}^J \) be a Harnack chain satisfying (ii) and (iii) of Proposition 2.1 and let \( E \) be a closed set in \( B_0 \cup \cdots \cup B_J \). For future purposes, we allow \( E \) may intersect \( B_0 \cup \cdots \cup B_J \). Let \( \Omega_j = B_0 \cup \cdots \cup B_j \setminus E \). Let \( b_j^* \) be an expansion of \( b_j \) such that \( b_j \subset b_j^* \subset B_j \setminus \Omega_j \) and \( \text{diam}(b_j^*) \approx \text{diam}(B_j) \approx \text{dist}(b_j, \partial B_j) \approx \text{diam}(b_j^*, \partial (B_{j-1} \cup B_j)) \) for \( 0 \leq j \leq J \). One of our main tools is an application of the scale invariant boundary Harnack principle.

**Lemma 5.1.** There exists a positive constant \( A \) depending only on \( A_0 \) with the following property: Let \( 2 \leq j \leq J - 1 \). Suppose \( u \) is a bounded positive harmonic function on \( \Omega_j \cup B_j \) vanishing q.e. on \( \partial (\Omega_j \cup B_j) \setminus B_j \). Suppose \( v \) is a bounded positive superharmonic function on \( \Omega_j \cup B_j \) harmonic on \( B_j \setminus b_j \) and vanishing q.e. on \( \partial B_j \setminus (B_{j-1} \cup B_{j+1}) \). Then
\[
\frac{u}{\sup_{b_j^*} u} \leq A \frac{v}{\inf_{b_j^*} v} \quad \text{on } \Omega_{j-1},
\]
where \( A > 0 \) depends only on \( A_0 \) and \( \tau \).

**Proof.** In this proof \( A \) and the constant comparison depend only on \( A_0 \) and \( \tau \). We may assume that \( \sup_{b_j^*} u = \inf_{b_j^*} v = 1 \) by normalization. Let \( I_j = B_j \cap \partial B_j^* \). Since \( B_{j-1} \cap B_j \neq \emptyset \), \( B_j \cap B_{j+1} \neq \emptyset \), and \( B_j^* \cap B_j^* = \emptyset \), it follows that \( I_j \neq \emptyset \) and
\[
\text{dist}(I_j, B_{j-1}) \approx \text{dist}(I_j, B_{j+1}) \approx \text{diam}(B_j).
\]
Hence the geometric nature yields the following scale invariant boundary Harnack principle over \( B_j \setminus b_j \):
\[
(5.1) \quad u \leq Av \quad \text{on } I_j.
\]
See Figure 3. (Actually, the scale invariant boundary Harnack principle for the smooth domain \( B_j \setminus b_j \) is an easy consequence of comparison with the distance function \( \delta_{\partial B_j}(x) \). We see that both \( u(x) \) and \( v(x) \) are comparable to \( \delta_{\partial B_j}(x) \) on \( I_j \).) In view of the assumption on \( u \) we have
\[
\{ x \in \partial (\Omega_{j-1} \cup (B_j^* \cap B_j)) : u(x) > 0 \} \subset I_j.
\]
Hence the maximum principle and (5.1) yield \( u \leq Av \) on \( \Omega_{j-1} \cup (B_j \cap B^*_j) \). This proves the lemma.

The next lemma provides the first three steps of the induction.

**Lemma 5.2.** There exists a positive constant \( A \) depending only on the dimension \( n \) and \( A_0 \) such that if \( E \) satisfies (3.1) for \( j = 1, 2 \) and

\[
5.2 \quad B^*_j R^j_1 E \cap B_0(x_0) < \varepsilon_0,
\]

with sufficiently small \( \varepsilon_0 \), then

\[
\omega^0(b_j, \Omega_j \cup B_j) \geq A \quad \text{for } j = 1, 2, 3
\]

**Proof.** Let \( 1 \leq j \leq 3 \). We claim

\[
5.3 \quad \omega(b_j, \Omega_j \cup B_j) + B_0 \cup \cdots \cup B_j \geq \omega(b_j, B_0 \cup \cdots \cup B_j) \quad \text{on } B_0 \cup \cdots \cup B_j.
\]

The inequality trivially holds q.e. on \( b_j \cup (E \cap (B_0 \cup \cdots \cup B_{j-1})) \), as \( \omega(b_j, \Omega \cup B_j) = 1 \) on \( b_j \) and \( B_0 \cup \cdots \cup B_{j-1} \). Observe that

\[
\omega(b_j, B_0 \cup \cdots \cup B_j) = 0 \quad \text{on } \partial(B_0 \cup \cdots \cup B_j);
\]

\[
\omega(b_j, \Omega_j \cup B_j) = 1 \quad \text{on } \partial b_j;
\]

\[
B_0 \cup \cdots \cup B_j \approx B^0 \cup \cdots \cup B_j = 1 \quad \text{q.e. on } E \cap (B_0 \cup \cdots \cup B_{j-1}).
\]

Hence the maximum principle over \( \Omega_j \cup B_j \setminus b_j \) yields (5.3). Substitute \( x_0 \) in (5.3). Then we have

\[
5.4 \quad \omega^0(b_j, \Omega_j \cup B_j) + B_0 \cup \cdots \cup B_j \geq \omega^0(b_j, B_0 \cup \cdots \cup B_j).
\]

Let us estimate the second term of (5.4). Let \( \tilde{B} \) be a ball containing \( B_0^* \cup \cdots \cup B_3^* \) with

\[
\text{diam}(\tilde{B}) \approx \text{dist}(\tilde{B}, B^*_0 \cup \cdots \cup B^*_2) \approx \text{diam}(B_0) \approx \cdots \approx \text{diam}(B_3).
\]

Monotonicity and subadditivity of regularized reduced functions yield

\[
B_0 \cup \cdots \cup B_j \approx B_0 \cup \cdots \cup B_j \approx B_0 \cup \cdots \cup B_j
\]

on \( B_0 \cup \cdots \cup B_j \). In view of Lemma 4.7 and (5.2), we have

\[
B_0 \cup \cdots \cup B_j \approx B_0 \cup \cdots \cup B_j < A \varepsilon_0.
\]

It follows from Lemma 4.4 and (3.1) that

\[
B_0 \cup \cdots \cup B_j \leq B_0 \cup \cdots \cup B_j \leq \frac{\text{Cap}(E \cap B_j)}{\text{Cap}(B_j)} \leq A \varepsilon_0.
\]

for \( j = 1, 2 \), where \( K \) is a closed ball in \( \mathbb{R}^n \setminus \tilde{B} \) such that \( \text{diam}(K) \approx \text{dist}(K, \tilde{B}) \approx \text{diam}(B_0) \).

Hence, letting \( 3A \varepsilon_0 < 2^{-1} \min_{j=1,2,3} \omega^0(b_j, B_0 \cup \cdots \cup B_j) \), we obtain from (5.4) that

\[
\omega^0(b_j, \Omega_j \cup B_j) \geq 2^{-1} \min_{j=1,2,3} \omega^0(b_j, B_0 \cup \cdots \cup B_j) = A,
\]

as required.

Lemma 3.2 readily follows from the following lemma. This is the main part of induction.

**Lemma 5.3.** There exist positive constants \( \varepsilon_0 < 1 \) and \( A_2 > 1 \) depending only on the dimension \( n \) and \( A_0 \) with the following property: if \( E \) satisfies (5.2) and (3.1) for \( j = 1, \ldots, J-1 \), then

\[
\omega^0(b_j, \Omega_j \cup B_j) \geq \exp(-A_2 j) \quad \text{for } j = 1, \ldots, J.
\]
Proof. Let us prove the lemma for \( n \geq 3 \). See Remark 5.4 for \( n = 2 \). For simplicity we write \( \omega_j = \omega_j(\Omega_j \cup B_j) \). It is sufficient to show that there exists \( 0 < \beta < 1 \) such that

\[
\omega_j \geq \beta \omega_{j-1}
\]

for \( j = 2, \ldots, J \). In view of Lemma 5.2, we have (5.5) for \( j = 2, 3 \) with some \( \beta = \beta_0 > 0 \).

Let \( j \geq 4 \). The maximum principle yields

\[
\omega(b_j, \Omega_j \cup B_j) + \omega(E \cap B_{j-1}, \Omega_j \cup B_{j-1} \cup B_j) \geq \omega(b_j, \Omega_j \cup B_{j-1} \cup B_j)
\]

on \( \Omega_j \cup B_{j-1} \cup B_j \). The Harnack inequality gives

\[
\omega(b_j, \Omega_j \cup B_{j-1} \cup B_j) \geq \omega(b_j, B_{j-1} \cup B_j) \geq A_3 \quad \text{on } B_{j-1}.
\]

Hence (5.6) yields

\[
\omega(b_j, \Omega_j \cup B_j) + \omega(E \cap B_{j-1}, \Omega_j \cup B_{j-3} \cup \cdots \cup B_j) \geq A_3 \omega(b_{j-1}, \Omega_{j-1} \cup B_{j-1}) \quad \text{on } \Omega_{j-1} \cup B_{j-1}
\]

with the aid of the obvious inclusion \( \Omega_j \cup B_{j-1} \cup B_j \subset \Omega_j \cup B_{j-3} \cup \cdots \cup B_j \).

Let us estimate the second term of the left hand side of (5.7). By (3.1) and Corollary 4.3 with \( B = B_{j-1} \) and \( B' = B'_{j-1} \), we have

\[
\omega(E \cap B_{j-1}, \Omega_j \cup B_{j-3} \cup \cdots \cup B_j) \leq \omega(E \cap B_{j-1}, \mathbb{R}^n) \leq A \frac{\text{Cap}_{B'_{j-1}}(E \cap B_{j-1})}{\text{Cap}_{B_{j-1}}(B_{j-1})} \leq A \varepsilon_0
\]

on \( \mathbb{R}^n \setminus B'_{j-1} \), so, in particular, on \( B_{j-3} \) by Proposition 2.1 (iii). On the other hand, the geometric nature shows

\[
\omega(b_{j-3}, \Omega_{j-3} \cup B_{j-3}) \geq \omega(b_{j-3}, B_{j-3}) \geq A \quad \text{on } B'_{j-3}.
\]

Apply Lemma 5.1 with \( j - 3 \) in place of \( j \). Let \( u \) be the restriction of \( \omega(E \cap B_{j-1}, \Omega_j \cup B_{j-3} \cup \cdots \cup B_j) \) on \( \Omega_{j-3} \cup B_{j-3} \) and let \( v = \omega(b_{j-3}, \Omega_{j-3} \cup B_{j-3}) \). Observe that \( u \) is a bounded positive harmonic function on \( \Omega_{j-3} \cup B_{j-3} \), vanishing q.e. on \( \partial(\Omega_{j-3} \cup B_{j-3}) \setminus B_{j-2} \), and that \( v \) is a bounded positive superharmonic function on \( \Omega_{j-3} \cup B_{j-3} \), harmonic in \( B_{j-3} \setminus B_{j-3} \) and vanishing q.e. on \( \partial B_{j-3} \setminus (B_{j-4} \cup B_{j-2}) \). By (5.8) and (5.9) we have \( \sup_{B'_{j-3}} u \leq A \varepsilon_0 \) and \( \inf_{B'_{j-3}} v \geq A \). Hence

\[
u \leq A_4 \varepsilon_0 \quad \text{on } \Omega_{j-4}
\]

where \( A_4 \) is independent of \( j \). Plug in the inequality into (5.7). We obtain

\[
\omega(b_j, \Omega_j \cup B_j) + A_4 \varepsilon_0 \omega(b_{j-3}, \Omega_{j-3} \cup B_{j-3}) \geq A_3 \omega(b_{j-1}, \Omega_{j-1} \cup B_{j-1}) \quad \text{on } \Omega_{j-4}.
\]

Since \( j \geq 4 \), we evaluate the inequality at \( x_0 \) to obtain

\[
\omega_j + A_4 \varepsilon_0 \omega_{j-3} \geq A_3 \omega_{j-1}.
\]

As was observed at the beginning of the proof, we have \( \omega_j \geq \beta_0 \omega_{j-1} \) for \( j = 2, 3 \) with some \( 0 < \beta_0 < 1 \). Let \( \beta = \min\{\beta_0, A_3/2\} \) and make \( \varepsilon_0 > 0 \) so small that

\[
A_4 \varepsilon_0 \beta^{-2} \leq A_3/2.
\]

Then an induction argument shows (5.5) for \( j \geq 4 \). In fact, let \( m \geq 3 \) and suppose (5.5) holds for \( 2 \leq j \leq m \). Then (5.10) with \( j = m + 1 \) yields

\[
\omega_{m+1} + A_4 \varepsilon_0 \omega_{m-2} \geq A_3 \omega_m.
\]

By induction hypothesis we have \( \omega_{m-2} \leq \beta^{-1} \omega_{m-1} \leq \beta^{-2} \omega_m \). Hence

\[
\omega_{m+1} + A_4 \varepsilon_0 \beta^{-2} \omega_m \geq A_3 \omega_m,
\]
so that (5.11) yields
\[ \omega_{m+1} \geq \frac{A_3}{2} \omega_m \geq \beta \omega_m. \]
Thus (5.5) holds for \(2 \leq j \leq m + 1\), and hence for \(2 \leq j \leq J\) by induction. The lemma is proved.

Remark 5.4. In case \(n = 2\), we modify the proof as follows. By the third condition of (2.1) \(B_j \setminus B_{j-1}\) includes a ball of radius comparable to \(r_j\). In view of the geometry, we find a closed ball \(K_j \subset B_j \setminus (B_j \cup B_{j-1})\) whose radius is comparable to \(r_j\). Making the radius of \(K_j\) smaller, we find an expansion \(B_{j-1}'\) of \(B_{j-1}\) such that \(B_j - 1 \subset B_{j-1}' \subset B_{j-1}\) and
\[ \text{diam}(B_{j-1}) \approx \text{diam}(B_{j-1}') \approx \text{diam}(K_j) \approx \text{dist}(K_j, B_{j-1}). \]
Apply Corollary 4.5 with \(B = B_{j-1}, B^* = B_{j-1}'\) and \(K = K_j\) instead of Corollary 4.3. We have
\[ \omega(E \cap B_{j-1}, \mathbb{R}^2 \setminus K_j) \leq A \frac{\text{Cap}_{B_{j-1}}(E \cap B_{j-1})}{\text{Cap}_{B_{j-1}}(B_{j-1})} \quad \text{on } \mathbb{R}^2 \setminus B_{j-1}'. \]
It follows from Lemma 4.7 that
\[ \frac{\text{Cap}_{B_{j-1}}(E \cap B_{j-1})}{\text{Cap}_{B_{j-1}}(B_{j-1})} \approx \frac{\text{Cap}_{B_{j-1}}(E \cap B_{j-1})}{\text{Cap}_{B_{j-1}}(B_{j-1})}. \]
In view of Proposition 2.1 (iii), we have \(B_0 \cup \cdots \cup B_j \subset \mathbb{R}^2 \setminus K_j\) and \(B_{j-3} \subset \mathbb{R}^2 \setminus B_{j-1}'\). Hence (5.12) becomes
\[ \omega(E \cap B_{j-1}, \Omega_j \cup B_{j-3} \cup \cdots \cup B_j) \leq \omega(E \cap B_{j-1}, \mathbb{R}^2 \setminus K_j) \leq A \frac{\text{Cap}_{B_{j-1}}(E \cap B_{j-1})}{\text{Cap}_{B_{j-1}}(B_{j-1})} \leq A \varepsilon_0 \quad \text{on } B_{j-3}. \]
Replace (5.8) by this inequality. The rest of the argument is the same.

Lemma 5.3 yields a stronger version of Theorem 3.1, which will be needed for the proof of Theorem 1.3.

Theorem 5.5. Let \(\{B_j\}_{j=0}^J\) be a Harnack chain satisfying (ii) and (iii) of Proposition 2.1. Then there exist positive constants \(\varepsilon_0 < 1\) and \(A_1 > 1\) depending only on \(n\), \(A_0\), and \(\tau\) with the following property: Let a closed set \(E\) satisfy \(E \cap B_j = \emptyset\) \((3.1)\) for \(j = 1, \ldots, J - 1\) and (5.2). If \(h\) is a positive harmonic function in \(B_0 \cup \cdots \cup B_j \setminus E\), then (3.2) holds.

6. Lower estimate of the Green function for a \(\psi\)-Hölder domain

The validity of the boundary Harnack principle on a domain \(D\) heavily depends on a lower estimate of the Green function for \(D\). In this section, we establish a very precise estimate of the Green function for a \(\psi\)-Hölder domain (Lemma 6.3). The proof is based on Theorem 5.5.

Let \(\psi(t)\) be a nondecreasing continuous function for \(t \geq 0\) with \(\psi(0) = 0\) satisfying (1.6). Without loss of generality we may assume that \(\psi(t) \geq t\) for \(t > 0\). Let us study the capacitary width of a subset of a \(\psi\)-Hölder domain \(D\). In view of the local nature, for a moment, we assume that \(D\) is above the graph of a \(\psi\)-Hölder continuous function \(\varphi\) in \(\mathbb{R}^{n-1}\), i.e., \(D = \{(x', x^n) : x^n > \varphi(x')\}\) and \(x' = (x^1, \ldots, x^{n-1})\). For a point \(x \in D\) we define \(d(x) = x^n - \varphi(x')\), where \(x = (x', x^n)\).

Lemma 6.1. Let \(\psi\) and \(D\) be as above. If \(x \in D\) and \(d(x) = 4\psi(r)\) for some \(0 < r \leq 1\), then \(\delta_D(x) \geq r\).
Proof. Let

$$
\Psi(t) = \psi(t) + \frac{1}{4\psi(1)} \int_0^t \psi(\tau) d\tau.
$$

Then $\Psi$ is a strictly increasing continuous function such that

$$
\psi(t) \leq \Psi(t) \leq \psi(t) + \frac{t}{4\psi(1)} \psi(t) \leq 2\psi(t) \quad \text{for } 0 < t \leq 4\psi(1).
$$

Hence $2\Psi(r) \leq 4\psi(r) = d(x) \leq 4\psi(1)$, so that $r \leq \Psi^{-1}(d(x)/2)$. Observe that if $|z' - x'| < r$, then

$$
|\psi(z') - \psi(x')| \leq \psi(|z' - x'|) \leq \psi(r) \leq \Psi\left(\Psi^{-1}\left(\frac{d(x)}{2}\right)\right) = \frac{d(x)}{2}.
$$

This implies that $B'(x', r) \times (x_n - \frac{1}{2}d(x), x_n + \frac{1}{2}d(x))$ lies in $D$, where $B'(x', r)$ stands for the $(n - 1)$-dimensional open ball with center at $x'$ and radius $r$. Hence

$$
\delta_D(x) \geq \min\left\{r, \frac{d(x)}{2}\right\} = \min\{r, 2\psi(r)\} = r.
$$

The lemma follows. \hfill \Box

Let us estimate the quasihyperbolic metric for a $\psi$-Hölder domain under an additional assumption on $\psi$.

**Lemma 6.2.** Let $\psi$ and $D$ be as in Lemma 6.1. Suppose that there exist constants $M$ and $M'$ such that $1 < M' < M$ and

$$
\psi(Mt) \leq M' \psi(t) \quad \text{for } 0 < t \leq 1.
$$

(6.1)

Let $y \in D$ with $d(y) = 4\psi(r)$ for some $0 < r \leq 1/M$. For a nonnegative integer $i$ we let $y_i$ be the point in $D$ such that $y_i' = y'$ and $d(y_i) = 4\psi(M'r)$. See Figure 4. Let $I$ be the integer such that $M'Ir \leq 1 < M'^{i+1}r$. Then

$$
k_D(y, y_i) \leq A \frac{\psi(r)}{r},
$$

where $A$ is independent of $r$.

**Proof.** Let $0 \leq i \leq I - 1$. Let $y_iy_{i+1}$ be the line segment connecting $y_i$ and $y_{i+1}$. By Lemma 6.1 we have $\delta_D(z) \geq M'Ir$ for $z \in y_iy_{i+1}$. Since the length of $y_iy_{i+1}$ is less than $4\psi(M'^{i+1}r)$, and since $M'^{i+1}r \leq M'Ir \leq 1$, it follows from (6.1) that

$$
k_D(y_i, y_{i+1}) \leq \frac{4\psi(M'^{i+1}r)}{M'I} \leq 4M'(\frac{M'}{M})^i \cdot \frac{\psi(r)}{r}.
$$

\[\text{Figure 4. Quasihyperbolic metric in a } \psi\text{-Hölder domain.}\]
Hence
\[ k_D(y, y_1) \leq \sum_{i=0}^{l-1} k_D(y_i, y_{i+1}) \leq 4M' \sum_{i=0}^{l-1} \left( \frac{M'}{M} \right)^i \frac{\psi(r)}{r} \leq \frac{4M'}{1 - M'/M} \cdot \frac{\psi(r)}{r}. \]

The lemma follows. \( \square \)

Now let us give a lower estimate of the Green function for a \( \psi \)-Hölder domain \( D \). Let \( g = G_D(\cdot, O) \) with fixed reference point \( O \in D \). Let \( O' \in D \) be a point near \( O \). Then \( g(O') = G_D(O', O) \approx 1 \). Let us estimate \( g \) near \( \partial D \). First consider a point \( y \in D \) represented as \( y = (y', y_n) \) and \( d(y) = y^n - \varphi(y') = 4\psi(r) \) with sufficiently small \( r > 0 \). Let \( y_I \) be as in Lemma 6.2. We may assume that \( k_D(y_I, O') \approx 1 \). Hence Lemma 6.2 yields \( k_D(y, O') \leq A \psi(r)/r \), so that
\[ g(y) \geq \exp \left( -A \frac{\psi(r)}{r} \right) \]
by (1.2). See Figure 5 (left).

As was observed in Lemma 6.1 we have \( \delta_D(y) \geq r \). So, the above inequality is considered to be a lower estimate of \( g(y) \) in terms of \( \delta_D(y) \). If \( x \in D \) is a point near \( \partial D \), then this estimate is not sufficient. The following crucial lemma asserts that \( g(x) \) has an appropriate lower bound, provided \( \mathbb{R}^n \setminus D \) is small near \( x \), even if \( \delta_D(x) \) is close to 0.

**Lemma 6.3.** Let \( D \) be a \( \psi \)-Hölder domain with \( \psi \) satisfying (1.7) for some \( M > 1 \). Then there exist positive constants \( A_5 > 1, \eta < 1 \) and \( T < 1 \) depending only on \( D \) with the following property: Let \( 0 < t \leq T \) and
\[ r = \frac{A_5}{\log(1/t)} \psi \left( \frac{1}{\log(1/t)} \right). \]
If \( x \in D \) and \( \cap_{B(x,2r)} B(x,r) \setminus D(x) < \eta \), then \( g(x) \geq t. \)

**Figure 5.** \( k_D(y, O') \) (left). \( \cap_{B(x_j,2r)}(B(x_j, r) \setminus D) \) decreases as \( j \) increases (right).

**Proof.** In view of the local nature, we may assume that (6.1) holds for some \( 1 < M' < M \), and that \( x \) is above the graph of a \( \psi \)-Hölder function \( \varphi \). Note that \( d(x) = x^n - \varphi(x') > 0 \) may be very small, so that Lemma 6.2 may not be applicable to \( r \) satisfying (6.2). Let us place a sequence
of points $x_j$ “over $x$” so that $d(x_j)$ increases with $j$. Let $1 < \tau < 2$. For a nonnegative integer $j$ we let $x_j$ be the point in $D$ such that $x'_j = x'$ and $d(x_j) = d(x) + \tau r j$, where $r > 0$ is defined by (6.2). Observe that

$$|x_i - x_j| = |i - j|\tau r.$$  

Hence, if $|i - j| \geq 2$, then $|x_i - x_j| \geq 2\tau r$, i.e., $B(x_i, \tau r) \cap B(x_j, \tau r) = \emptyset$. We also see that $B(x_{j-1}, r) \cap B(x_j, r)$ contains an open ball with radius $(1 - \tau / 2)r$; and that $B(x_j, (\tau - 1)r) \subset B(x_j, r) \setminus B(x_{j-1}, r)$. Thus $\{B(x_j, r)\}$ is a Harnack chain satisfying the conditions in Proposition 2.1.

Let $J$ be the integer such that $\tau r(J - 1) < 4\psi(r) \leq \tau r J$. Then $J \leq A\psi(r)/r$. Lemma 6.1 implies that $\delta_D(x_j) \geq r$, in other words, $B(x_j, r) \setminus D = \emptyset$. Moreover

$$(6.3) \quad g(x_j) \geq \exp\left(- A\frac{\psi(r)}{r}\right),$$

as was observed before the lemma. On the other hand, $B(x_j, r) \setminus D$ may be a nonempty set for $0 \leq j < J$. In view of the geometry and translation invariance of capacity, we observe that $\Cap_{B(x_j, r)}(B(x_j, r) \setminus D)$ decreases as $j$ increases. See Figure 5 (right).

By Lemma 4.7 and Lemma 4.6 we have

$$\eta > B(x_r, 2r) \Bar{B}(x_r, 1) \Cap_{B(x_j, r)}(B(x_j, r) \setminus D) \geq B(x_r, 2r) \Bar{B}(x_r, 1) \Cap_{B(x_j, r)}(B(x_j, r) \setminus D) \geq A \frac{\Cap_{B(x_j, \tau r)}(B(x_j, \tau r) \setminus D)}{\Cap_{B(x_j, \tau r)}(B(x_j, \tau r))}.$$ 

Hence

$$\frac{\Cap_{B(x_j, \tau r)}(B(x_j, r) \setminus D)}{\Cap_{B(x_j, \tau r)}(B(x_j, r))} \leq \frac{\Cap_{B(x_j, \tau r)}(B(x_j, r) \setminus D)}{\Cap_{B(x_j, \tau r)}(B(x_j, r))} < \epsilon_0 \quad \text{for } 0 \leq j < J,$$

provided $\eta > 0$ is sufficiently small. Here, $\epsilon_0$ is the positive constant in Theorem 5.5. Apply Theorem 5.5 to $\{B(x_j, r)\}_{j=0}^J$ and $E = B(x_0, r) \cup \cdots \cup B(x_J, r) \setminus D$. Observe that $B(x_j, r) \subset D$, so that $E \cap B(x_j, r) = \emptyset$. Then

$$g(x) = g(x_0) \geq g(x_j) \exp\left(- A\frac{\psi(r)}{r}\right).$$

This, together with (6.3), yields

$$g(x) \geq \exp\left(- 2A\frac{\psi(r)}{r}\right).$$

Now let $A_5 = 2A$, and let $s = \frac{1}{\log(1/t)}$ for simplicity. Then (6.2) yields

$$2A\frac{\psi(r)}{r} = 2A \cdot \frac{\psi(A_5 s \psi(s))}{A_5 s \psi(s)} = \frac{\psi(A_5 s \psi(s))}{s \psi(s)}.$$ 

Let $T > 0$ be so small that $A_5 \psi\left(\frac{1}{\log(1/T)}\right) \leq 1$. If $0 < t \leq T$, then $A_5 \psi(s) \leq 1$, so that

$$2A\frac{\psi(r)}{r} \leq \frac{\psi(s)}{s \psi(s)} = \frac{1}{s}.$$ 

Hence, (6.4) becomes

$$g(x) \geq \exp\left(- \frac{1}{s}\right) = \exp(- \log(1/t)) = t,$$

as required.
7. Capacitary width

In [Aik98] we defined the notion of capacitary width and used it for the proof of the boundary Harnack principle in [Aik01] and [Aik09]. See [GSC11] for further applications. This section is devoted to a variant of capacitary width which plays an important role for the proof of Theorem 1.3. The crucial estimate is given in Lemma 7.6 with the aid of Lemma 6.3. Let us recall the definition of capacitary width in [Aik98].

Definition 7.1. Let $0 < \eta < 1$. For an open set $U$ we define the capacitary width $w_\eta(U)$ by

$$w_\eta(U) = \inf \left\{ r > 0 : \frac{\text{Cap}_{B(x, 2r)}(B(x, r) \setminus U)}{\text{Cap}_{B(x, 2r)}(B(x, r))} \geq \eta \text{ for all } x \in U \right\}.$$ 

We note that the constant $\eta$ has no significance. In fact, if $0 < \eta_1 < \eta_2 < 1$, then

$$w_{\eta_1}(U) \leq w_{\eta_2}(U) \leq A w_{\eta_1}(U),$$

where $A$ depends only on the dimension $n$, $\eta_1$ and $\eta_2$ ([Aik98, Proposition 2]).

Capacitary width is useful to estimate harmonic measure ([Aik98, Proposition 1] and [Aik01, Lemma 1]).

Lemma 7.2. Let $U$ be an open set, $x \in U$ and $R > 0$. Then

$$\omega^s(U \cap S(x, R), U \cap B(x, R)) \leq \exp \left( 2 - \frac{A_6 R}{w_\eta(U)} \right),$$

where $A_6 > 0$ depends only on the dimension $n$ and $\eta$.

We define a modified capacitary width.

Definition 7.3. Let $0 < \eta < 1$. For an open set $U$ we define

$$w^*_\eta(U) = \inf \left\{ r > 0 : \frac{\text{Cap}_{B(x, 2r)}(B(x, r) \setminus U)}{\text{Cap}_{B(x, 2r)}(B(x, r))} \geq \eta \text{ for all } x \in U \right\}.$$ 

Proposition 7.4. There is a constant $A > 1$ depending only on the dimension $n$ such that

$$2^{-1} w_{\eta/A}(U) \leq w^*_\eta(U) \leq A w_{\eta/A}(U)$$

for all open sets $U$ in $\mathbb{R}^n$.

The second inequality readily follows from Lemma 4.6. For the first inequality, we prepare the following lemma. For a real function on $f$ on a set $V$ we put $E_f(t) = \{ x \in V : f(x) < t \}$ and define

$$\text{q.e.inf}_V f = \sup \{ t : E_f(t) \text{ is polar} \}.$$ 

By definition $\text{inf}_V f \leq \text{q.e.inf}_V f$; if $f \geq t$ q.e. on $V$, then $\text{q.e.inf}_V f \geq t$. The following lemma is essentially the same as [Aik98, Lemma 2].

Lemma 7.5. Let $0 < r < R$. If $E$ is a relatively compact subset of $B(x, R)$, then

$$\text{q.e.inf}_{y \in B(x, r)} \left( \frac{\text{Cap}_{B(x, R)}(E)}{\text{Cap}_{B(x, R)}(B(x, r))} \right) \leq \frac{\text{Cap}_{B(x, R)}(E)}{\text{Cap}_{B(x, R)}(B(x, r))}.$$ 

Proof. For simplicity we let $B = B(x, R)$ and let $G_B$ be the Green function for $B$. Let $\mu_E$ and $\mu_{B(x, r)}$ be the capacitary distributions of $E$ and $B(x, r)$ in $B$, respectively, i.e., $B \mu_E = G_B \mu_E$ and

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Let \( D \) and \( g \) be as in Lemma 7.6. There exist positive constants \( \gamma \) and \( \delta \) such that

\[
\text{Cap}_B(E) = \|\mu_E\| \quad \text{and} \quad \text{Cap}_B(B(x, r)) = \|\mu_{B(x, r)}\|.
\]

Observe that a polar set has null \( \mu_{B(x, r)} \) measure. Since \( \text{supp} \mu_{B(x, r)} \subset \overline{B(x, r)} \), it follows Fubini’s theorem that

\[
\text{Cap}_B(E) = \|\mu_E\| \geq \int G_B \mu_{B(x, r)} d\mu_E = \int G_B \mu_{B(x, r)} d\mu_E = \int \frac{\mathbb{B}_{E}^{B(x, r)}}{\mathbb{B}_{E}^{B(x, r)}} d\mu_{B(x, r)}
\]

\[
\geq \text{q.e.inf}_{y \in \overline{B(x, r)}} \|\mu_{B(x, r)}\| = \text{q.e.inf}_{y \in \overline{B(x, r)}} \left( \frac{\mathbb{B}_{E}^{B(x, r)}}{\mathbb{B}_{E}^{B(x, r)}} \right) \text{Cap}_B(B(x, r)).
\]

Hence the lemma follows. \( \square \)

**Proof of Proposition 7.4 (the first inequality).** Let \( 3r > w_\eta^*(U) \). Then

\[
\mathbb{B}_{(y, 6r)} \overline{\mathbb{B}_{(y, 3r)} \setminus U}(y) \geq \eta \quad \text{for all} \ y \in U.
\]

Let \( x \in U \). Observe that if \( |x - y| \leq 3r \), then

\[
B(y, 6r) \subset B(x, 9r) \quad \text{and} \quad B(y, 3r) \subset B(x, 6r).
\]

Hence

\[
\mathbb{B}_{(x, 12r)} \overline{\mathbb{B}_{(x, 6r)} \setminus U}(y) \geq \mathbb{B}_{(y, 9r)} \overline{\mathbb{B}_{(y, 6r)} \setminus U}(y) \geq \mathbb{B}_{(y, 6r)} \overline{\mathbb{B}_{(y, 3r)} \setminus U}(y) \quad \text{for} \ y \in \overline{B(x, 3r)}.
\]

Taking the infimum with respect to \( y \in \overline{B(x, 3r)} \), we obtain from (7.1) that

\[
\text{q.e.inf}_{y \in \overline{B(x, 3r)}} \left( \frac{\mathbb{B}_{(x, 12r)} \overline{\mathbb{B}_{(x, 6r)} \setminus U}(y)}{\mathbb{B}_{(x, 6r)} \overline{\mathbb{B}_{(x, 3r)} \setminus U}(y)} \right) \geq \eta,
\]

since \( \mathbb{B}_{(x, 12r)} \overline{\mathbb{B}_{(x, 6r)} \setminus U} = 1 \) q.e. on \( B(x, 6r) \setminus U \). Hence Lemma 7.5 yields

\[
\eta \leq \frac{\text{Cap}_{B(x, 12r)}(B(x, 6r) \setminus U)}{\text{Cap}_{B(x, 12r)}(B(x, 3r))} \leq A \frac{\text{Cap}_{B(x, 12r)}(B(x, 6r) \setminus U)}{\text{Cap}_{B(x, 12r)}(B(x, 6r))},
\]

where \( A \) depends only on the dimension. This shows that \( w_\eta(A)(U) \leq 6r \), so that \( 2^{-1}w_\eta(A)(U) \leq w_\eta^*(U) \). \( \square \)

**Lemma 6.3** readily yields an estimate of the modified capacity width in a \( \psi \)-Hölder domain.

**Lemma 7.6.** Let \( D \) be a \( \psi \)-Hölder domain with \( \psi \) satisfying (1.7) for some \( M > 1 \) and let \( g(x) = G_D(x, O) \) be the Green function with pole at \( O \in D \). Let \( A_5 > 1 \), \( \eta < 1 \) and \( T < 1 \) be as in Lemma 6.3. Then

\[
w_\eta^*(\{x \in D : g(x) < t\}) \leq \frac{A_5}{\log(1/t)} \psi \left( \frac{1}{\log(1/t)} \right) \quad \text{for} \ 0 < t \leq T.
\]

**Proof.** Let \( U = \{x \in D : g(x) < t\} \) and let \( r \) be the right hand side of the required inequality. Lemma 6.3 says that if \( x \in D \) satisfies \( \mathbb{B}_{(x, 2r)} \overline{\mathbb{B}_{(x, r)} \setminus U}(x) < \eta \), then \( g(x) \geq t \), in other words \( x \notin U \). This means that \( \mathbb{B}_{(x, 2r)} \overline{\mathbb{B}_{(x, r)} \setminus U}(x) \geq \eta \) for every \( x \in U \). Hence \( w_\eta^*(U) \leq r \) by definition. \( \square \)

In view of Proposition 7.4, modifying \( \eta > 0 \), we obtain the following estimate of capacity width.

**Corollary 7.7.** Let \( D \) and \( g \) be as in Lemma 7.6. There exist positive constants \( A_7, \eta \) and \( T \) such that

\[
w_\eta^*(\{x \in D : g(x) < t\}) \leq \frac{A_7}{\log(1/t)} \psi \left( \frac{1}{\log(1/t)} \right) \quad \text{for} \ 0 < t \leq T.
\]
8. Proof of Theorem 1.3

Combining the estimate of capacitary width (Corollary 7.7) and the box argument, we prove the global boundary Harnack principle by establishing an equivalent property, i.e., the global Carleson estimate in terms of Green function (Theorem C). In [Aik08] and [Aik09] we have observed that boundary Harnack principles are equivalent to Carleson estimates, and that they can be formulated in terms of Green function.

**Definition 8.1.** We say that a domain $D$ enjoys the **global Carleson estimate** if for each pair $(V, K)$ with (1.5) and $O \in D \cap K$, the following property holds: If $u$ is a positive superharmonic function on $D$ such that

(i) $u$ is bounded, positive and harmonic in $D \cap V$,

(ii) $u$ vanishes q.e. on $\partial D \cap V$,

then

$$u(x) \leq Au(O) \quad \text{for } x \in D \cap K,$$

where $A > 1$ depends only on $D, V, K$ and $O$.

Let us reformulate the above definition. Let $\overline{B}$ be a closed ball including $D$. Observe that $F = \overline{B} \setminus V$ is a compact set and $D \setminus V = D \cap F$ and $K \cap F = \emptyset$. Thus, instead of a pair of a bounded open set $V$ and a compact set $K$, we can consider a pair of disjoint compact sets $K$ and $F$ with

(8.1) $\partial D \cap K \neq \emptyset, D \cap K \neq \emptyset, \partial D \cap F \neq \emptyset,$ and $D \cap F \neq \emptyset$.

The Riesz decomposition theorem says that, in $D \cap V$, $u$ can be represented as the Green potential $G_D \mu$ of a measure $\mu$ on $D \cap \partial V$. Therefore, the global Carleson estimate is restated in terms of Green function.

**Definition 8.2.** We say that a domain $D$ enjoys the **global Carleson estimate in terms of Green function** if its Green function $G_D$ satisfies the following:

$$G_D(x, y) \leq AG_D(O, y) \quad \text{for } x \in D \cap K \text{ and } y \in D \cap F,$$

whenever a pair of disjoint compact sets $K$ and $F$ satisfies (8.1) and $O \in D \cap K$. Here $A$ depends only on $D, K, F$ and $O$.

Similarly, the global boundary Harnack principle in terms of Green function is defined as follows.

**Definition 8.3.** We say that a domain $D$ enjoys the **global boundary Harnack principle in terms of Green function** if its Green function $G_D$ satisfies the following:

$$\frac{G_D(x, y)}{G_D(x', y')} \leq A \quad \text{for } x, x' \in D \cap K \text{ and } y, y' \in D \cap F,$$

whenever a pair of disjoint compact sets $K$ and $F$ satisfies (8.1). Here $A$ depends only on $D, K, F$.

**Theorem C** ([Aik09, Theorem 2.3]). Let $D$ be a bounded domain in $\mathbb{R}^n$. Then the following statements are equivalent:

(i) $D$ enjoys the global boundary Harnack principle.

(ii) $D$ enjoys the global boundary Harnack principle in terms of Green function.

(iii) $D$ enjoys the global Carleson estimate.
(iv) $D$ enjoys the global Carleson estimate in terms of Green function.

In view of the above theorem, we complete the proof of Theorem 1.3 by showing that the domain enjoys the global Carleson estimate in terms of Green function. We shall employ the box argument ([BBB91], [BBB91], [Aik01], [Aik09]).

Proof of Theorem 1.3. Let $D$ be a $\psi$-Hölder domain with (1.6) and (1.7). In view of the local nature, we may assume that $\psi(t) \geq t$ for $0 < t \leq 1$ and

$$\sum_{j=0}^{\infty} \psi(2^{-j}) < \infty. \quad (8.2)$$

Take a pair of disjoint compact sets $K$ and $F$ satisfying (8.1). Let $R = \frac{1}{2} \text{dist}(K, F) > 0$. For $r > 0$ we put $U(r) = \{x \in \mathbb{R}^n : \text{dist}(x, F) < r\}$. Let $U = U(R)$ and $\omega_0 = \omega(D \cap \partial U, D \cap U)$. Without loss of generality, we may assume that $O \in D \cap K$. Let $g(y) = G_D(O, y)$. We claim

$$\omega_0 \leq Ag \quad \text{on } D \cap F. \quad (8.3)$$

Before proving the claim, we complete the proof of Theorem 1.3 by showing that $D$ enjoys the global Carleson estimate in terms of Green function. Let $x \in D \cap K$. Observe that $G_D(x, y) \leq A$ for $y \in D \cap U$, where $A$ is independent of $x$ and $y$. By the maximum principle applied to the harmonic function $G_D(x, \cdot)$ in $D \cap U$, we have $G_D(x, y) \leq A\omega_0(y)$ for $y \in D \cap U$. Hence (8.3) yields

$$G_D(x, y) \leq AG_D(O, y) \quad \text{on } D \cap F.$$

This shows that $D$ enjoys the global Carleson estimate in terms of Green function, and hence the global boundary Harnack principle by Theorem C.

Now let us prove (8.3) by the box argument. In view of (8.2), we find a nonnegative integer $j_0$ such that

$$\sum_{j=j_0}^{\infty} \psi(2^{-j}) \leq \frac{A_6 R}{6A_7}, \quad (8.4)$$

where $A_6$ and $A_7$ are the constants in Lemma 7.2 and Corollary 7.7, respectively. Let $R_{j_0} = R$ and

$$R_j = \left\{1 - \frac{1}{2S_0} \sum_{i=j_0}^{j-1} \psi(2^{-i})\right\}R \quad \text{for } j > j_0,$$

where $S_0$ is the summation of (8.4). Observe that $\{R_j\}_{j=j_0}^{\infty}$ decreases to $R/2$, and that

$$R_j - R_{j+1} = \frac{\psi(2^{-j})R}{2S_0}. \quad (8.5)$$

Put

$$U_j = \{y \in D \cap U(R_j) : 0 < g(y) < \exp(-2^j)\},$$

$$T_j = \{y \in D \cap U(R_{j+1}) : \exp(-2^{j+1}) \leq g(y) < \exp(-2^j)\}.$$

We may assume that $g < \exp(-2^{j_0})$ on $D \cap U$. Hence, it follows from the choice of $\{R_j\}$ that $D \cap F \subset \bigcup_{j=j_0}^{\infty} T_j$. See Figure 6.

Let

$$q_j = \begin{cases} \sup_{T_j} \frac{\omega_0}{g} & \text{if } T_j \neq \emptyset, \\ 0 & \text{if } T_j = \emptyset. \end{cases}$$

\[\text{Id: ehi-final.tex,v 1.4 2014/09/10 01:08:16 aik Exp T\TeX\!ed at September 10, 2014 10:08}\]
In order to prove (8.3), it is sufficient to show that \( q_j \) is bounded. By definition \( q_{j_0} \leq \exp(2^{j+1}) \).
Let \( j > j_0 \). Let us invoke the maximum principle over \( U_j \). Observe that
\[
D \cap \partial U_j \subset (D \cap \partial U_j \cap \partial U(R_j)) \cup \{ y \in D \cap \partial U_j : g(y) = \exp(-2^j) \}
\subset (D \cap \partial U_j \cap \partial U(R_j)) \cup (D \cap \partial U_j \cap T_{j-1}).
\]
The maximum principle yields
\[
\omega_0 \leq \omega(D \cap \partial U_j \cap \partial U(R_j), U_j) + q_{j-1}g \quad \text{on } U_j,
\]
so that
\[
(8.6) \quad q_j = \sup_{T_j} \frac{\omega_0}{g} \leq \exp(2^{j+1}) \sup_{T_j} \omega(D \cap \partial U_j \cap \partial U(R_j), U_j) + q_{j-1}.
\]

Let us estimate the harmonic measure on the right hand side. Let \( x \in T_j \). Then \( B(x, R_j - R_{j+1}) \subset U(R_j) \) by definition. Hence the maximum principle yields
\[
\omega(D \cap \partial U_j \cap \partial U(R_j), U_j) \leq \omega(S(x, R_j - R_{j+1}) \cap U_j, B(x, R_j - R_{j+1}) \cap U_j)
\]
on \( B(x, R_j - R_{j+1}) \cap U_j \). Evaluating at \( x \), and then applying Lemma 7.2, we obtain
\[
\omega^*(D \cap \partial U_j \cap \partial U(R_j), U_j) \leq \omega^*(S(x, R_j - R_{j+1}) \cap U_j, B(x, R_j - R_{j+1}) \cap U_j)
\leq \exp\left(2 - \frac{A_6(R_j - R_{j+1})}{w_\eta(U_j)}\right).
\]
Hence (8.6) becomes
\[
q_j \leq \exp(2^{j+1}) \exp\left(2 - \frac{A_6(R_j - R_{j+1})}{w_\eta(U_j)}\right) + q_{j-1} = \exp\left(2j\left(2 - \frac{A_6(R_j - R_{j+1})}{2^{j+1}w_\eta(U_j)}\right) + 2\right) + q_{j-1}.
\]
Observe from (8.5), Corollary 7.7 with \( t = \exp(-2^j) \), and (8.4) that
\[
\frac{A_6(R_j - R_{j+1})}{2^{j+1}w_\eta(U_j)} = \frac{A_6R}{2S_0} \cdot \psi(2^{-j}) \geq \frac{A_6R}{2S_0} \cdot \frac{1}{A_7} \geq 3,
\]
so that
\[
q_j \leq \exp\left(2j\left(2 - \frac{A_6(R_j - R_{j+1})}{2^{j+1}w_\eta(U_j)}\right) + 2\right) + q_{j-1} \leq \exp(-2^j + 2) + q_{j-1}.
\]
The convergence of \( \sum_{j=j_0}^{\infty} \exp(-2^j + 2) \) shows the boundedness of \( \{q_j\} \), which yields (8.3) and hence the theorem. \( \square \)
REFERENCES


DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN
E-mail address: aik@math.sci.hokudai.ac.jp
Constants, Equations and Theorems etc (Labeled Only)

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