

Sets of determination for harmonic functions in an NTA domain

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1. Introduction

Let D be an NTA domain in \mathbb{R}^N ($N \geq 2$) with Green function $G(x, y)$. Without loss of generality we may assume that D contains the origin 0. It is proved that the Martin compactification of D is homeomorphic to the Euclidean closure of D and that every boundary point is minimal ([15]). Thus the ratio

$$K(x, y) = \frac{G(x, y)}{g(y)} \quad \text{with } g(y) = G(y, 0)$$

has a continuous extension on $D \times \overline{D}$. By the same symbol we denote the continuous extension. If $y \in \partial D$, then $K(\cdot, y)$ is a minimal harmonic function on D . Sometimes we write K_y for $K(\cdot, y)$. By definition $K_y(0) = 1$. For every nonnegative superharmonic function u on D which is harmonic near the origin, there is a unique measure μ_u on D such that

$$u = K\mu_u = \int_{D \cup \partial D} K(\cdot, y) d\mu_u(y).$$

This measure μ_u is called the representing measure of u . For every nonnegative harmonic function h on D there is a unique measure μ_h concentrated on ∂D such that $h = K\mu_h$ and $\|\mu_h\| = h(0)$. Here, we say, in general, that a measure μ is concentrated on A if any Borel set outside A has μ measure zero.

Following Beurling [4] and Dahlberg [7], we introduce the notion of determination of a point measure.

Definition A. A set $E \subset D$ is said to determine the point measure at $y \in \partial D$ if for all positive harmonic functions h with representing measure μ_h we have $\mu(\{y\}) = \inf\{h(x)/K(x, y) : x \in E\}$.

Let $B(x, r)$ be the open ball with center at x and radius r . We write $\delta(x)$ for the distance between x and ∂D . For $0 < \rho < 1$ let

$$E_\rho = \bigcup_{x \in E} B(x, \rho\delta(x)).$$

For a smooth domain the following characterization of sets determining a point measure was given by Dahlberg [7] (see also [1, 2], [4], [9], [18] and [20]).

Theorem A. *Let D be a Liapunov-Dini domain. Suppose $E \subset D$ and $y \in \partial D$. Then the following are equivalent:*

- (i) E determines the point measure at y ;
- (ii) there is ρ , $0 < \rho < 1$, such that

$$(1) \quad \int_{E_\rho} |x - y|^{-N} dx = \infty;$$

- (iii) (1) holds for all ρ , $0 < \rho < 1$.

A set $E \subset D$ is said to be minimally thin at $y \in \partial D$ if the regularized reduced function $\widehat{R}_{K_y}^E$ is a Green potential. The minimal thinness is closely related to the notion of determination of a point measure. In fact, the essential part of Theorem A is based on the following theorem.

Theorem B. *Let D be a Liapunov-Dini domain. Suppose $y \in \partial D$. If E is a measurable subset of D such that*

$$(2) \quad \int_E |x - y|^{-N} dx = \infty,$$

then E is not minimally thin at y .

Using Theorems A and B, Gardiner [11] extended some results of [5, 6] and [13] to functions on balls. Essén [10] and Dudeley-Ward [9] also used Theorems A and B and gave generalizations. They dealt with smooth domains.

The aim of this paper is to consider a generalization to NTA domains. Hereafter we let D be an NTA domain. Since a general NTA domain may have wedges, Theorems A and B do not hold for an NTA domain. In [2, Section 4] we characterized minimal thinness and obtained integral characterizations corresponding to (1) and (2). For a Lipschitz domain, see Ancona [3, Theorem 7.4] and Zhang [21]. However, they are rather complicated (they depend on the boundary point y). In this paper, we generalize some results of [10] and [11] without using integral characterizations like (1) and (2).

It is not hard to see that $E \subset D$ is minimally thin at $y \in \partial D$ if and only if there is a finite measure μ on \overline{D} such that $\mu(\{y\}) = 0$ and $K_y \leq K\mu$ on E (see [5], [8] and [19]). From this observation, we introduce sets “minimally thin for harmonic functions” as follows.

Definition 1. A set $E \subset D$ is said to be minimally thin at $y \in \partial D$ for harmonic functions if there is a finite measure μ concentrated on ∂D such that $\mu(\{y\}) = 0$ and $K_y \leq K\mu$ on E .

Remark. In view of Definitions 1 and A, it follows that E is minimally thin at y for harmonic functions if and only if E does not determine the point measure at y . Let us remark that Hayman [12, p.481 and Theorem 7.37] defined sets “rarefied for harmonic functions”, which correspond to rarefied sets given first by Lelong-Ferrand [17].

By definition if E is minimally thin at y for harmonic functions, then E is minimally thin at y . In view of the Harnack principle, more is true. If E is minimally thin at y for harmonic functions, then E_ρ , $0 < \rho < 1$, is minimally thin at y for harmonic functions, and hence minimally thin at y . Let us prove that the converse is true. This is the key theorem for the succeeding argument.

Theorem 1. *Let $y \in \partial D$ and $E \subset D$. Then the following are equivalent:*

- (i) E is minimally thin at y for harmonic functions.
- (ii) E_ρ is minimally thin at y for some ρ , $0 < \rho < 1$.
- (iii) E_ρ is minimally thin at y for all ρ , $0 < \rho < 1$.

The following theorem is well-known as the minimal fine limit theorem (see [5], [8] and [19]).

Theorem C. *Let $h = K\mu_h$ and $H = K\mu_H$ be positive harmonic functions on D . Let u be a Green potential. Then, for μ_h almost every boundary point y , there are sets E and F which are minimally thin at y such that*

$$\lim_{\substack{x \rightarrow y \\ x \in D \setminus E}} \frac{H(x)}{h(x)} = \frac{d\mu_H}{d\mu_h}(y) \quad \text{and} \quad \lim_{\substack{x \rightarrow y \\ x \in D \setminus E}} \frac{u(x)}{h(x)} = 0.$$

Theorem 1 improves the first assertion of Theorem C.

Corollary 1. *Let $h = K\mu_h$ and $H = K\mu_H$ be positive harmonic functions on D . Then, for μ_h almost every boundary point y , there exists a set E minimally thin at y for harmonic functions such that*

$$\lim_{\substack{x \rightarrow y \\ x \in D \setminus E}} \frac{H(x)}{h(x)} = \frac{d\mu_H}{d\mu_h}(y).$$

For two measures μ and ν on ∂D we say $\nu \leq \mu$ if $\nu(A) \leq \mu(A)$ for every Borel subset A of ∂D . It is easy to see that $\nu \leq \mu$ if and only if $d\mu/d\nu \geq 1$ for ν almost every point on ∂D . Since D is not minimally thin at any point $y \in \partial D$, it follows from Theorem C (or Corollary 1) that $\nu \leq \mu$ if and only if $K\nu \leq K\mu$ on D .

Definition 2. Let ν be a finite measure on ∂D . A set $E \subset D$ is said to determine the measure ν if for every finite measure μ concentrated on ∂D , the inequality $K\nu \leq K\mu$ on E implies that $\nu \leq \mu$, or equivalently $K\nu \leq K\mu$ on D .

The following theorem is a generalization of [11, Theorem 2].

Theorem 2. *Let ν be a finite measure on ∂D and $E \subset D$. Then the following are equivalent:*

- (i) E determines the measure ν .
- (ii) E determines the point measure at y for ν almost every boundary point y .

- (iii) E is not minimally thin at y for harmonic functions for ν almost every boundary point y .
- (iv) E_ρ is not minimally thin at y for ν almost every boundary point y and for some (or all) ρ , $0 < \rho < 1$.
- (v) $\inf_{x \in E} \frac{H(x)}{K\nu(x)} = \inf_{x \in D} \frac{H(x)}{K\nu(x)}$ for all positive harmonic functions H .

Let $\omega(x, A)$ be the harmonic measure at $x \in D$ of $A \subset \partial D$. We write simply ω for the harmonic measure at the origin 0. We observe that $K\omega \equiv 1$ on D . In case ν is the harmonic measure ω we can give further equivalent condition to Theorem 2. Let $\alpha > 0$. For each $y \in \partial D$ we associate the nontangential region $\Gamma_\alpha(y) = \{x \in D : |x - y| < (1 + \alpha)\delta(x)\}$. We say that $\{x_j\}$ is a nontangential sequence converging to y if $x_j \rightarrow y$ and $x_j \in \Gamma_\alpha(y)$ for some $\alpha > 0$.

Corollary 2. *Let $E \subset D$. Then each statement with $\nu = \omega$ in Theorem 2 is equivalent to*

- (vi) E includes a nontangential sequence converging to y for ω almost every boundary point $y \in \partial D$.

The above corollary as well as further equivalent conditions are given by Gardiner [11, Corollary 2] for the unit ball. We observe that the harmonic measure ω can be replaced by the surface measure if D is a Lipschitz domain.

Definition 3. Let A be a closed subset of ∂D . A set $E \subset D$ is said to determine the closed set A if E determines the measure ν for all measures ν concentrated on A .

Bonsall and Walsh defined the notion of positive Poisson basic (P.P.B.). In our situation their definition is generalized as follows.

Definition 4. Let A be a closed subset of ∂D . A set $E \subset D$ is said to be a positive Martin basic (abbreviated to P.M.B.) set for A if, for every positive continuous function f on A , there exist sequences $\{\lambda_j\}$ and $\{x_j\}$ with λ_j positive and x_j in E such that $f(y) = \sum_j \lambda_j K(x_j, y)$ for $y \in A$.

Let $\mathcal{H}(D, A)$ be the class of functions $h = h_1 - h_2$, where $h_j = K\mu_{h_j}$ is a positive harmonic function with measure μ_{h_j} concentrated on A . Then [6, Theorem 10] becomes the following form.

Theorem D. *Let A be a closed subset of ∂D and $E \subset D$. Then the following are equivalent:*

- (i) $\sup_{x \in E} h(x) = \sup_{x \in D} h(x)$ for all $h \in \mathcal{H}(D, A)$.
- (ii) E is a P.M.B. set for A .
- (iii) For each $x \in D$ there is a measure λ_x concentrated on E such that $\|\lambda_x\| = 1$ and $K(x, y) = \int_E K(\xi, y) d\lambda_x(\xi)$ for $y \in A$.

It is easy to see that if E determines A , then E is a P.M.B. set for A . The converse is not true in general. In fact, let A be a singleton $\{y\}$ with $y \in \partial D$. Then any nonempty

set E (even a compact subset of D) is a P.M.B. set for A . The following theorem shows that this is rather an exceptional case. For the unit disk the theorem was proved by Essén [10, Theorem 2].

Theorem 3. *Let A be a closed subset of ∂D and assume that A contains at least two points. Suppose $E \subset \bigcup_{y \in A} \Gamma_\alpha(y)$ for some $\alpha > 0$. Then the following are equivalent:*

- (i) $\sup_{x \in E} h(x) = \sup_{x \in D} h(x)$ for all $h \in \mathcal{H}(D, A)$.
- (ii) E is a P.M.B. set for A .
- (iii) E determines A .
- (iv) E determines the point measure at y for every $y \in A$.
- (v) E is not minimally thin at any $y \in A$ for harmonic functions.
- (vi) E_ρ is not minimally thin at any $y \in A$ for some (or all) ρ , $0 < \rho < 1$.

Let us consider the case when $A = \partial D$. Since $\bigcup_{y \in \partial D} \Gamma_\alpha(y)$ includes a neighborhood of ∂D , we readily obtain a generalization of [11, Theorem 1].

Corollary 3. *The following are equivalent:*

- (i) $\sup_{x \in E} h(x) = \sup_{x \in D} h(x)$ for all $h \in \mathcal{H}(D, \partial D)$.
- (ii) E is a P.M.B. set for ∂D .
- (iii) E determines the boundary ∂D .
- (iv) E determines the point measure at y for every $y \in \partial D$.
- (v) E is not minimally thin at any $y \in \partial D$ for harmonic functions.
- (vi) E_ρ is not minimally thin at any $y \in \partial D$ for some (or all) ρ , $0 < \rho < 1$.

Finally we add below a further equivalent condition to Theorem 1.

Theorem 4. *Let $y \in \partial D$ and $E \subset D$. Then E is minimally thin at y for harmonic functions if and only if there is a positive harmonic function H such that*

$$\liminf_{\substack{x \rightarrow y \\ x \in D}} \frac{H(x)}{g(x)} < \liminf_{\substack{x \rightarrow y \\ x \in E}} \frac{H(x)}{g(x)}.$$

If D is the unit ball $B(0, 1)$, then it is easy to see that $g(x)/(1 - |x|)$ has a positive limit as $x \rightarrow y$ for every boundary point y . Hence, for the unit ball, Theorem 3 is the same result given by Gardiner [11, Theorem 3].

So far, we have observed that many results of [5, 6, 10, 11, 13] can be extended to an NTA domain without Theorems A and B. Therefore We raise

Question. *Can one extend these result to a general Martin space?*

Our argument here depends on the estimates of the Martin kernel and the boundary Harnack principle, so it is not applicable to a general Martin space.

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2. Proof of Theorem 1 and Corollary 1

By the symbol M we denote an absolute positive constant whose value is unimportant and may change from line to line. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $M \geq 1$ such that $M^{-1}f_1 \leq f_2 \leq Mf_1$. The constant M will be called the constant of comparison. First we recall the well-known Harnack inequality.

Lemma 1. *For $0 < \rho < 1$ there exists a positive constant $M(\rho)$ with the following properties:*

- (i) $M(\rho) \downarrow 0$ as $\rho \downarrow 0$.
- (ii) *Suppose $x \in D$ and $x' \in B(x, \rho\delta(x))$. Then for any positive harmonic functions h and H on D*

$$(1 - M(\rho)) \frac{H(x')}{h(x')} \leq \inf_{B(x, \rho\delta(x))} \frac{H}{h} \leq \sup_{B(x, \rho\delta(x))} \frac{H}{h} \leq (1 + M(\rho)) \frac{H(x')}{h(x')}.$$

For the proof of Theorem 1 we use the following estimate of the Martin kernel, whose proof will be given in Section 5.

Lemma 2. *Let $0 < \rho < 1 < \gamma$. Suppose $x, y \in D$. If $|x - y| \geq \rho\delta(x)$, then*

$$K(x, y) \leq MK(x, z) \text{ for all } z \in B(y, \gamma\delta(y)) \cap \partial D$$

with M depending only on D , ρ and γ .

The following lemma includes the crucial part of the proof of Theorem 1. For the lemma we are motivated by the argument of Sjögren [20, Proof of Theorem 2].

Lemma 3. *Let h be a positive harmonic function on D . Suppose $E \subset D$, $0 < \rho < 1$ and there is a Green potential u such that $u \geq h$ on E_ρ . Suppose F is a subset of ∂D and there is $\eta > 1$ such that*

$$(3) \quad B(x, \eta\delta(x)) \cap F \neq \emptyset \text{ for } x \in E.$$

Then there exists a finite measure μ concentrated on a countable subset of F such that $K\mu \geq h$ on E_ρ .

Remark. Let F be a dense subset of ∂D . Then (3) holds for any $E \subset D$ and $\eta > 1$.

Proof of Lemma 3. By the covering lemma ([16, Lemma 3.2] and [22, Theorem 1.3.5]) we can find sequences $\{x_j^i\}_j \subset E$, $i = 1, \dots, N$, such that N depends only on the dimension, $\{B(x_j^i, \rho\delta(x_j^i))\}_j$ is mutually disjoint and

$$(4) \quad E \subset \bigcup_{i=1}^N \bigcup_j B(x_j^i, \rho\delta(x_j^i)).$$

Obviously, the right hand side is included in E_ρ and hence the Green potential u majorizes h on $E_\rho^i = \bigcup_j B(x_j^i, \rho\delta(x_j^i))$. Therefore $\widehat{R}_h^{E_\rho^i}$ is a Green potential, which is represented as $K\nu_i$ with finite measure ν_i concentrated on $\partial E_\rho^i \cap D$. Note that

$$(5) \quad K\nu_i \geq h \text{ on } E_\rho^i.$$

Since $\{B(x_j^i, \rho\delta(x_j^i))\}_j$ do not accumulate in D for each $i = 1, \dots, n$, it follows that

$$\partial E_\rho^i \cap D = \bigcup_j \partial B(x_j^i, \rho\delta(x_j^i)).$$

Thus the measure ν_i is concentrated on the union of spheres $\partial B(x_j^i, \rho\delta(x_j^i))$.

By (3) we can take a point $z_j^i \in F \cap B(x_j^i, \eta\delta(x_j^i))$. Define the measure μ_i by the summation of point masses at z_j^i of magnitude $\nu_i(\partial B(x_j^i, \rho\delta(x_j^i)))$. Then $\|\mu_i\| = \|\nu_i\| < \infty$, and hence $K\mu_i$ is a positive harmonic function. Let x be one of $\{x_j^i\}_j$, say x_j^i . If $y \in \partial B(x_j^i, \rho\delta(x_j^i))$, then $|x - y| \geq \rho\delta(x)$ and

$$|y - z_j^i| \leq |y - x_j^i| + |x_j^i - z_j^i| \leq (\rho + \eta)\delta(x_j^i) \leq \frac{\rho + \eta}{1 - \rho}\delta(y).$$

Hence Lemma 2 with $\gamma = (\rho + \eta)/(1 - \rho) > 1$ yields

$$K(x, y) \leq MK(x, z_j^i)$$

with $M > 0$ independent of x, y and z_j^i . By the definition of the integration we have $K\nu_i(x) \leq MK\mu_i(x)$. Therefore (5) and Lemma 1 show that

$$K\mu_i \geq Mh \text{ on } E_\rho^i.$$

Hence, by (4), the finite measure $\mu = M^{-1} \sum_{i=1}^N \mu_i$ satisfies

$$K\mu \geq h \text{ on } E.$$

Moreover μ is concentrated on $\bigcup_{i,j} \{z_j^i\}$, which is a countable subset of F . The lemma is proved.

Proof of Theorem 1. We have only to prove (ii) \implies (i). Suppose E_ρ is minimally thin at y , in other words there is a Green potential which majorizes the harmonic function K_y on E_ρ . Observe that $F = \partial D \setminus \{y\}$ is a dense subset of ∂D . Hence, it follows from Lemma 3 and its remark that there is a finite measure μ concentrated on F such that $K\mu \geq K_y$ on E_ρ . This implies that E_ρ is minimally thin at y for harmonic functions. Thus (i) follows.

The following lemma follows easily from Theorem 1 (cf. [5, Theorem II, 9 and Theorem XV, 9]).

Lemma 4. *Let $y \in \partial D$. Suppose $E_j \subset D$ are minimally thin at y for harmonic functions for $j = 1, \dots$. Then there is a sequence of positive numbers r_j such that $E = \bigcup_j E_j \cap B(y, r_j)$ is minimally thin at y for harmonic functions.*

Proof of Corollary 1. By Theorem C, it is sufficient to show that if $\lim_{\substack{x \rightarrow y \\ x \in D \setminus F}} H(x)/h(x) = \alpha$ with a set F minimally thin at y , then there is a set E minimally thin at y for harmonic functions such that $\lim_{\substack{x \rightarrow y \\ x \in D \setminus E}} H(x)/h(x) = \alpha$.

For $0 < \rho < 1$ we let $F(\rho) = \{x \in F : B(x, \rho\delta(x)) \subset F\}$. We observe that if $x \in D \setminus F(\rho)$, then there is a point $x' \in B(x, \rho\delta(x)) \setminus F$. Hence Lemma 1 yields

$$\alpha(1 - M(\rho)) \leq \liminf_{\substack{x \rightarrow y \\ x \in D \setminus F(\rho)}} \frac{H(x)}{h(x)} \leq \limsup_{\substack{x \rightarrow y \\ x \in D \setminus F(\rho)}} \frac{H(x)}{h(x)} \leq \alpha(1 + M(\rho)).$$

Since $\bigcup_{x \in F(\rho)} B(x, \rho\delta(x)) \subset F$, it follows from Theorem 1 that $F(\rho)$ is minimally thin at y for harmonic functions. From Lemma 4 we can find $r_j \rightarrow 0$ such that

$$E = \bigcup_j F\left(\frac{1}{j}\right) \cap B(y, r_j)$$

is minimally thin at y for harmonic functions. By the construction of E we see that $\lim_{\substack{x \rightarrow y \\ x \in D \setminus E}} H(x)/h(x) = \alpha$. The corollary is proved.

3. Proof of Theorem 2 and Corollary 2

We shall show Theorem 2. The essential part will be (i) \implies (iii). For the proof we shall invoke Lemma 3.

Proof of Theorem 2. The equivalence (ii) \iff (iii) \iff (iv) follows from Theorem 1. Let us prove (iii) \implies (i), (i) \iff (v) and (i) \implies (iii).

(iii) \implies (i): Suppose (iii) holds, i.e. E is not minimally thin at y for harmonic functions for ν almost every $y \in \partial D$. Let μ be a finite measure on ∂D and suppose $K\nu \leq K\mu$ on E . Then Corollary 1 yields

$$\frac{d\mu}{d\nu} \geq 1 \text{ } \nu\text{-a.e. on } \partial D.$$

Thus $\nu \leq \mu$ and (i) follows.

(i) \iff (v): Suppose (i) holds. Take a positive harmonic function $H = K\mu_H$ on D and let $\alpha = \inf_E H/K\nu$. Then $\alpha K\nu \leq K\mu_H$ on E . Since E determines the measure ν , it follows that $\alpha K\nu \leq K\mu_H$ on D , which implies that $\alpha = \inf_D H/K\nu$. Thus (v) follows.

Next suppose (v) holds. Take a finite measure μ on ∂D such that $K\nu \leq K\mu$ on E . Theorem C says that the ratio $K\mu/K\nu$ has minimal fine limit $d\mu/d\nu$ for ν -almost every boundary point. This limit is greater than or equal to 1 since

$$1 \leq \inf_E \frac{K\mu}{K\nu} = \inf_D \frac{K\mu}{K\nu}.$$

Hence $\nu \leq \mu$ and (i) follows.

(i) \implies (iii): Suppose (iii) does not hold, i.e. there is $A \subset \partial D$ with $\nu(A) > 0$ such that E is minimally thin at any $y \in A$ for harmonic functions, or equivalently E_ρ is minimally thin at any $y \in A$.

Suppose first that there is $y \in A$ with $\nu(\{y\}) > 0$. Since E is not minimally thin at y for harmonic functions, we can find a measure μ_y on ∂D such that $\mu_y(\{y\}) = 0$ and $K\mu_y \geq K\nu$ on E . Observe that the measure

$$\mu = \nu|_{\partial D \setminus \{y\}} + \nu(\{y\})\mu_y$$

satisfies $\nu \not\leq \mu$ and $K\mu \geq K\nu$ on E . Thus E does not determine the measure ν .

Suppose next that $\nu(\{y\}) = 0$ for any $y \in A$. Let $\nu' = \nu|_A$. Then $\widehat{R}_{K\nu'}^{E_\rho}$ is a Green potential ([5, Corollary to Theorem XV, 11]); in other words there is a Green potential which majorizes the harmonic function $K\nu'$ on E_ρ . By Lemma 3 we can find a finite measure ν^* concentrated on a countable subset A^* of ∂D such that $K\nu^* \geq K\nu'$ on E . Then the measure $\mu = \nu^* + \nu|_{\partial D \setminus A}$ satisfies $K\mu \geq K\nu$ on E . Observe that

$$\begin{aligned} \nu(A \setminus A^*) &= \nu(A) > 0, \\ \mu(A \setminus A^*) &= \nu^*(A \setminus A^*) = 0. \end{aligned}$$

This implies $\nu \not\leq \mu$. Thus E does not determine the measure ν . Therefore (i) does not hold. Thus (i) \implies (iii) follows.

In view of [2, Theorem 4], we obtain the following lemma. Actually, this lemma is not so difficult. For a Lipschitz domain, Hunt and Wheeden [14] used this fact as the property of *n. t. B* sets.

Lemma 5. *Let $y \in \partial D$. Then a nontangential sequence converging to y is not minimally thin at y for harmonic functions.*

Proof of Corollary 2. By Lemma 5 we readily have (vi) \implies (iii). We prove the corollary by showing (v) \implies (vi). Suppose (vi) does not hold. Then there is $\alpha > 0$ and $r > 0$ such that $\omega(A) > 0$ with $A = \{y \in \partial D : \Gamma_\alpha(y) \cap B(y, r) \cap E = \emptyset\}$. Let

$$H(x) = \omega(x, \partial D \setminus A) = \int_{\partial D \setminus A} K(x, y) d\omega(y).$$

We observe from the minimal fine limit theorem (Theorem C) that for ω a.e. $y \in A$ there is a set F minimally thin at y such that $\lim_{x \in D \setminus F, x \rightarrow y} H(x) = 0$. Since $\omega(A) > 0$, it follows, in particular, that

$$\inf_{x \in D} H(x) = 0.$$

Hence, it is sufficient to show that $\inf_{x \in E} H(x) > 0$.

For $x \in D$ we let $P_\alpha(x) = \{y \in \partial D : x \in \Gamma_\alpha(y)\}$. Let $D_0 = \{x \in D : \delta(x) < r/(1+\alpha)\}$. By definition if $x \in E \cap D_0$, then $P_\alpha(x) \subset \partial D \setminus A$. Hence $H(x) \geq \omega(x, P_\alpha(x))$. On the other hand, in view of the definition of $\Gamma_\alpha(y)$, $P_\alpha(x) = B(x, (1+\alpha)\delta(x)) \cap \partial D$, and hence one of the Carleson estimates reads

$$\omega(x, P_\alpha(x)) \geq M \text{ for } x \in D_0,$$

where M is a positive constant independent of x (cf. [15, Lemma 4.2]). Therefore,

$$\inf_{x \in E} H(x) \geq \min\left\{ \inf_{x \in E \cap D_0} H(x), \inf_{x \in D \setminus D_0} H(x) \right\} \geq \min\{M, \inf_{x \in D \setminus D_0} H(x)\} > 0.$$

Thus the corollary is proved.

Remark. Let $E \subset D$ and $A \subset \partial D$. Then E determines the point measure at y for ω a.e. $y \in A$ if and only if E includes a nontangential sequence converging to y for ω a.e. $y \in A$.

4. Proof of Theorem 3

Let us prove Theorem 3. The essential part will be (i) \implies (v). For the proof we shall invoke Lemma 3 and the assumption that $E \subset \bigcup_{y \in A} \Gamma_\alpha(y)$.

Proof of Theorem 3. We have observed in Theorem D that (i) \iff (ii). By definition (iii) \implies (iv) and by Theorem 2 (iv) \implies (iii). In view of Theorem 1 and the remark after Definition 1 we have (iv) \iff (v) \iff (vi). Let us prove (iii) \implies (i) and (i) \implies (v).

(iii) \implies (i): Let ω be the harmonic measure at the origin as before Corollary 2. We observe that $K\omega \equiv 1$ on D . Take $h = K\mu_1 - K\mu_2 \in \mathcal{H}(D, A)$ with measures μ_1 and μ_2 concentrated on A . Put $\alpha = \sup_E h$. Then

$$K\mu_1 \leq K(\mu_2 + \alpha\omega) \text{ on } E.$$

Since E determines A , it follows from the discussion before Definition 2 that the same inequality holds on D ; in other words

$$h = K\mu_1 - K\mu_2 \leq \alpha K\omega = \alpha \text{ on } D.$$

Hence $\sup_D h \leq \alpha$. Thus (i) follows.

(i) \implies (v): Let us prove the implication by contradiction. Suppose (v) does not hold, i.e. E is minimally thin at some point $y \in A$ for harmonic functions. It is sufficient to show that there is a measure μ concentrated on $A \setminus \{y\}$ such that $K_y \leq K\mu$ on E since $h = K_y - K\mu \in \mathcal{H}(D, A)$ satisfies $\sup_E h \leq 0$ and yet, by Theorem C,

$$\lim_{\substack{x \rightarrow y \\ x \in D \setminus F}} \frac{h(x)}{K_y(x)} = 1 \quad \text{with } F \text{ being minimally thin at } y,$$

which in particular implies that $\sup_D h > 0$.

Suppose first y is an isolated point of A . Since E is minimally thin at y for harmonic functions, it follows from Lemma 5 that E does not contain a nontangential sequence converging to y . From the assumption that $E \subset \bigcup_{y \in A} \Gamma_\alpha(y)$ it follows that $\text{dist}(E, y) > 0$. Therefore there is $r > 0$ such that $B(y, r) \cap (A \setminus \{y\}) = \emptyset$ and $B(y, r) \cap E = \emptyset$. Let $y' \in A \setminus \{y\}$. By the boundary Harnack principle (see Lemma 6 below) we can show

$$K_y(x) \leq MK_{y'}(x) \text{ for } x \in D \cap \partial B(y, \frac{r}{2})$$

with M depending on y, y' and r but not on x . Hence the maximum principle yields the same inequality for $x \in D \setminus B(y, \frac{r}{2})$. Thus $K_y \leq K\mu$ on E with the point mass μ at y' of magnitude M .

Suppose next y is not an isolated point of A . Then we can find a sequence y'_j in A converging to y . We observe that

$$\Gamma_\alpha(y) \subset \bigcup_j \Gamma_{2\alpha}(y'_j).$$

Hence

$$E \subset \bigcup_{y' \in A \setminus \{y\}} \Gamma_{2\alpha}(y'),$$

which implies (3) with $F = A \setminus \{y\}$ and $\eta = 1 + 2\alpha$. Let $0 < \rho < 1$. By Theorem 1 E_ρ is minimally thin at y , and hence we find a Green potential which majorizes K_y on E_ρ . Therefore Lemma 3 yields a measure μ on $A \setminus \{y\}$ such that $K\mu \geq K_y$ on $E_\rho \supset E$. The theorem is proved.

5. Proof of Lemma 2

In this section we prove Lemma 2. Let us recall the definition of an NTA domain. A bounded domain is called NTA when there exist positive constants M and r_0 such that

- (a) Corkscrew condition. For any $z \in \partial D$, $r < r_0$ there exists a point $A_r(z) \in D$ such that $M^{-1}r < |A_r(z) - z| < r$ and $\delta(A_r(z)) > M^{-1}r$.
- (b) The complement of D satisfies the corkscrew condition.
- (c) Harnack chain condition. If $\varepsilon > 0$ and x_1 and x_2 belong to D , $\delta(x_j) > \varepsilon$ and $|x_1 - x_2| < C\varepsilon$, then there exists a Harnack chain from x_1 and x_2 whose length depends on C , but not ε .

Without loss of generality we may assume that $r_0 = 1$ and $B(0, 2) \subset D$. The boundary Harnack principle ([15, Lemma 4.10]) is crucial.

Lemma 6. *Let $z \in \partial D$ and let $0 < r < 1$. Suppose u and v are positive harmonic functions on $B(z, 2r) \cap D$ and that u and v vanish continuously on $B(z, 2r) \cap \partial D$. Then*

$$\frac{u}{u(A_r(z))} \approx \frac{v}{v(A_r(z))} \text{ on } B(z, r) \cap D$$

with constant of comparison independent of z , r , u and v .

The boundary Harnack principle is a powerful tool and produces many results. The following is an easy corollary.

Lemma 7. *Let $z \in \partial D$ and let $0 < r < 1$. Suppose u is a positive harmonic function on $B(z, 2r) \cap D$ and that u vanishes continuously on $B(z, 2r) \cap \partial D$. Then*

$$\sup_{B(z, r) \cap D} u \approx u(A_r(z))$$

with constant of comparison independent of z , r and u .

Applying Lemma 7 to $g = G(\cdot, 0)$, we obtain

Lemma 8. *Let $z \in \partial D$ and let $0 < r < R < 1$. Then $g(A_r(z)) \leq Mg(A_R(z))$.*

Let $\Theta(x, y) = \frac{K(x, y)}{g(x)}$. It is known that $\Theta(x, y)$ has a continuous extension on $\bar{D} \times \bar{D}$.

By the same symbol we denote the continuous extension. The kernel Θ is referred to as the Naïm's Θ kernel for D . By definition Θ is symmetric.

Lemma 9. For $z \in \partial D$ we let $\theta_z(r) = \Theta(A_r(z), z)$. Then

$$\theta_z(R) \leq M\theta_z(r) \text{ for } 0 < r < R < 1,$$

where M depends only on D . Moreover, if $x, y \in \overline{D}$ and $2|y - z| \leq |x - z| < 1$, then

$$\Theta(x, y) \approx \Theta(x, z) \approx \theta_z(|x - z|)$$

with constant of comparison depending only on D .

Proof. Let $0 < r < R < 1$. The maximum principle yields

$$\sup_{D \cap \partial B(z, R)} K(\cdot, z) \leq \sup_{D \cap \partial B(z, r)} K(\cdot, z).$$

By Lemma 7 we have

$$K(A_R(z), z) \leq MK(A_r(z), z)$$

with M independent of z , r and R . This, together with Lemma 8, yields

$$\theta_z(R) = \frac{K(A_R(z), z)}{g(A_R(z))} \leq M \frac{K(A_r(z), z)}{g(A_r(z))} = M\theta_z(r).$$

Suppose $x, y \in \overline{D}$ and $2|y - z| \leq |x - z| < 1$. By the continuity we may assume that $x, y \in D$. Let $r = |x - z|$. Observe that $K(\cdot, y)$ and g are harmonic on $D \cap B(z, 2r) \setminus B(z, \frac{1}{2}r)$ and vanish on $\partial D \cap B(z, 2r) \setminus B(z, \frac{1}{2}r)$. By an elementary geometrical observation and the Lemma 6 we have

$$\frac{K(\cdot, y)}{K(A_r(z), y)} \approx \frac{g}{g(A_r(z))}$$

on $D \cap \partial B(z, r)$. Hence

$$\Theta(x, y) = \frac{K(x, y)}{g(x)} \approx \frac{K(A_r(z), z)}{g(A_r(z))} = \Theta(A_r(z), z) = \theta_z(r).$$

The above comparison holds particularly for $y = z$, whence the lemma follows.

Lemma 10. Let $z \in \partial D$, $x \in D$ and $y \in \overline{D}$.

- (i) If $2|y - z| \leq |x - z| < 1$, then $K(x, y) \approx K(x, z)$ with constant of comparison independent of z , x and y .
- (ii) If $2|x - z| \leq |y - z| < 1$, then $K(x, y) \leq MK(x, z)$ with M independent of z , x and y .

Proof. The first assertion readily follows from Lemma 9 and the definitions of K and Θ . Suppose $2|x - z| \leq |y - z| < 1$. Changing the roles of x and y , we obtain from Lemma 9 that

$$\Theta(x, y) \approx \Theta(y, z) \approx \theta_z(|y - z|) \leq M\theta_z(|x - z|) \approx \Theta(x, z).$$

Hence $K(x, y) \leq MK(x, z)$. The lemma follows.

Proof of Lemma 2. Let $x, y \in D$ and $z \in \partial D$. Suppose

$$(6) \quad \begin{aligned} |x - y| &\geq \rho\delta(x), \\ |y - z| &\leq \gamma\delta(y). \end{aligned}$$

Without loss of generality we may assume that $|x - z| < 1$ and $|y - z| < 1$. Suppose first $2|y - z| \leq |x - z|$. Then Lemma 10 (i) yields $K(x, y) \approx K(x, z)$. In particular we have the required estimate. Suppose next $2|x - z| \leq |y - z|$. Then Lemma 10 (ii) yields $K(x, y) \leq MK(x, z)$. Thus we have again the required estimate. Finally suppose $\frac{1}{2}|y - z| \leq |x - z| \leq 2|y - z|$. In view of (6), we have

$$(7) \quad |x - y| \geq M|y - z|.$$

Let $r = 4M|y - z|$ and $y_0 = A_r(z)$. Then $4|y - z| = M^{-1}r \leq |y_0 - z| \leq r = 4M|y - z|$ by the Corkscrew condition. Hence (6) and (7) yield

$$(8) \quad |y - y_0| \leq (4M + 1)|y - z| \leq M' \min\{|x - y|, \delta(y)\}.$$

Observe that $G(x, \cdot)$ and g are positive and harmonic in $D' = D \setminus \{x, 0\}$. In view of (8) and the Harnack chain condition, we can find a Harnack chain from y to y_0 in D' with length independent of x, y and y_0 . Hence it follows from the Harnack principle that $K(x, y) \approx K(x, y_0)$. Since $2|x - z| \leq 4|y - z| \leq |y_0 - z|$, it follows from Lemma 10 (ii) that

$$K(x, y) \approx K(x, y_0) \leq MK(x, z).$$

The lemma is proved.

6. Proof of Theorem 4

First we note the following characterization of minimal thinness which readily follows from [5, Theorem XV.6, Theorem XV.8 and Theorem XV.9].

Theorem E. *Let $y \in \partial D$ and $E \subset D$. Then the following are equivalent:*

- (i) E is minimally thin at y .
- (ii) There is a nonnegative superharmonic function u on D such that

$$\liminf_{\substack{x \rightarrow y \\ x \in D}} \frac{u(x)}{g(x)} < \liminf_{\substack{x \rightarrow y \\ x \in E}} \frac{u(x)}{g(x)}.$$

- (iii) There is a Green potential p on D such that

$$\liminf_{\substack{x \rightarrow y \\ x \in D}} \frac{p(x)}{g(x)} < \liminf_{\substack{x \rightarrow y \\ x \in E}} \frac{p(x)}{g(x)}.$$

- (iv) There is a Green potential p on D such that

$$\liminf_{\substack{x \rightarrow y \\ x \in D}} \frac{p(x)}{g(x)} < \liminf_{\substack{x \rightarrow y \\ x \in E}} \frac{p(x)}{g(x)} = \infty.$$

Proof of Theorem 4 (Sufficiency). Suppose there is a positive harmonic function H such that

$$\liminf_{\substack{x \rightarrow y \\ x \in D}} \frac{H(x)}{g(x)} < \liminf_{\substack{x \rightarrow y \\ x \in E}} \frac{H(x)}{g(x)}.$$

Then by Lemma 1 there is ρ , $0 < \rho < 1$, such that the above inequality with E_ρ replacing E holds. Hence, Theorem E, (ii) \implies (i), implies that E_ρ is minimally thin at y . By Theorem 1 E is minimally thin at y for harmonic functions.

For the necessity of Theorem 4 we need to consider a version of the above theorem in the context of harmonic functions. To this end we give a refinement of Lemma 3. We extend the notation $E_\rho = \bigcup_{x \in E} B(x, \rho\delta(x))$ for $\rho \geq 1$.

Lemma 11. *Let h , E , ρ , u , F and η be as in Lemma 3. Then the measure μ given in Lemma 3 satisfies $K\mu \leq M_0u$ on $D \setminus E_{3\eta+2\rho}$.*

Proof. Let $x \in D \setminus E_{3\eta+2\rho}$. Let x_j^i and z_j^i be as in the proof of Lemma 3. We observe that

$$|x - z_j^i| \geq |x - x_j^i| - |x_j^i - z_j^i| \geq (3\eta + 2\rho - \eta)\delta(x_j^i) \geq 2(\eta + \rho)\delta(x_j^i) > 2|y - z_j^i|.$$

Hence, Lemma 10 (i) with $z = z_j^i$ yields $K(x, y) \approx K(x, z_j^i)$. Let ν_i , μ_i and μ be the measures as in the proof of Lemma 3. The above inequality implies that $K\mu_i(x) \approx K\nu_i(x)$. Hence

$$K\mu(x) \leq M \sum_{i=1}^N K\nu_i(x) \leq M \sum_{i=1}^N \widehat{R}_h^{E_\rho^i}(x) \leq M_0 u(x).$$

The lemma follows.

Proof of Theorem 4 (Necessity). Suppose E is minimally thin at y for harmonic functions. Let $0 < \rho < 1 < \eta$. We take $\alpha > 6\eta + 4\rho$ and let $\beta = (\alpha - 6\eta - 4\rho)/(1 + 3\eta + 2\rho)$. By Lemma 5 we may assume that $E \subset D \setminus \Gamma_\alpha(y)$. We observe that

$$(9) \quad E_{3\eta+2\rho} \cap \Gamma_\beta(y) = \emptyset.$$

In fact, if $z \in E_{3\eta+2\rho}$, then by an elementary calculation $|z - y| \geq (1 + \alpha - 3\eta - 2\rho)(1 + 3\eta + 2\rho)^{-1}\delta(z) = (1 + \beta)\delta(z)$. This means that $z \notin \Gamma_\beta(y)$. Thus (9) follows. Since E_ρ is minimally thin at y by Theorem 1, it follows from Theorem E, (i) \implies (iv), that there is a Green potential p such that

$$(10) \quad \liminf_{\substack{x \rightarrow y \\ x \in D}} \frac{p(x)}{g(x)} < \liminf_{\substack{x \rightarrow y \\ x \in E_\rho}} \frac{p(x)}{g(x)} = \infty.$$

By Lemma 5 the nontangential region $\Gamma_\beta(y)$ is not minimally thin at y . Hence it follows from Theorem E, (iii) \implies (i), that

$$\liminf_{\substack{x \rightarrow y \\ x \in \Gamma_\beta(y)}} \frac{p(x)}{g(x)} = \liminf_{\substack{x \rightarrow y \\ x \in D}} \frac{p(x)}{g(x)} < \infty.$$

Let

$$c > M_0 \liminf_{\substack{x \rightarrow y \\ x \in \Gamma_\beta(y)}} \frac{p(x)}{g(x)},$$

where M_0 is the constant in Lemma 11. By (10) we can choose $\delta > 0$ such that $p \geq cg$ on $E_\rho \cap B(y, \delta)$. By Lemmas 3 and 11 with $u = p$ and (9) we find a positive harmonic function H such that $H \geq cg$ on $E_\rho \cap B(y, \delta)$ and $H \leq M_0 p$ on $\Gamma_\beta(y)$. Hence

$$\liminf_{\substack{x \rightarrow y \\ x \in D}} \frac{H(x)}{g(x)} \leq M_0 \liminf_{\substack{x \rightarrow y \\ x \in \Gamma_\beta(y)}} \frac{p(x)}{g(x)} < c \leq \liminf_{\substack{x \rightarrow y \\ x \in E_\rho}} \frac{H(x)}{g(x)} \leq \liminf_{\substack{x \rightarrow y \\ x \in E}} \frac{H(x)}{g(x)}.$$

Thus Theorem 4 is proved.

REFERENCES

1. H. Aikawa, *Quasiadditivity of Riesz capacity*, Math. Scand. **69** (1991), 15–30.
2. H. Aikawa, *Quasiadditivity of capacity and minimal thinness*, Ann. Acad. Sci. Fenn. Ser. A. I. Mathematica **18** (1993), 65–75.
3. A. Ancona, *Positive harmonic functions and hyperbolicity*, Potential Theory — Surveys and Problems, Lecture Notes in Math. 1344, Springer Verlag, 1987, pp. 1–23.
4. A. Beurling, *A minimum principle for positive harmonic functions*, Ann. Acad. Sci. Fenn. Ser. AI **372** (1965).
5. M. Brelot, *On topologies and boundaries in potential theory*, Lecture Notes in Math. 174, Springer Verlag, 1971.
6. F. F. Bonsall and D. Walsh, *Vanishing ℓ^1 -sums of the Poisson kernel and sums with positive coefficients*, Proc. Edinburgh Math. Soc. **32** (1989), 431–447.
7. B. E. J. Dahlberg, *A minimum principle for positive harmonic functions*, Proc. London Math. Soc. (3) **33** (1976), 238–250.
8. J. L. Doob, *Classical potential theory and its probabilistic counterpart*, Springer Verlag, 1984.
9. N. F. Dudley Ward, *Atomic decompositions of integrable or continuous functions*, PhD thesis (1991), University of York.
10. M. Essén, *On minimal thinness, reduced functions and Green potentials*, Proc. Edinburgh Math. Soc. **36** (1992), 87–106.
11. S. J. Gardiner, *Sets of determination for harmonic functions*, Trans. American Math. Soc **338** (1993), 233–243.
12. W. K. Hayman, *Subharmonic functions, Vol. 2*, Academic Press, 1989.
13. W. K. Hayman and T. J. Lyons, *Bases for positive continuous functions*, J. London Math. Soc. **42** (1990), 292–308.
14. R. R. Hunt and R. L. Wheeden, *Positive harmonic functions on Lipschitz domains*, Trans. Amer. Math. Soc. **147** (1970), 507–527.
15. D. S. Jerison and C. E. Kenig, *Boundary behavior of harmonic functions in non-tangentially accessible domains*, Adv. in Math. **46** (1982), 80–147.
16. N. S. Landkof, *Foundations of modern potential theory*, Springer Verlag, 1972.
17. J. Lelong-Ferrand, *Étude au voisinage de la frontière des fonctions surharmoniques positive dans un demi-espace*, Ann. Sci. École Norm. Sup. **66** (1949), 125–159.
18. V. G. Maz'ya, *Beurling's theorem on a minimum principle for positive harmonic functions*, (Russian), Zapiski Nauchnykh Seminarov LOMI **30** (1972), 76–90; (English translation), J. Soviet Math. **4** (1975), 367–379.
19. L. Naïm, *Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel*, Ann. Inst. Fourier (Grenoble) **7** (1957), 183–281.
20. P. Sjögren, *Weak L_1 characterization of Poisson integrals, Green potentials and H^p spaces*, Trans. Amer. Math. Soc. **233** (1977), 179–196.
21. Y. Zhang, *Ensembles équivalents à un point frontière dans un domaine lipschitzien*, Séminaire de théorie du potentiel Paris No. 9, Lecture Notes in Math. 1393, Springer Verlag, 1989, pp. 256–265.
22. W. P. Ziemer, *Weakly differentiable functions*, Springer, 1989.

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