Carleson type estimates for $p$-harmonic functions and the Conformal Martin boundary of John domains in metric measure spaces

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Dedicated to Professor Saburou Saitoh on the occasion of his 60th birthday

Abstract

A Carleson type estimate provides control of the bound of positive harmonic functions vanishing on a portion of the boundary. Such an estimate is well-known for harmonic functions in certain Euclidean domains. In this note we prove a Carleson type estimate for $p$-harmonic functions on bounded John domains in a complete metric space equipped with an Ahlfors $Q$-regular measure supporting a $(1, p)$-Poincaré inequality for some $1 < p \leq Q$. This result is new even in the Euclidean setting when $p \neq 2$. We then use the Carleson estimate to study the conformal Martin boundary of bounded John domains in metric measure spaces of $Q$-bounded geometry.

1 Introduction

In the study of the local Fatou theorem for harmonic functions, Carleson [Ca] proved the following crucial estimate for positive harmonic functions, now referred to as the Carleson estimate. Given a bounded Lipschitz domain $D$ in the Euclidean space $\mathbb{R}^n$ there exist constants $K, C > 1$, depending only on $D$, with the following property: If $\xi \in \partial D$, $r > 0$ sufficiently small, and $x_r$ is a point in $D$ with $|x_r - \xi| = r$ and $\text{dist}(x_r, \partial D) \geq r/C$, then

$$u \leq Ku(x_r) \quad \text{on } D \cap B(\xi, r).$$

whenever $u$ is a positive harmonic function in $D \cap B(\xi, Cr)$ vanishing continuously on $\partial D \cap B(\xi, Cr)$. Here $B(\xi, r)$ denotes the open ball with center $\xi$ and radius $r$.

The Carleson estimate has been verified for more general Euclidean domains such as NTA domains, and plays an important role in the study of harmonic analysis on nonsmooth domains. There are at least three different proofs of the Carleson estimate based on: (i) uniform barriers, (ii) the boundary Harnack principle, and (iii) the mean value inequality of subharmonic functions:

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(i) Carleson’s original proof as well as the extension to NTA domains due to Jerison–Kenig [JK] are based on uniform barriers. This method was used also by the second author, together with Holopainen and Tyson, in [HST] to study conformal Martin boundaries of bounded uniform domains in metric measure spaces of bounded geometry. This approach requires the notion of uniform fatness of the boundary introduced by Lewis in [L].

(ii) In [Ai1], the first named author proved the boundary Harnack principle directly and verified the Carleson estimate as a corollary. This method does not rely on uniform barriers and is applicable to uniform domains with a small boundary, and more generally even to an irregular uniform domain. However, this method does not seem to be applicable to nonlinear equations.

(iii) In the study of Denjoy domains, Benedicks [Be] employed Domar’s argument [D] based on the mean value inequality of subharmonic functions. His approach was generalized to Lipschitz Denjoy domains by Chevallier [Chv]. The first author, together with Hirata and Lundh, utilized Domar’s argument in [AHL] to prove a version of the Carleson estimate for John domains in Euclidean spaces.

The first goal of this note is to show that Domar’s argument applies not only to harmonic functions on an Euclidean domain but also to solutions of certain nonlinear equations on metric measure spaces. Throughout this paper, we assume that $(X, d, \mu)$ is a proper metric measure space with at least two points and that $\mu$ a doubling Borel measure. Here we say that $X$ is proper if closed and bounded subsets of $X$ are compact; and that $\mu$ is doubling if there is a constant $C_d \geq 1$ such that

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)),$$

where $B(x, r) = \{ y \in X : d(x, y) < r \}$ is the open ball with center $x$ and radius $r$. Moreover, we fix $1 < p < \infty$, and assume that $X$ supports a $(1, p)$-Poincaré inequality (see Definition 2.1 below). We shall establish a Carleson type estimate (Theorem 5.2 below) for John domains in the setting of such metric measure spaces by adapting the version of Domar’s argument found in [AHL].

Our second goal is the study of conformal Martin boundaries of bounded John domains whose boundaries may not be uniformly fat. Under the additional assumption that the measure $\mu$ is Ahlfors $Q$-regular (for some $Q > 1$), we will use the above Carleson type estimate to extend the results of [HST] and [Sh3] to greater generality. One of our main results is Theorem 6.1, which describes the behavior of the conformal Martin kernels. The growth estimate (Theorem 6.1 (ii)) is new.

In the general setting of metric measure spaces, it is not clear whether there exists even one bounded uniform domain in $X$. However, if $X$ is a geodesic space, then every ball in $X$ is a John domain with the center of the ball acting as a John center. It is a well-known fact that any doubling metric measure space supporting a $(1, p)$-Poincaré inequality is quasi-convex; that is, there is a constant $q \geq 1$ such that for every pair of points $x, y \in X$ there is a rectifiable curve $\gamma_{xy}$ in $X$ connecting $x$ to $y$ with the property that the length $\ell(\gamma_{xy})$ of $\gamma_{xy}$ satisfies

$$\ell(\gamma_{xy}) \leq q \, d(x, y).$$

Thus in our situation there are a plethora of bounded John domains in $X$ even if $X$ is not a geodesic space. It is therefore desirable to study the conformal Martin boundary of bounded John domains
in $X$. The theory developed in [HST] indicates that the conformal Martin boundary is conformally invariant. The results developed in this note are therefore useful in the study of a Fatou type property of conformal mappings between two bounded John domains in metric spaces; see [Sh3].

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2 Definitions and Notations

Unless otherwise stated, $C$ denotes a positive constant whose exact value is unimportant, can change even within the same line, and depends only on fixed parameters such as $X$, $d$, $\mu$ and $p$. If necessary, we will specify its dependence on other parameters.

In the setting of metric measure spaces that may not have a Riemannian structure, the following notion of upper gradients, first formulated by Heinonen and Koskela in [HeK1], replaces the notion of distributional derivatives (in [HeK1] upper gradients are referred to as very weak gradients). A Borel function $g$ on $X$ is an upper gradient of a real-valued function $f$ on $X$ if for all non-constant rectifiable paths $\gamma : [0, l_\gamma] \to X$ parameterized by arc length,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds,$$

where the above inequality is interpreted as saying also that $\int_\gamma g \, ds = \infty$ whenever at least one of $|f(\gamma(0))|$ and $|f(\gamma(l_\gamma))|$ is infinite. See [HeK1] and [KoMc] for more on this notion.

**Definition 2.1.** We say that $X$ supports a $(1, p)$-Poincaré inequality if there are constants $\kappa \geq 1$ and $C_p \geq 1$ such that for all balls $B(x, r) \subset X$, all measurable functions $f$ on $X$, and all $p$-weak upper gradients $g$ of $f$,

$$\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C_p \kappa \left( \int_{B(x,r)} g^p \, d\mu \right)^{1/p},$$

where

$$f_{B(x,r)} := \int_{B(x,r)} f \, d\mu := \frac{1}{\mu(B(x, r))} \int_{B(x,r)} d\mu.$$

Following [Sh1], we consider a version of Sobolev spaces on $X$.

**Definition 2.2.** Let

$$\|u\|_{N^{1,p}} = \left( \int_X |u|^p \, d\mu \right)^{1/p} + \inf_g \left( \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of $u$. The Newtonian space on $X$ is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}} = 0$. 

3
The space $N^{1,p}(X)$ equipped with the norm $\| \cdot \|_{N^{1,p}}$ is a Banach space and a lattice, see [Sh1]. An alternative definition of Sobolev spaces given by Cheeger in [Ch] yields the same space as $N^{1,p}(X)$ whenever $p > 1$; see Theorem 4.10 in [Sh1]. Cheeger’s definition yields the notion of partial derivatives in the following theorem (Theorem 4.38 in [Ch]).

**Theorem 2.3** (Cheeger). Let $X$ be a metric measure space equipped with a positive doubling Borel regular measure $\mu$ admitting a $(1, p)$-Poincaré inequality for some $1 < p < \infty$. Then there exists a countable collection $(U_\alpha, X^\alpha)$ of measurable sets $U_\alpha$ and Lipschitz “coordinate” functions $X^\alpha : X \to \mathbb{R}^{k(\alpha)}$ such that $\mu(X \setminus \bigcup_\alpha U_\alpha) = 0$, and for each $\alpha$ the following conditions hold.

The measure of $U_\alpha$ is positive and $1 \leq k(\alpha) \leq N$, where $N$ is a constant depending only on the doubling constant of $\mu$ and the constant from the Poincaré inequality. If $f : X \to \mathbb{R}$ is Lipschitz, then there exist unique bounded measurable vector-valued functions $d^\alpha f : U_\alpha \to \mathbb{R}^{k(\alpha)}$ such that for $\mu$-a.e. $x_0 \in U_\alpha$,

$$
\lim_{r \to 0^+} \sup_{x \in B(x_0, r)} \frac{|f(x) - f(x_0) - d^\alpha f(x_0) \cdot (X^\alpha(x) - X^\alpha(x_0))|}{r} = 0.
$$

We can assume that the sets $U_\alpha$ are pairwise disjoint, and extend $d^\alpha f$ by zero outside $U_\alpha$. Regarding $d^\alpha f(x)$ as vectors in $\mathbb{R}^N$, we let $df = \sum_\alpha d^\alpha f$. The differential mapping $d : f \mapsto df$ is linear and it is shown in page 460 of [Ch] that there is a constant $C > 0$ such that for all Lipschitz functions $f$ and $\mu$-a.e. $x \in X$,

$$
\frac{1}{C} |df(x)| \leq g_f(x) := \inf_g \limsup_{r \to 0^+} \int_{B(x, r)} g \, d\mu \leq C|df(x)|.
$$

Here $|df(x)|$ is a norm coming from a measurable inner product on the tangent bundle of $X$ created by the above Cheeger derivative structure (see the discussion in [Ch]), and the infimum is taken over all upper gradients $g \in L^p(X)$ of $f$; $g_f$ is in some sense the minimal upper gradient of $f$ (see Corollary 3.7 in [Sh2]). Also, by Proposition 2.2 in [Ch], $df = 0$ $\mu$-a.e. on every set where $f$ is constant.

By [Ch, Theorem 4.47] or [Sh1, Theorem 4.1], the Newtonian space $N^{1,p}(X)$ is equal to the closure in the $N^{1,p}$-norm of the collection of Lipschitz functions on $X$ with finite $N^{1,p}$-norm. By [FKH, Theorem 10], there exists a unique “gradient” $du$ satisfying (2) for every $u \in N^{1,p}(X)$. Moreover, if $\{u_j\}_{j=1}^\infty$ is a sequence in $N^{1,p}(X)$, then $u_j \to u$ in $N^{1,p}(X)$ if and only if as $j \to \infty$, $u_j \to u$ in $L^p(X, \mu)$ and $du_j \to du$ in $L^p(X, \mu; \mathbb{R}^N)$. Hence the differential structure extends to all functions in $N^{1,p}(X)$. Throughout this note we will use this structure, see for example Definition 2.5 below.

**Definition 2.4.** The $p$-capacity of a Borel set $E \subset X$ is the number

$$
\text{Cap}_p(E) := \inf_u \left( \int_X |u|^p \, d\mu + \int_X |du|^p \, d\mu \right),
$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on $E$. A property is said to hold $p$-quasieverywhere in $X$ if the set on which the property does not hold has zero $p$-capacity. The relative $p$-capacity $\text{Cap}_p(K; \Omega)$ of a compact set $K$ with respect to an open set $\Omega \supset K$ is given by

$$
\text{Cap}_p(K; \Omega) = \inf \int_\Omega |du|^p \, d\mu,
$$
where the infimum is taken over all functions \( u \in N^{1,p}(X) \) for which \( u|_{K} \geq 1 \) and \( u|_{X \setminus \Omega} = 0 \). If no such function exists, then we set \( \text{Cap}_{p}(K; \Omega) = \infty \). For more on capacity, see [AO], [HeK2], [KiMa0], [HKM, Chapter 2], and the references therein.

Corollary 3.9 in [Sh1] implies that \( N^{1,p}(\Omega) \) equipped with the \( N^{1,p} \)-norm is a closed subspace of \( N^{1,p}(X) \). By [Sh2, Theorem 4.8], if \( \Omega \) is relatively compact, then the space \( \text{Lip}_0(\Omega) \) of Lipschitz functions with compact support in \( \Omega \) is dense in \( N^{1,p}_0(\Omega) \). In the rest of this paper, \( \Omega \subset X \) will always denote a bounded domain in \( X \) with \( \text{Cap}_{p}(X \setminus \Omega) > 0 \).

**Definition 2.5.** Let \( \Omega \subset X \) be a domain. A function \( u : X \to [-\infty, \infty] \) is said to be \( p \)-**harmonic** in \( \Omega \) if \( u \in N^{1,p}_{\text{loc}}(\Omega) \) and for all relatively compact subsets \( U \) of \( \Omega \) and for every function \( \varphi \in N^{1,p}_0(U) \),

\[
\int_{U} |du|^p \, d\mu \leq \int_{U} |d(u + \varphi)|^p \, d\mu. \tag{3}
\]

We say that \( u \) is a \( p \)-subsolution in \( \Omega \) if (3) holds for every non-positive function \( \varphi \in N^{1,p}_0(U) \). We say that \( u \) is a \( p \)-**quasiminimizer** if there is constant \( C_{qm} \geq 1 \) such that for all relatively compact subsets \( U \) of \( \Omega \) and every function \( \varphi \in N^{1,p}_0(U) \),

\[
\int_{U} |du|^p \, d\mu \leq C_{qm} \int_{U} |d(u + \varphi)|^p \, d\mu. \tag{4}
\]

Furthermore, we say that \( u \) is a \( p \)-**quasipseudominimizer** if (4) holds true whenever \( \varphi \) is a non-positive function in \( N^{1,p}_0(U) \).

**Remark 2.6.** It is easily seen that \( p \)-harmonic functions are \( p \)-quasiminimizers and that \( p \)-subsolution are \( p \)-quasipseudominimizers. See [KiMa2] and [KiSh] for more on quasiminimizers.

**Definition 2.7.** By \( H_{p}^{U}f \) we denote the solution to the \( p \)-Dirichlet problem on the open set \( U \) with boundary data \( f \in N^{1,p}(U) \), i.e., \( H_{p}^{U}f \) is \( p \)-harmonic in \( U \) and \( H_{p}^{U}f - f \in N^{1,p}_0(U) \). An upper semicontinuous function \( u \) is said to be \( p \)-subharmonic in \( \Omega \) if the comparison principle holds, i.e., if \( f \in N^{1,p}(U) \) is continuous up to \( \partial U \) and \( u \leq f \) on \( \partial U \), then \( u \leq H_{p}^{U}f \) on \( U \) for all relatively compact subsets \( U \) of \( \Omega \).

**Definition 2.8.** Let \( \Omega \) be a relatively compact domain in \( X \) and let \( y \in \Omega \). An extended real-valued function \( g = g(\cdot, y) \) on \( \Omega \) is said to be a \( p \)-**singular function with singularity at \( y \)** if it satisfies the following four criteria:

(i) \( g \) is \( p \)-harmonic in \( \Omega \setminus \{y\} \) and \( g > 0 \) on \( \Omega \);

(ii) \( g|_{\chi \Omega} = 0 \) \( p \)-q.e. and \( g \in N^{1,p}(X \setminus B(y, r)) \) for each \( r > 0 \);

(iii) \( y \) is a singularity, i.e., \( \lim_{x \to y} g(x) = \infty \);
whenever \( 0 \leq a < b < \infty \),
\[
\text{Cap}_p(\Omega^b; \Omega_a) = (b - a)^{1/p},
\]
where \( \Omega^b = \{ x \in \Omega : g(x) \geq b \} \) and \( \Omega_a = \{ x \in \Omega : g(x) > a \} \).

In [HoSh] it was shown that every relatively compact domain in a metric measure space equipped with a doubling measure supporting a \((1, q)\)-Poincaré inequality with \( q < p \) has a \( p \)-singular function which plays a role analogous to the Green function of the Euclidean \( p \)-Laplace operator. It was shown by Keith–Zhong in [KZ] that a complete metric space supporting a \((1, p)\)-Poincaré inequality for some \( p > 1 \) also supports a \((1, q)\)-Poincaré inequality for some \( 1 \leq q < p \). Hence to apply the results of [KiSh] and [HoSh] we do not need to apriori assume the better Poincaré inequality.

Note that we have the doubling property on the measure \( \mu \) as a standing assumption. As a consequence of this doubling property, it can be shown that there are constants \( Q > 0 \) and \( C_1 \) such that for all \( x \in X \), \( 0 < \rho < R \), and \( y \in B(x, R) \),
\[
\frac{1}{C_1} \left( \frac{\rho}{R} \right)^Q \leq \frac{\mu(B(y, \rho))}{\mu(B(x, R))}. \tag{6}
\]
The book [He] has a proof of this fact. The measure \( \mu \) is said to be Ahlfors \( Q \)-regular if there is a constant \( C \geq 1 \) such that for every \( x \in X \) and for every \( r > 0 \),
\[
\frac{r^Q}{C} \leq \mu(B(x, r)) \leq C r^Q. \tag{7}
\]

For the rest of this section we will assume that \( X = (X, d, \mu) \) is of \( Q \)-bounded geometry, i.e., \( \mu \) is Ahlfors \( Q \)-regular and \( X \) supports a \((1, Q)\)-Poincaré inequality ([BHK, Section 9] or [HST]). It was shown in [HeK1] that metric spaces of \( Q \)-bounded geometry possess a Loewner type property related to the \( Q \)-modulus of curve families connecting compacta. Therefore we can use the techniques of [Ho] to show that for each \( y \in \Omega \) there is exactly one \( Q \)-singular function for \( \Omega \) with singularity at \( y \) satisfying equation (5). This enables us to define a boundary in a manner similar to the classical potential theoretic Martin boundary.

**Definition 2.9.** Fix \( x_0 \in \Omega \). Given a sequence \( (x_n) \) of points in \( \Omega \), we say that the sequence is fundamental (relative to \( x_0 \)) if it has no accumulation point in \( \Omega \) and the sequence of normalized singular functions
\[
M_{x_n}(x) = M(x, x_n) := \frac{g(x, x_n)}{g(x_0, x_n)}
\]
is locally uniformly convergent in \( \Omega \). Here \( g \) is the \( Q \)-singular function for \( \Omega \). We set \( M(x, x_0) = 0 \) when \( x \neq x_0 \), and set \( M(x_0, x_0) = 1 \).

Given a fundamental sequence \( \xi = (x_n) \), we denote the corresponding limit function
\[
M(x) = M_\xi(x) := \lim_{n \to \infty} M(x, x_n),
\]
and call it a conformal Martin kernel function. We say that two fundamental sequences \( \xi \) and \( \zeta \) are equivalent (relative to \( x_0 \)), \( \xi \sim \zeta \), if \( M_\xi = M_\zeta \). It is worth noting that \( M_\xi \) is a non-negative
measure. It therefore follows from a particular choice of \( x \) for every \( k \) class \( \text{DG} \) to absorb the terms involving \( B \) ity. In this section we assume that \( \Omega \) \( X \) Recall that \( 3 \) Domar’s argument \( \text{KiSh} \) Martin boundary, whereas the conformal Martin boundary can be constructed immediately as in \( \text{KiSh} \) in \((1)\). The collection of all equivalence classes of fundamental sequences in \( \Omega \) (or equivalently, the collection of all conformal Martin kernel functions) is the conformal Martin boundary \( \partial_c \text{M} \Omega \) of the domain \( \Omega \). This collection is endowed with the local uniform topology: a sequence \( \xi_n \) in this boundary is said to converge to a point \( \xi \) if the sequence of functions \( M_{\xi_n} \) with singularity at \( y_n \) converges locally uniformly to \( M \).

**Definition 2.10.** The collection of all equivalence classes of fundamental sequences in \( \Omega \) (or equivalently, the collection of all conformal Martin kernel functions) is the conformal Martin boundary \( \partial_c \text{M} \Omega \) of the domain \( \Omega \). This collection is endowed with the local uniform topology: a sequence \( \xi_n \) in this boundary is said to converge to a point \( \xi \) if the sequence of functions \( M_{\xi_n} \) converges locally uniformly to \( M_\xi \).

The classical Martin boundary theory can be extended to general domains in metric measure spaces under certain circumstances. However, there are examples of metric measure spaces with Ahlfors \( Q \)-regular measure, \( Q > 2 \), supporting a \((1, Q)\)-Poincaré inequality but not \((1, 2)\)-Poincaré inequality. For domains in such a metric space, \( 2 \)-singular functions of Definition 2.8 may not exist and it is not easy to say what kind of ideal boundary should correspond to the classical Martin boundary, whereas the conformal Martin boundary can be constructed immediately as in \cite{HST}.

## 3 Domar’s argument

Recall that \( X \) is a proper metric space and that \( \mu \) is doubling and supports a \((1, p)\)-Poincaré inequality. In this section we assume that \( \Omega \subset X \) is a bounded open set. Increasing the value of \( C_1 \) in (6) to absorb the terms involving \( B(x, 2 \text{diam}(\Omega)) \) and \((2 \text{diam}(\Omega))^Q \), we obtain the lower mass bound:

\[
\mu(B(y, r)) \geq \frac{r^Q}{C_1} \quad \text{for } y \in \overline{\Omega} \text{ and } 0 < r \leq 2 \text{diam}(\Omega).
\]

(8)

Let \( u \) be a non-negative, locally bounded \( p \)-subharmonic function or a \( p \)-quasisubminimizer in \( \Omega \). Then \( u \) is a \( p \)-quasisubminimizer (see \cite[Corollary 7.8]{KiMa1}) and hence \( u \) is in the De Giorgi class \( \text{DG}_p(\Omega) \) (see \cite[Lemma 5.1]{KiMa2}). This means that if \( B(x, R) \subset \Omega \), then

\[
\int_{\{y \in B(x, \rho) : u(y) > k\}} g_u^p \, d\mu \leq \frac{C}{(r^p)^p} \int_{\{y \in B(x, r) : u(y) > k\}} (u - k)^p \, d\mu
\]

for every \( k \in \mathbb{R} \) and \( 0 < \rho < r < R/\kappa \), where \( \kappa \) is the scaling constant from the Poincaré inequality. It therefore follows from \cite[Theorem 4.2]{KiSh} (with \( k_0 = 0 \)) and the doubling property of the measure \( \mu \) that if \( B(x, R) \subset \Omega \), then

\[
u(x) \leq C_x \left( \int_{B(x, R)} u^p \, d\mu \right)^{1/p},
\]

(9)
where \( C_s \geq 1 \) is independent of \( x, R \) and \( u \), but depends on the quasisubminimizing constant \( C_{qu} \). If a function \( u \) on an open set \( U \) satisfies (9) for every ball \( B(x,R) \subset U \), then we say that \( u \) enjoys the weak sub-mean value property in \( U \). Using the weak sub-mean value property, we shall give the following modification of Domar’s theorem (see [D] and [AHL]). Observe that the weak sub-mean value property (9) holds for more general classes of functions than the class of \( p \)-harmonic functions. Indeed, [KiSh] proved this property for \( p \)-quasiminimizers and more generally for functions in the De Giorgi class; see [KiSh] and [KiMa2]. Hence the following lemma is phrased for the class of all functions satisfying the weak sub-mean value property, though, in this note, it will later be applied only to \( p \)-quasisubminimizers. For \( u > 0 \) we write

\[
\log^+ u = \begin{cases} \log u & \text{if } u \geq 1, \\ 0 & \text{otherwise}. \end{cases}
\]

**Lemma 3.1.** Let \( \Omega \) be a bounded open set in \( X \) and let \( \delta_\Omega(x) = \text{dist}(x, X \setminus \Omega) \). Suppose \( u \) is a locally bounded non-negative function on \( \Omega \) satisfying the weak sub-mean value property (9) in \( \Omega \). If there is a positive real number \( \varepsilon \) with

\[
I := \int_\Omega (\log^+ u)^{Q-1+\varepsilon} \, d\mu < \infty,
\]

where \( Q \) is the lower mass bound in (8), then there exists a constant \( C > 0 \) independent of \( u \) such that

\[
u(x) \leq 4C_s^2 \exp(CI^{1/\varepsilon} \delta_\Omega(x)^{-Q/\varepsilon}) \quad \text{for all } x \in \Omega.
\] (10)

Note that in the Euclidean setting we have \( \delta_\Omega(x) = \text{dist}(x, \partial \Omega) \), but in the general setting of metric measure spaces this may not be the case.

**Proof.** We first prove the following estimate.

Let \( x \in X \) and \( R > 0 \) such that \( u \) satisfies the \( p \)-submeanvalue property on \( B(x, R) \). If \( u(x) \geq t > 0, \alpha \geq 2C_s \), and

\[
\mu(\{y \in B(x,R) : \frac{t}{\alpha} < u(y) \leq at\}) \leq \frac{\mu(B(x,R))}{\alpha^{2p}},
\] (11)

then there exists \( x_2 \in B(x,R) \) such that \( u(x_2) > at \). Here \( C_s \) is the constant in (9). To see this, suppose that \( u(y) \leq at \) for every \( y \in B(x,R) \). Then by (9),

\[
t \leq u(x) \leq C_s \left( \int_{B(x,R)} u^p \, d\mu \right)^{1/p}
\]

\[
\leq C_s \left( \frac{1}{\mu(B(x,R))} \left( \frac{t}{\alpha} \right)^p \mu(B(x,R)) + \int_{\{y \in B(x,R) : u(y) > t/\alpha \}} u^p \, d\mu \right)^{1/p} \leq \frac{2^{1/p}C_s}{\alpha^p} t.
\]

Since \( C_s/\alpha \leq 1/2 \) and \( p > 1 \) by assumption, we have \( 1 \leq 2^{1/p}C_s/\alpha < 1 \), which is not possible. Hence there must be a point \( x_2 \in B(x,R) \) such that \( u(x_2) > at \).

Whatever the value of \( C \) might be, we have \( C > 0 \). Hence \( \exp(CI^{1/\varepsilon} \delta_\Omega(x)^{-Q/\varepsilon}) \geq 1 \). Therefore in order to prove (10), it suffices to show that there exists a constant \( C > 0 \) such that whenever \( u(x) > 4C_s^2 \),

\[
u(x) \leq \exp(CI^{1/\varepsilon} \delta_\Omega(x)^{-Q/\varepsilon}).
\] (12)
Fix $x \in \Omega$ such that $u(x) > 4C^2$. Then $u(x) > 1$ and hence $\log^+ u(x) = \log u(x)$. Therefore, demonstrating (12) is equivalent to showing that there is a constant $C > 0$ independent of $x$ so that

$$\delta_\Omega(x) \leq C \Omega^{1/2} (\log^+ u(x))^{-c/\Omega}.$$  \hspace{1cm} (13)

To this end, let us choose $a = 2C$. This sets us up to use (11). For $j \in \mathbb{N}$ let

$$R_j = \left[ C_1 a^{2p} \mu(\{y \in \Omega : a^{j-2}u(x) < u(y) \leq a^j u(x)\}) \right]^{1/\Omega},$$

where $C_1$ is the constant in (8). We claim

$$\delta_\Omega(x) \leq 2 \sum_{j=1}^{\infty} R_j.$$ \hspace{1cm} (14)

In order to prove (14), we now construct a sequence of points in $\Omega$, finite or infinite depending on the situation, as follows. Let $x_1 = x$. If $\delta_\Omega(x_1) < R_1$, then consider the singleton sequence $(x_1)$. Suppose $\delta_\Omega(x_1) \geq R_1$. Since $B(x_1, R_1) \subset \Omega$, it follows that

$$\left[ C_1 a^{2p} \mu(\{y \in B(x_1, R_1) : a^{-1}u(x_1) < u(y) \leq au(x_1)\}) \right]^{1/\Omega} \leq R_1.$$ 

It therefore follows from (8) that

$$\mu(\{y \in B(x_1, R_1) : a^{-1}u(x_1) < u(y) \leq au(x_1)\}) \leq \frac{R_1^2}{C_1 a^{2p}} \leq \frac{\mu(B(x_1, R_1))}{a^{2p}}.$$ 

Now, by (11) with $t = u(x_1)$, there is a point $x_2 \in B(x_1, R_1)$ such that $u(x_2) > au(x_1)$. If $\delta_\Omega(x_2) < R_2$, then consider the sequence $(x_1, x_2)$. Otherwise, $B(x_2, R_2) \subset \Omega$, and as before we can apply (11) with $t = au(x_1)$ to get $x_3 \in B(x_2, R_2)$ such that $u(x_3) > au(x_2) > a^2 u(x_1)$. Inductively, we may construct $x_j$ given $(x_1, x_2, \ldots, x_{j-1})$ such that for $j = 1, \ldots, J - 1$,

$$\delta_\Omega(x_j) \geq R_j, \quad d(x_j, x_{j-1}) < R_{j-1}, \quad \text{and} \quad u(x_j) > au(x_{j-1}) > a^{j-1} u(x_1).$$

If $\delta_\Omega(x_{j-1}) < R_{j-1}$, then we stop here. Otherwise, we may use (11) to find a point $x_j \in B(x_{j-1}, R_{j-1})$ such that $u(x_j) > au(x_{j-1}) > a^{j-1} u(x_1)$. We will now show that $\delta_\Omega(x_1) \leq 2 \sum_{j=1}^{\infty} R_j$. To do so, we consider two cases.

**Case 1:** the sequence is finite. Then there is a positive integer $J$ such that $\delta_\Omega(x_J) < R_J$. Then

$$\delta_\Omega(x_1) \leq \sum_{j=1}^{J-1} d(x_j, x_{j+1}) + \delta_\Omega(x_J) < \sum_{j=1}^{J-1} R_j + R_J \leq \sum_{j=1}^{\infty} R_j.$$ 

**Case 2:** the sequence is infinite. Then for every $j \in \mathbb{N}$ we have $u(x_j) > a^{j-1} u(x_1)$; that is, \( \lim_{j \to \infty} u(x_j) = \infty \). But then as $u$ is locally bounded on $\Omega$, the infinite sequence $(x_j)_j$ has no accumulation point in $\Omega$. Since, by assumption, $X$ is proper and hence $\overline{\Omega}$ is compact, there is a subsequence converging to a point in $\partial \Omega$. Hence there exists some $J \in \mathbb{N}$ for which $\delta_\Omega(x_J) < \frac{1}{2} \delta_\Omega(x_1)$. Hence

$$\delta_\Omega(x_1) \leq \sum_{j=1}^{J-1} d(x_j, x_{j+1}) + \delta_\Omega(x_J) < \sum_{j=1}^{J} R_j + \frac{1}{2} \delta_\Omega(x_1).$$
we can then conclude that
\[ \delta_\Omega(x_1) \leq 2 \sum_{j=1}^{J} R_j \leq 2 \sum_{j=1}^{\infty} R_j. \]

Hence (14) follows whether the sequence constructed above is finite or not.
Therefore to show (13) it suffices to prove that
\[ \sum_{j=1}^{\infty} R_j \leq C t^{1/Q} (\log^+ u(x))^{-\epsilon/Q}. \] (15)

Let \( j_0 \) be the unique positive integer such that \( a^{j_0} < u(x) \leq a^{j_0+1} \). Then \( j_0 \geq 2 \), since we have assumed that \( u(x) > 4C_2^2 = a^2 \). Recall that for \( j \in \mathbb{N} \),
\[ R_j = \left[ C_1 a^{2j} \mu(\{ y \in \Omega : a^{j-2} u(x) < u(y) \leq a^j u(x) \}) \right]^{1/Q}. \]

We obtain from Hölder’s inequality that
\[ \sum_{j=1}^{\infty} R_j \leq C \sum_{j=1}^{\infty} \left[ \mu(\{ y \in \Omega : a^{j_0+j-2} < u(y) \leq a^{j_0+j+1} \}) \right]^{1/Q} \]
\[ \leq C \sum_{j=j_0-1}^{\infty} \left[ \mu(\{ y \in \Omega : a^j < u(y) \leq a^{j+3} \}) \right]^{1/Q} \]
\[ = C \sum_{j=j_0-1}^{\infty} \frac{j^{(Q-1+\epsilon)/Q}}{j^{(Q-1+\epsilon)/Q}} \left[ \mu(\{ y \in \Omega : a^j < u(y) \leq a^{j+3} \}) \right]^{1/Q} \]
\[ \leq C \left[ \sum_{j=j_0-1}^{\infty} \frac{1}{j^{(Q-1+\epsilon)/(Q-1)}} \right]^{(Q-1)/Q} \left[ \sum_{j=j_0-1}^{\infty} \int_{\{ y \in \Omega : a^j < u(y) \leq a^{j+3} \}} j^{Q-1+\epsilon} d\mu \right]^{1/Q} \]
\[ \leq C j_0^{-\epsilon/Q} \left[ \sum_{j=j_0-1}^{\infty} \int_{\{ y \in \Omega : a^j < u(y) \leq a^{j+3} \}} \left( \frac{\log^+ u}{\log a} \right)^{Q-1+\epsilon} d\mu \right]^{1/Q}. \]

Note that each \( y \in \Omega \) belongs to at most three of the sets \( \{ y \in \Omega : a^j < u(y) \leq a^{j+3} \} \). Hence we see that
\[ \sum_{j=1}^{\infty} R_j \leq C j_0^{-\epsilon/Q} I^{1/Q}. \]

By the choice of \( j_0 \) it can be seen that \( a^{j_0} < u(x) \leq a^{j_0+1} \) and \( j_0 \geq 2 \). We therefore have \( j_0 \leq \log u(x)/\log a \leq j_0 + 1 \leq 2 j_0 \), and hence
\[ \sum_{j=1}^{\infty} R_j \leq C \left( \frac{\log^+ u(x)}{2 \log a} \right)^{-\epsilon/Q} I^{1/Q}, \]
which is (15). Note that \( C \) is independent of \( x \) and \( u \). This completes the proof of Lemma 3.1. □

It should be emphasized that we do not require in the above two lemmata that \( \Omega \) be a domain. It is sufficient to assume merely that \( \Omega \) be a bounded open subset of \( X \) such that \( X \setminus \Omega \neq \emptyset \).
4 Geometry of bounded John domains in $X$

Let $\Omega \subset X$ be a domain. For $0 < c < 1$ a rectifiable curve $\gamma$ connecting $x, y \in \Omega$ is said to be a $c$-John curve in $\Omega$ if $\delta(\gamma, z) \geq c \ell(\gamma, z)$ for every $z \in \gamma$, where $\gamma_{xz}$ is the subcurve of $\gamma$ having $x$ and $z$ as its two endpoints. We say that $\Omega$ is a John domain with John center $x_0 \in \Omega$ and John constant $c$ if every point $x \in \Omega$ can be connected to $x_0$ by a $c$-John curve in $\Omega$. For $A > 1$, a rectifiable curve $\gamma$ connecting $x, y \in \Omega$ is said to be a $A$-uniform curve in $\Omega$ if $\ell(\gamma) \leq A\d(x, y)$ and $\min\{\ell(\gamma_{xz}), \ell(\gamma_{zy})\} \leq A\delta(\gamma)$ for every $z \in \gamma$. We say that $\Omega$ is an $A$-uniform domain if every pair of distinct points $x, y \in \Omega$ can be joined by an $A$-uniform curve $\gamma$ in $\Omega$. Obviously, a uniform domain is a John domain; but the converse is not necessarily true.

Given $x \in X$ and $R > 0$ we let $B(x, R) = \{y \in X : d(x, y) \leq R\}$. Note that in general this may be a larger set than the closure of the open ball $B(x, R)$ itself. By $S(x, R)$ we mean the sphere centered at $x$ of radius $R$: $S(x, R) = \{y \in X : d(x, y) = R\}$. This set contains $\partial B(x, R)$, but in general could be a larger set.

For domains $V \subset X$, we let $k_V$ denote the following quasi-hyperbolic “metric” on $V$. If $x, y \in V$, then

$$k_V(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds(t)}{\delta_V(y(t))},$$

where the infimum is taken over all rectifiable curves connecting $x$ to $y$ in $V$. If no such curve exists in $V$, then we set $k_V(x, y) = \infty$. It is worth noting that if $V$ is a John domain, then $k_V$ is always finite-valued; that is, it is indeed a metric. In fact, if $\gamma$ is a $c$-John curve connecting $x$ to $y$ in $\Omega$, then

$$k_\Omega(x, y) \leq \int_{\gamma} \frac{ds(t)}{\delta_\Omega(y(t))} \leq \int_0^{\delta_\Omega(x)/2} \frac{ds}{\delta_\Omega(x)/2} + \int_{\delta_\Omega(x)/2}^{\ell(\gamma)} \frac{ds}{cs} \leq 1 + \frac{\log 2/c}{c} + \frac{1}{c} \log \left(\frac{\delta_\Omega(y)}{\delta_\Omega(x)}\right). \quad (16)$$

Suppose for a moment that $\Omega$ is a uniform domain. Let $\xi \in \partial \Omega$ and $x \in \Omega \cap \overline{B}(\xi, R/2)$. By the uniformity we find a point $y \in S(\xi, R)$ with $\delta_\Omega(y) \geq R/(3A)$ and a uniform curve $\gamma \subset \Omega$ connecting $x$ and $y$. We observe that $\gamma \subset \Omega \cap B(\xi, CR)$ with $C > 1$ depending only on the uniformity. For this point $y$ we have

$$k_\Omega(x, y) = k_{\Omega \cap B(\xi, 2CR)}(x, y) \leq C \left[\log \left(\frac{R}{\delta_\Omega(x)}\right) + 1\right].$$

This property can be generalized to a John domain. Following [AHL], we give the following definition.

**Definition 4.1.** Let $N \in \mathbb{N}$. We say that $\xi \in \partial \Omega$ has a system of local reference points of order $N$ if there exist constants $R_\xi > 0$, $A_\xi > 1$ and $A_\xi > 1$ such that whenever $0 < R < R_\xi$ we can find $y_1, \ldots, y_N \in \Omega \cap S(\xi, R)$ with the properties that

(i) $A_\xi^{-1}R \leq \delta_\Omega(y_i) \leq R$ for each $i = 1, \ldots, N$,

(ii) for every $x \in \Omega \cap \overline{B}(\xi, R/2)$ we have some $i \in \{1, \ldots, N\}$ for which

$$k_\Omega(x, y_i) = k_{\Omega \cap B(\xi, A_\xi R)}(x, y_i) \leq A_\xi \left[\log \left(\frac{R}{\delta_\Omega(x)}\right) + 1\right].$$
Remark 4.2. If $\Omega$ is a uniform domain, then every $\xi \in \partial \Omega$ has a system of local reference points of order 1. The constants $R_{\xi}, \lambda_{\xi}$ and $A_{\xi}$ depend only on the uniformity of $\Omega$.

Lemma 4.3. Let $\Omega \subset X$ be a John domain with John center $x_0$ and John constant $c$. Then there exists $N_0 \in \mathbb{N}$ depending only on $x_0$, $c$ and the data of $X$ such that every $\xi \in \partial \Omega$ has a system of local reference points of order at most $N_0$ with constants $R_{\xi} = R_0 := \min(\delta_\Omega(x_0)/2, \text{diam}(\Omega)/10)$, $\lambda_{\xi} = 8$ and $A_{\xi} = A_0 := \max(2/c, 3/2 + c^{-1}\log 2/c)$.

Proof. Let $\xi \in \partial \Omega$ and $0 < R < R_0$ and suppose $x \in \Omega \cap \overline{B}(\xi, R/2)$. Let $\gamma_x$ be a $c$-John curve connecting $x$ to $x_0$ and let $y_x$ be the point of $\gamma_x$ at which $\gamma_x$ leaves the ball $B(\xi, R)$ for the first time. Then

$$R \geq \delta_\Omega(y_x) \geq c \ell(\beta_x) \geq cR/2 \geq R/A_0,$$

where $\beta_x$ is the subcurve of $\gamma_x$ that terminates at $y_x$. Let $E = \{ y_x : x \in \Omega \cap \overline{B}(\xi, R/2) \}$. By the 5-covering lemma (see [He] for example) and the doubling property of $\mu$ we find finitely many points $y_1 = y_{x_1}, \ldots, y_N = y_{x_N} \in E$ such that $N \leq N_0$,

$$E \subset \bigcup_{i=1}^N B(y_i, cR/(50q)) \subset B(\xi, R)$$

and $\{B(y_i, cR/(50q))\}_{i=1}^N$ is pairwise disjoint, where $q \geq 1$ is the quasi-convexity constant in (1).

Let us demonstrate that $y_1, \ldots, y_N$ satisfy the conditions in Definition 4.1. Obviously, (i) holds true by (17). For the proof of (ii) take $x \in \Omega \cap \overline{B}(\xi, R/2)$, and the point $y_x$ and the subcurve $\beta_x$ as above. Then, $\delta_{\Omega \cap B(\xi, 5R)}(z) = \delta_\Omega(z)$ for $z \in \beta_x$, so that (16) gives

$$k_{\Omega \cap B(\xi, 5R)}(y_x, y_i) \leq \int_{\beta_x} \frac{ds(t)}{\delta_{\Omega}(\beta_x(t))} \leq 1 + \frac{\log 2/c}{c} + \frac{1}{c} \log\left(\frac{R}{\delta_{\Omega}(x)}\right).$$

Since $y_x \in E \subset \bigcup B(y_i, cR/(10q))$, we can find a positive integer $i \leq N$ such that $y_x \in B(y_i, cR/(10q))$. By the quasi-convexity of $X$ there is a rectifiable curve $\gamma$ connecting $y_x$ to $y_i$ with $\ell(\gamma) \leq q d(y_x, y_i) \leq cR/10$. By (17),

$$\delta_{\Omega \cap B(\xi, 5R)}(z) = \delta_\Omega(z) \geq \delta_{\Omega}(y_x) - cR/10 > cR/5 \quad \text{for } z \in \gamma,$$

so that

$$k_{\Omega \cap B(\xi, 5R)}(y_x, y_i) \leq \int_{\gamma} \frac{ds(t)}{\delta_{\Omega}(\gamma(t))} \leq \frac{cR/10}{cR/5} = \frac{1}{2}.$$ Hence

$$k_{\Omega \cap B(\xi, 5R)}(x, y_i) \leq k_{\Omega \cap B(\xi, 5R)}(x, y_x) + k_{\Omega \cap B(\xi, 5R)}(y_x, y_i) \leq A_0 \left[ \log\left(\frac{R}{\delta_{\Omega}(x)}\right) + 1 \right].$$

This completes the proof of Lemma 4.3. \qed

Lemma 4.4. Let $\Omega \subset X$ be a bounded John domain with John center $x_0$ and John constant $c$. Then there exist positive constants $C$ and $\tau$ depending only on the data $X$ and of $\Omega$ such that for each $\xi \in \partial \Omega$ and $0 < R < 2c \delta_{\Omega}(x_0)/(10q)$,

$$\int_{\Omega \cap B(\xi, R)} \left( \frac{R}{\delta_{\Omega}(x)} \right)^{\tau} \mu(x) \leq C \mu(B(\xi, R)).$$
Hajłasz and Koskela [HaK1, Lemma 6] demonstrated the lemma for Euclidean domains by an indirect proof using Sobolev extension and embedding theorems (see [HaK1, Remark 11]). The following proof (a modification of the proof from [AHL]) is more geometric, and holds true for all relatively compact John domains in quasiconvex metric measure spaces equipped with a doubling measure, irrespective of whether $X$ supports a Poincaré inequality.

**Proof.** For each nonnegative integer $j$ let $E_j = \cup_{i=j+1}^{\infty} V_i$ with

$$V_i = \{ x \in \Omega \cap B(\xi, R) : 2^{-(t+1)}R \leq \delta_\Omega(x) < 2^{-j}R \}.$$  

We claim that for every $x \in E_j$ there exist two points $y_x, y'_x \in \Omega$ such that

(i) $x \in B(y_x, (q + 1)2^{-j}R/c)$,

(ii) $B(y'_x, 2^{-(j+2)}R) \subset V_j \cap B(y_x, (q + 1)2^{-j}R/c)$.

To see this let $x \in E_j$. Let $\gamma_x$ be a $c$-John curve connecting $x$ and the John center $x_0$. Observe that we can choose $y_x \in \gamma_x$ such that $\delta_\Omega(y_x) = 2^{-j}R \geq c d(x, y_x)$, thus satisfying (i). As $X$ is proper, there is a point $y'_x \in X \setminus \Omega$ with $\delta_\Omega(y_x) = d(x, y'_x)$. Since $X$ is quasi-convex, there is a curve $\beta$ connecting $y_x$ and $y'_x$ such that $d(\beta) \leq q 2^{-j}R$. Let $y'_x \in B \cap \Omega$ be a point satisfying $\delta_\Omega(y'_x) = (2^{-(j+1)} + 2^{-j})R/2$. Then for $z \in B(y'_x, 2^{-(j+2)}R)$,

$$d(y_x, z) \leq d(y_x, y'_x) + d(y'_x, z) < \ell(\beta) + 2^{-(j+2)}R < \frac{(q + 1)2^{-j}R}{c},$$

$$d(\xi, z) \leq d(\xi, x) + d(x, y_x) + d(y_x, z) < R + \frac{(q + 1)2^{-j}R}{c} + \frac{2^{-j}R}{c} + \frac{(q + 1)2^{-j}R}{c} = R + \frac{(3q + 4)2^{-j}R}{4c},$$

$$2^{-(j+1)}R < \delta_\Omega(y'_x) - d(y'_x, z) \leq \delta_\Omega(z) \leq \delta_\Omega(y'_x) + d(y'_x, z) < 2^{-j}R.$$  

The above three inequalities together yield (ii).

In view of (i), the collection $\{B(y_x, (q + 1)2^{-j}R/c)\}_{x \in E_j}$ forms a covering of $E_j$. By the 5-covering lemma we have a pairwise disjoint sub-collection $\{B(y_k, (q + 1)2^{-j}R/c)\}_{k \in \mathbb{N}}$ such that $E_j \subset \{B(y_k, 5(q + 1)2^{-j}R/c)\}_{k \in \mathbb{N}}$. Since $\mu$ is a doubling measure, we can find $C_2 > 1$, which depends solely on $C_d$, such that

$$\mu(B(y_k, 5(q + 1)2^{-j}R/c)) \leq C_2 \mu(B(y'_k, 2^{-(j+2)}R)) \leq C_2 \mu(V_j \cap B(y_k, (q + 1)2^{-j}R/c)).$$

Here $y'_k$ is associated with $y_k$ in the same manner that $y'_x$ is paired with $y_x$, and the second inequality above follows from (ii). Therefore,

$$\sum_{i=j+1}^{\infty} \mu(V_i) = \mu(E_j) \leq \sum_{k \in \mathbb{N}} \mu(B(y_k, 5(q + 1)2^{-j}R/c)) \leq C_2 \sum_{k \in \mathbb{N}} \mu(V_j \cap B(y_k, (q + 1)2^{-j}R/c)) \leq C_2 \mu(V_j).$$

Let $1 < t < 1 + 1/C_2$ and $\tau = \log_2 t$. Then proceeding as in [AHL, Proof of Lemma 6] or [Ai1, Lemma 5] we can deduce from the above inequality that

$$\int_{\Omega \cap B(\xi, R)} \left( \frac{R}{\delta_\Omega(z)} \right)^\tau d\mu(z) \leq \sum_{j=0}^{\infty} \int_{V_j} \left( \frac{R}{\delta_\Omega(z)} \right)^\tau d\mu(z) \leq \sum_{j=0}^{\infty} \mu(V_j) \leq C \mu(B(\xi, R)).$$

\[\square\]
**Definition 4.5.** A finite collection \( \{ B(x_i, r_i) \}_{i=1}^k \) of balls is said to be a Harnack chain connecting two points \( x, y \in \Omega \) with **length** \( k \) if

(i) for \( i = 1, \ldots, k - 1 \), \( B(x_i, r_i) \cap B(x_{i+1}, r_{i+1}) \neq \emptyset \),

(ii) for \( i = 1, \ldots, k \), \( B(x_i, 2kr_i) \subset \Omega \),

(iii) \( x \in B(x_1, r_1) \) and \( y \in B(x_k, r_k) \),

where \( \kappa \) is the scaling constant from the Poincaré inequality (see Definition 2.1).

In light of the definition of \( k_{\Omega}(x, y) \) we observe that \( x \) and \( y \) are connected by a Harnack chain of length no more than \( 1 + C k_{\Omega}(x, y) \). Hence the Harnack inequality for \( p \)-quasiminimizers (see [KiSh]) gives the following.

**Lemma 4.6.** If \( h \) is a positive \( p \)-quasiminimizer on \( \Omega \), then

\[
\frac{1}{A_H} \exp \left( -A_H k_{\Omega}(x, y) \right) \leq \frac{h(x)}{h(y)} \leq A_H \exp \left( A_H k_{\Omega}(x, y) \right)
\]

for \( x, y \in \Omega \),

where \( A_H > 1 \) is independent of \( h, x \) and \( y \).

5 A Carleson type estimate for \( p \)-harmonic functions on a John domain in Ahlfors regular spaces

We will henceforth assume that \( \mu \) is Ahlfors \( Q \)-regular (see (7)). We will also assume from now on that \( \Omega \subset X \) is a bounded John domain with John center \( x_0 \in \Omega \) and John constant \( 0 < c < 1 \) and that \( X \setminus \Omega \neq \emptyset \). If \( x \in \Omega \) and \( y_x \) is a John curve connecting \( x \) to a local reference point \( y_i \) associated with \( x \) as in the proof of Lemma 4.3, then

\[
k_{\Omega}(x, y_i) \leq C_3 \left[ \log \left( \frac{\delta_\Omega(y_i)}{\delta_\Omega(x)} \right) + 1 \right].
\]

Therefore by Lemma 4.6, for every positive \( p \)-quasiminimizer \( h \) on \( \Omega \) and \( x \in \Omega \) we have

\[
\frac{h(x)}{h(y_i)} \leq A_H \exp \left( A_H C_3 \left[ \log \left( \frac{\delta_\Omega(y_i)}{\delta_\Omega(x)} \right) + 1 \right] \right) \leq C \left( \frac{\delta_\Omega(y_i)}{\delta_\Omega(x)} \right)^d,
\]

where \( C, d > 0 \) depend only on \( A_H \) and \( C_3 \) but not on \( x \) nor on \( h \). Thus we have a weak estimate

\[
\frac{h(x)}{h(y_i)} \leq C \left( \frac{\delta_\Omega(y_i)}{\delta_\Omega(x)} \right)^d. \tag{18}
\]

Let \( N_0 \) be as in Lemma 4.3 and \( R_0 \) be as in the proof of Lemma 4.3. We recall again the standing assumption that \( X \) is a proper metric space.
Proposition 5.1. Let \( \xi \in \partial \Omega \), \( 0 < R < R_0/16 \), and \( h \) be a positive \( p \)-quasiminimizer in \( \Omega \cap B(\xi, 16R) \) vanishing \( p \)-quasieverywhere on \( \partial \Omega \cap B(\xi, 16R) \). If \( h \) is bounded in \( \Omega \cap B(\xi, R/2) \setminus \overline{B(\xi, R/16)} \), then for all \( x \in \Omega \cap S(\xi, R/4) \),

\[
h(x) \leq C \sum_{i=1}^{N} h(y_i),
\]

where \( y_1, \ldots, y_N \) is a system of local reference points for \( \xi \) (\( N \leq N_0 \)).

Proof. By (18), for all \( x \in \Omega \cap \overline{B(\xi, R/2)} \) there is an integer \( i \in \{1, \ldots, N\} \) such that

\[
\frac{h(x)}{h(y_i)} \leq C \left( \frac{R}{\delta(\xi)} \right)^\lambda.
\]

Hence, for every \( x \in \Omega \cap \overline{B(\xi, R/2)} \),

\[
h(x) \leq C \left( \frac{R}{\delta(\xi)} \right)^\lambda \sum_{i=1}^{N} h(y_i).
\]

(19)

Let

\[
u := \frac{1}{C \sum_{i=1}^{N} h(y_i)} h.
\]

Then \( \nu \) is non-negative and locally bounded on the bounded open set \( \Omega_0 \) given by \( \Omega_0 := B(\xi, R/2) \setminus \overline{B(\xi, R/16)} \). Moreover, \( \nu \) is a \( p \)-quasisubminimizer in \( \Omega_0 \), because it is a \( p \)-quasiminimizer in \( \Omega_0 \cap \Omega \) and vanishes in \( \Omega_0 \cap \partial \Omega \); see Lemma 3.11 of [BBS1]. By (19),

\[
u(x) \leq \left( \frac{R}{\delta(\xi)} \right)^\lambda \text{ for } x \in \Omega_0 \cap \Omega.
\]

Let \( \tau > 0 \) be as in Lemma 4.4. Choose \( \varepsilon > 0 \) such that \( Q - 1 + \varepsilon > 1 \) and apply the elementary inequality

\[
(\log t)^{Q-1+\varepsilon} \leq \left( \frac{Q - 1 + \varepsilon}{\tau} \right)^{Q-1+\varepsilon} t^\tau \text{ whenever } t \geq 1
\]

to the quantity \( t = R/\delta(\xi) \geq 1 \) whenever \( x \in \Omega \cap \Omega_0 \) to obtain the following estimate:

\[
I = \int_{\Omega \cap \Omega_0} (\log^+ \nu(x))^{Q-1+\varepsilon} d\mu(x) \leq \int_{\Omega \cap \Omega_0} \left[ \lambda \log^+ \left( \frac{R}{\delta(\xi)} \right) \right]^{Q-1+\varepsilon} d\mu(x)
\]

\[
\leq \int_{\Omega \cap \Omega_0} C \left( \frac{R}{\delta(\xi)} \right)^\tau d\mu(x).
\]

By Lemma 4.4 and the Ahlfors regularity of \( \mu \),

\[
I \leq C_{\lambda, Q, \varepsilon} \int_{\Omega \cap B(\xi, R/2)} \left( \frac{R}{\delta(\xi)} \right)^\tau d\mu(x) \leq C R^{Q} < \infty.
\]

(20)

Therefore (10) of Lemma 3.1 yields

\[
u(x) \leq C \exp \left( C t^{1/\delta} \right) \quad \text{whenever } x \in \Omega \cap \Omega_0.
\]
On the other hand, if \( x \in \Omega \cap S(\xi, R/4) \), then \( \delta_{\Omega}(x) \approx R \). Hence by (20),

\[
\begin{align*}
u(x) \leq C \exp \left( C I^{1/e} R^{-Q/e} \right) \leq C.
\end{align*}
\]

Now in view of the definition of \( u \), we have the desired result. \( \square \)

As a corollary to Proposition 5.1 we have the following Carleson estimate. Such an estimate follows from Proposition 5.1 and the strong maximum principle (see [KiSh]) for \( p \)-quasiminimizers.

**Theorem 5.2.** Let \( \Omega \subset X \) be a bounded John domain, \( \xi \in \partial \Omega \), and \( 0 < R < R_0/16 \) where \( R_0 \) is as in the proof of Lemma 4.4. If \( h \) is a positive \( p \)-quasiminimizer in \( \Omega \cap B(\xi, 16R) \) vanishing \( p \)-quasieverywhere on \( \partial \Omega \cap B(\xi, 16R) \) and bounded in \( \Omega \cap B(\xi, R/2) \), then

\[
\begin{align*}
h(x) \leq C \sum_{i=1}^{N} h(y_i) \quad \text{for every } x \in \Omega \cap B(\xi, R/4),
\end{align*}
\]

where \( y_1, \ldots, y_N \in \Omega \cap S(\xi, R) \) are a system of local reference points for \( \xi \). The constant \( C > 1 \) is independent of \( x, \xi, R, \) and \( h \).

**Corollary 5.3.** Let \( \Omega \subset X \) be a uniform domain, \( \xi \in \partial \Omega \), and \( 0 < R < R_0/16 \) where \( R_0 \) is as in the proof of Lemma 4.4. If \( h \) is a positive \( p \)-quasiminimizer in \( \Omega \cap B(\xi, 16R) \) vanishing \( p \)-quasieverywhere on \( \partial \Omega \cap B(\xi, 16R) \) and bounded in \( \Omega \cap B(\xi, R/2) \), then

\[
\begin{align*}
h(x) \leq Ch(y) \quad \text{for every } x \in \Omega \cap B(\xi, R/4),
\end{align*}
\]

where \( y \in \Omega \cap S(\xi, R) \) is such that \( \delta_{\Omega}(y) \geq R/C \). The constant \( C > 1 \) is independent of \( x, \xi, R, \) and \( h \).

**Remark 5.4.** In light of Remark 2.6, the above theorem and corollary hold if \( h \) is \( p \)-harmonic.

## 6 The conformal Martin boundary of a bounded John domain in \( X \)

In this section we assume that \( X \) is of \( Q \)-bounded geometry and apply the results of the previous sections with \( p = Q \). Given a Cheeger derivative structure on \( X \), the corresponding conformal Martin boundary has been defined in [HST] (see Definition 2.9). In the following, we let \( \lambda > 0 \) be as in (18).

**Theorem 6.1.** Let \( \Omega \subset X \) be a bounded John domain such that \( \text{Cap}_Q(X \setminus \Omega) > 0 \). Let \( M \) be a conformal Martin kernel for \( \Omega \) with fundamental sequence \( (y_k) \). Then \( (y_k) \) converges to a point \( \xi_M \in \partial \Omega \) and the following statements hold:

(i) \( M \) is bounded in a neighborhood of \( \xi \) for each \( \xi \in \partial \Omega \setminus \{\xi_M\} \);

(ii) there is a constant \( C \geq 1 \) such that \( M(x) \leq C d(x, \xi_M)^{-\lambda} \) for all \( x \in \Omega \);

(iii) \( M \) vanishes continuously \( Q \)-quasieverywhere on \( \partial \Omega \);
(iv) there is a sequence \((x_n)\) in \(\Omega\) converging to \(\xi_M\) such that \(\lim_n M(x_n) = \infty\), and hence the point \(\xi_M\) is uniquely determined from \(M\) but does not depend on the fundamental sequence that gave rise to it.

A crucial result needed to prove (iv) of the above theorem is Corollary 6.2 of [BBS2]. Observe that Corollary 6.2 of [BBS2] is invalid if \(\text{Cap}_Q(X \setminus \Omega) = 0\). The proof of this result uses the fact that solutions to \(p\)-Dirichlet problems with boundary data belonging to \(N^{1,Q}(X)\) satisfy a comparison theorem; see [Sh2] for a proof of this fact under the assumption that \(\text{Cap}_Q(X \setminus \Omega) > 0\). However, if \(\text{Cap}_Q(X \setminus \Omega) = 0\) then this comparison theorem obviously fails and as a consequence Corollary 6.2 of [BBS2] also fails to hold.

Let \(U\) be an open subset of \(X\). Recall that \(H^U_f\) stands for the solution to the \(Q\)-Dirichlet problem on the open set \(U\) with boundary data \(f \in N^{1,Q}(X)\). Given a continuous function \(f : \partial U \to \mathbb{R}\), the authors of [BBS2] construct a Perron solution in \(U\) with boundary data \(f\) (in fact, [BBS2] also shows that continuous functions are resolutive); let \(P^U_Q f\) denote such a solution. If such a function \(f\) also happens to belong to \(N^{1,Q}(X)\), then \(P^U_Q f = H^U_Q f\). We say that \(\xi \in \partial U\) is a \(Q\)-regular boundary point for \(U\) if \(\lim_{U} U \ni \xi \ P^U_Q f(x) = f(\xi)\) whenever \(f\) is a bounded and continuous function on \(\partial U\). To prove Theorem 6.1 we need the following lemma.

**Lemma 6.2.** Let \(\Omega \subset X\) be a bounded domain such that \(\text{Cap}_Q(X \setminus \Omega) > 0\). Then \(Q\)-quasi every point \(\xi \in \partial \Omega\) is a \(Q\)-regular boundary point for \(\Omega \cap B(\xi, r)\) whenever \(r > 0\).

**Proof.** Let \((r_n)\) be an enumeration of all positive rational numbers. Moreover, let \((x_k)\) be a sequence of points in \(\Omega\) that is dense in \(\Omega\). By Theorem 3.9 of [BBS1], the set \(J_{k,n}\) of all points in \(B(x_k, r_n) \cap \partial \Omega\) that are \(Q\)-irregular points for \(\Omega \cap B(x_k, r_n)\) is of zero \(Q\)-capacity. Hence the set \(J := \bigcup_{k,n} J_{k,n}\) is a zero \(Q\)-capacity subset of \(\partial \Omega\). Let \(\xi \in \partial \Omega \setminus J\). We will demonstrate that \(\xi\) satisfies the requirements of the above lemma.

Suppose \(\xi\) does not satisfy the requirements of the above lemma; that is, suppose there is \(r < \text{diam}(\Omega)\) such that \(\xi\) is not a \(Q\)-regular boundary point for \(\Omega \cap B(\xi, r)\). Then, following the notation set up in the proof of Theorem 3.9, [BBS1], we can find a ball \(B_{l,m}\) centered at some point in \(\partial \Omega\) and radius \(p_{l,m}\) such that \(\xi \in B_{l,m} \subseteq 2B_{l,m} \subseteq B(\xi, r)\), and a Lipschitz function \(\varphi_{l,m}\) that is compactly supported in \(2B_{l,m}\) and takes on the value of 1 in \(B_{l,m}\) such that

\[
\liminf_{\Omega \cap B(\xi, r) \ni y \to \xi} H^\Omega_Q B(\xi, r) \varphi_{l,m}(y) < 1.
\]

On the other hand, as \(2B_{l,m} \subseteq B(\xi, r)\) and \(\xi \in B_{l,m}\), we can find \(k, n\) such that \(\xi \in B(x_k, r_n) \subseteq B(x_k, r_n) \subseteq B(\xi, r)\). As \(\xi \notin J_{k,n}\), we know that \(\xi\) is a \(Q\)-regular boundary point of the open set \(\Omega \cap B(x_k, r_n)\). Hence we have by the comparison theorem that

\[
\liminf_{\Omega \cap B(\xi, r) \ni y \to \xi} H^\Omega_Q B(\xi, r) \varphi_{l,m}(y) \geq \liminf_{\Omega \cap B(x_k, r_n) \ni y \to \xi} H^\Omega_Q B(x_k, r_n) \varphi_{l,m}(y) = 1,
\]

contradicting the above inequality. Hence such \(r\) does not exist, and consequently \(\xi\) satisfies the lemma. \(\Box\)

**Proof of Theorem 6.1.** Let \((y_k)\) be a fundamental sequence in \(\Omega\) giving rise to the kernel \(M\); see Definition 2.9. We will show that there is a point \(\xi_M \in \partial \Omega\) such that \(\lim_k y_k = \xi_M\). Since \((y_k)\) is a fundamental sequence, it has no accumulation point in \(\Omega\). Since \(\Omega \subset F\) is compact, there is a
point $\xi_M \in \partial \Omega$ and a subsequence $(y_{k_n})$ such that $\lim y_{k_n} = \xi_M$. We will demonstrate that for every $\xi \in \partial \Omega \setminus \{\xi_M\}$ the function $M$ is bounded in a neighborhood of $\xi$ and that $M$ satisfies (iii) and (iv) of the theorem. As a consequence, this observation concludes that $\lim y_k = \xi_M$. Moreover, if $(w_n)$ is another fundamental sequence giving rise to $M$, then $\lim w_n = \xi_M$.

Let us begin with the proof of (i) of the theorem. Fix $\xi \in \partial \Omega \setminus \{\xi_M\}$. For ease of notation let us call the subsequence of $(y_k)$ that converges to $\xi_M$ also as $(y_k)$. Then there exists $r = 4r_{\xi} > 0$ such that for all $k \in \mathbb{N}$, $y_k \notin \Omega \cap B(\xi, 16(q + 1)r/c)$ and $16(q + 1)r/c < R_0$, where $R_0$ is as in the proof of Lemma 4.3. Let $M_k$ be the function given by $M_k(x) = g(x, y_k)/g(x_0, y_k)$. Then $M_k$ is positive and $Q$-harmonic in $\Omega \cap B(\xi, 16(q + 1)r/c)$, and vanishes $Q$-quasieverywhere on $\partial \Omega$. Moreover, $M_k$ is bounded on $\Omega \cap B(\xi, r)$, and therefore by Theorem 5.2, for $x \in \Omega \cap B(\xi, r/2)$ we have

$$M_k(x) \leq C \sum_{i=1}^N M_k(y_i),$$

where $y_1, \ldots, y_N \in \Omega \cap S(\xi, r/2)$ is a system of local reference points for $\xi$. Note that $\delta_{\Omega}(y_i) \geq r/A$. Thus we have

$$k_{\Omega}(y_i, x_0) \leq \frac{A}{cr}d(y_i, x_0) \leq \frac{A}{cr}\text{diam}(\Omega),$$

and as $M_k(x_0) = 1$, by Lemma 4.6 we have $M_k(y_i) \leq \exp \left(\frac{A}{cr}\text{diam}(\Omega)\right)$, and consequently for $x \in \Omega \cap B(\xi, r/4)$,

$$M_k(x) \leq \sum_{i=1}^N \exp \left(\frac{A}{cr}\text{diam}(\Omega)\right) \leq N_0 \exp \left(\frac{A}{cr}\text{diam}(\Omega)\right) =: C_\xi.$$

Since $M = \lim_k M_k$, it follows that $M$ is bounded in a neighborhood of $\xi$ for each $\xi \in \partial \Omega \setminus \{\xi_M\}$. Thus (i) in the theorem is satisfied.

Let us next prove (iii) of the theorem. Since by assumption $\text{Cap}_Q(X \setminus \Omega) > 0$, by the Poincaré inequality we see that $\text{Cap}_Q(\partial \Omega)$ is positive. By Lemma 6.2 we may assume that $\xi \in \partial \Omega$ is $Q$-regular for $\Omega \cap B(\xi, r)$ for every $r > 0$. We can easily see that whenever $\rho > 0$ we have $\text{Cap}_Q(\partial \Omega \setminus \Omega) > 0$. Let $r = r_{\xi}/4$. We have by the above argument that the family $\{M_k\}_k$ is a uniformly bounded family of $Q$-harmonic functions on $\Omega \cap B(\xi, r/4)$; $M_k \leq C_\xi$. Let $f$ be a compactly supported Lipschitz continuous function on $X$ such that $f = C_\xi$ on $S(\xi, r)$ and $f = 0$ on $B(\xi, r/2)$. Then we easily see that

$$M_k(y) \leq f(y) \quad \text{for every } y \in \partial(\Omega \cap B(\xi, r)),$$

since $M_k = 0$ on $X \setminus \Omega$ by the construction of $M_k$ in [HoSh]. Note that $\text{Cap}_Q(\overline{B}(\xi, 2r) \setminus \Omega) > 0$ and that both $f$ and $M_k$ are in the class $N^{1-Q}(\overline{B}(\xi, 2r))$ and are $Q$-harmonic in $B(\xi, r) \cap \Omega$ for sufficiently large $k$. Therefore by the regular comparison theorem (see [Sh2]) we have $M_k \leq H_{Q}^{\Omega \cap B(\xi, r)} f$ on $\Omega \cap B(\xi, r)$. Since $\xi$ is a $Q$-regular boundary point for $\Omega \cap B(\xi, r)$ and $f$ is a continuous boundary data, it follows that for every $\varepsilon > 0$ there can be found $\rho_\varepsilon > 0$ such that $0 < H_{Q}^{\Omega \cap B(\xi, r)} < \varepsilon$ on $\Omega \cap B(\xi, \rho_\varepsilon)$. Hence $0 < M_k(x) < \varepsilon$ on $\Omega \cap B(\xi, \rho_\varepsilon)$ for every $k$. Thus, as $M$ is the pointwise (and locally uniform) limit of $M_k$ in $\Omega$, we see that $M \leq \varepsilon$ on $\Omega \cap B(\xi, \rho_\varepsilon)$. This means that $M$ tends to zero continuously $Q$-quasieverywhere in $\partial \Omega$. This proves (iii) of the theorem.

Now we prove (iv) of the theorem. Suppose that there is no sequence $(x_n)$ in $\Omega$ converging to $\xi_M$ for which $\lim_n M(x_n) = \infty$. Then $M$ is also bounded in a neighborhood of $\xi_M$, and consequently
by (i) of the theorem (proven above), $M$ is a positive $Q$-harmonic function in $\Omega$ that is bounded on $\Omega$ and in addition vanishes $Q$-quasieverywhere in $\partial\Omega$. It then follows from Corollary 6.2 of [BBS2] that $M$ is identically zero on $\Omega$, contradicting the fact that $M(x_0) = 1$. Thus (iv) of the above theorem is also true.

It now only remains to prove (ii). We have already shown that whenever $R > 0$ the function $M$ is bounded in $\Omega \cap B(\xi_M, R/2) \setminus \overline{B}(\xi_M, R/16)$ and vanishes on $\partial \Omega \setminus \{\xi_M\}$. Therefore by Proposition 5.1, if $0 < R < R_0/32$, then $M$ satisfies

$$M(x) \leq C \sum_{i=1}^{N} M(y_i) \quad \text{for } x \in S(\xi_M, R) \cap \Omega,$$

where $y_1, \ldots, y_N \in \Omega \cap S(\xi_M, 2R)$ is a system of local reference points of order $N$ for $\xi_M$. By the comparison theorem we have

$$M(x) \leq C \sum_{i=1}^{N} M(y_i)$$

whenever $x \in \Omega \setminus B(\xi_M, R)$. Now an application of (18) to each of the reference points $y_i$ together with the estimate $\delta_{\Omega}(y_i) \geq R/C$ gives

$$M(y_i) \leq M(x_0) \left( \frac{\delta_{\Omega}(x_0)}{R} \right)^{d} = \left( \frac{\delta_{\Omega}(x_0)}{R} \right)^{d} = CR^{-d}.$$

Hence

$$M(x) \leq C N R^{-d} \leq C N_0 d(x, \xi_M)^{-d} \quad \text{for } x \in \Omega \cap B(\xi_M, 2R) \setminus B(\xi_M, R),$$

whenever $0 < R < R_0/32$. The desired result now follows from the fact that $\Omega$ is bounded.

This completes the proof of Theorem 6.1. \hfill \qed

To obtain Theorems 5.2 and 6.1 as well as other results in this and the previous sections, it suffices to only know that $Q$-quasievery boundary point of $\Omega$ is a “John point”. More specifically, a point $\xi \in \partial \Omega$ is said to be a John point if there is a radius $R_\xi > 0$, a point $x_\xi \in \Omega$, and a constant $C_\xi > 0$ such that for every $x \in B(\xi, R_\xi) \cap \Omega$ there is a compact rectifiable curve $\gamma$ connecting $x$ to $x_\xi$ with $\gamma \subset \Omega \cap B(\xi, C_\xi R_\xi)$ such that for every $y \in \gamma$ we have $\delta_{\Omega}(y) \geq \ell(y_{\gamma})/C_\xi$, see [Sh3]. Examples of domains satisfying the above weak condition include Euclidean domains obtained by pasting outward pointing cusps to a ball.

7 Compactness of $X$

The proof of the results discussed in this note required that the closure of $\Omega$ be compact. In this section we will demonstrate that instead of merely assuming the closure of $\Omega$ be compact we may assume the stronger condition that $X$ is a complete metric space. Since a metric space supporting a doubling measure is totally bounded, if $X$ is complete then it is proper as well and hence bounded domains in $X$ are relatively compact.

If $X$ is not a complete metric space, let $\widehat{X}$ denote the completion of $X$, $\widehat{d}$ the extension of $d$ and let $\widehat{\mu}$ be the extension of $\mu$ to $\widehat{X}$: $\widehat{\mu}(A) = \mu(A \cap X)$ whenever $A \subset \widehat{X}$. 

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Proposition 7.1. If $(X,d,\mu)$ is a metric measure space with doubling measure $\mu$ and supporting a $(1,p)$-Poincaré inequality, then so is $(\widehat{X}, d, \widehat{\mu})$. In this case, $N^{1,p}(X) = N^{1,p}(\widehat{X})$. In addition, if $\mu$ is Ahlfors $Q$-regular, then so is $\widehat{\mu}$.

Proof. We first prove the doubling property of $\widehat{\mu}$. Let $B(x, r)$ be a ball in $\widehat{X}$. If $x \in X$, then the doubling property of $\mu$ itself guarantees the doubling inequality for $\widehat{\mu}$: $\widehat{\mu}(B(x, 2r)) = \mu(B(x, 2r) \cap X) \leq C_{d} \mu(B(x, r) \cap X) = C_{d} \mu(B(x, r)).$ Suppose $x \in \widehat{X} \setminus X$. Then as $\widehat{X}$ is the completion of $X$, there is a point $x' \in X$ such that $d(x, x') < r/8$. Thus by the doubling property of $\mu$,

$$
\widehat{\mu}(B(x, 2r)) \leq \mu(B(x', 3r)) = \mu(B(x', 3r) \cap X) \leq C_{d}^{2} \mu(B(x', 3r/8) \cap X) \leq C_{d}^{2} \mu(B(x, r)) \cap X).$$

That is, $\widehat{\mu}$ is doubling with doubling constant $C_{d}^{3}$. A similar argument demonstrates that if $\mu$ is Ahlfors $Q$-regular then so is $\widehat{\mu}$.

Now we demonstrate that if $\mu$ is doubling and supports a $(1,p)$-Poincaré inequality then $\widehat{\mu}$ also supports a $(1,p)$-Poincaré inequality. To this end, let $u \in N^{1,p}(\widehat{X})$ and $g$ be an upper gradient of $u$ in $\widehat{X}$. Note that $g$ is also an upper gradient of $u$ in $X$ as $X \subset \widehat{X}$. Suppose $B(x, r) \subset \widehat{X}$. If $x \in X$, then by the definition of $\widehat{\mu}$ and by the fact that $\mu$ itself supports a $(1,p)$-Poincaré inequality we have the Poincaré inequality with respect to $\widehat{X}$. If $x \notin X \setminus \widehat{\mu}$, then taking $x' \in X$ such that $d(x, x_{0}) < r/2$, we obtain

$$
\inf_{c \in \mathbb{R}} \int_{B(x, r)} |u - c| \, d\hat{\mu} \leq \inf_{c \in \mathbb{R}} \int_{B(x', 2r)} |u - c| \, d\hat{\mu} \leq \int_{B(x', 2r) \cap X} |u - u_{B(x, 2r) \cap X}| \, d\mu \leq C_{p} 4r \int_{B(x', 2r) \cap X} g^{p} \, d\mu \leq Cr \int_{B(x, 2r) \cap X} g^{p} \, d\mu = Cr \int_{B(x, 3r) \cap X} g^{p} \, d\mu.
$$

Hence the pair $u, g$ satisfies a $(1,p)$-Poincaré inequality on $\widehat{X}$.

Finally, we prove that $N^{1,p}(\widehat{X}) \cap X = N^{1,p}(X)$. Observe that $N^{1,p}(\widehat{X}) \cap X \subset N^{1,p}(X)$. On the other hand, as $X$ supports a $(1,p)$-Poincaré inequality, we know from [Sh1] that Lipschitz functions are dense in $N^{1,p}(X)$. We may extend a Lipschitz function $u \in N^{1,p}(X)$ to $\widehat{X}$, for example via a McShane extension. Since the minimal $p$-weak upper gradient $g_{u}$ of such a function has the property that $g_{u}(x) \approx \text{Lip} \, u(x)$ for $\mu$-a.e. $x \in X$ (see for example [Ch]) and hence for $\widehat{\mu}$-a.e. $x \in \widehat{X}$, we see that $u \in N^{1,p}(\widehat{X})$ with the $N^{1,p}(\widehat{X})$-norm of $u$ equaling the $N^{1,p}(X)$-norm of $u$. Now the density of Lipschitz functions in $N^{1,p}(X)$ guarantees the coincidence of the two function spaces. \hfill \qed

The above result demonstrates that the requirement of $X$ being proper is not a stringent requirement as it appears to be, especially since we do not assume much about $X \setminus \Omega$ apart from the condition that $\text{Cap}_{Q}(X \setminus \Omega) > 0$. The following proposition further strengthens this claim. Note that for every domain $\Omega \subset X$ there is a domain $\Omega_{0} \subset \widehat{X}$ such that $\Omega = \Omega_{0} \cap \widehat{X}$. Let $\Omega_{0}$ denote the largest such domain in $\widehat{X}$.

Proposition 7.2. In the situation of Proposition 7.1, if $\text{Cap}_{Q}(\widehat{X}) \setminus X = 0$ then $\partial_{eM} \Omega = (\Omega_{0} \setminus \Omega) \cup \partial_{eM} \Omega_{0}.$

Proof. In this proof, balls $B(x, r)$ denote balls in $X$; $B(x, r) = \{y \in X : d(y, x) < r\}$. By $\widehat{B}(x, r)$ we then denote the closure of $B(x, r)$ in $\widehat{X}$. 

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Let \((r_i)\) be a sequence of positive real numbers such that \(\lim_i r_i = 0\). If \(y \in \Omega \subset \Omega_0\), the construction of a \(Q\)-singular function on \(\Omega\) with singularity at \(y\) is through a sequence of functions \(u_i \in N_0^{1,Q}(\Omega)\) that are \(Q\)-energy minimizers in \(\Omega \setminus \overline{B(y,r_i)}\), take on the value 1 in \(B(y,r_i)\), and \(0 \leq u_i \leq 1\) on \(X\); see [HoSh]. By a \(Q\)-energy minimizer we mean that for every \(\phi \in N_0^{1,Q}(\Omega \setminus \overline{B(y,r_i)})\),

\[
\int_{\Omega \setminus \overline{B(y,r_i)}} |du_i|^Q \, d\mu \leq \int_{\Omega \setminus \overline{B(y,r_i)}} |d(u_i + \phi)|^Q \, d\mu.
\]

Since by Proposition 7.1 we have \(N_0^{1,Q}(\Omega) \subset N^{1,Q}(X) = N^{1,Q}(\overline{X})\), we see that \(u_i\) is extendable to \(\Omega_0\) so that \(u_i \in N_0^{1,Q}(\Omega_0)\). We claim that \(u_i\) is \(Q\)-harmonic in \(\Omega_0 \setminus \overline{B(y,r_i)}\). Indeed, if \(\varphi \in N_0^{1,Q}(\Omega_0 \setminus \overline{B(y,r_i)})\), then \(\varphi|_X \in N_0^{1,Q}(\Omega)\), and hence

\[
\int_{\Omega_0 \setminus \overline{B(y,r_i)}} |du_i|^Q \, d\mu = \int_{\Omega \setminus \overline{B(y,r_i)}} |du_i|^Q \, d\mu \\
\leq \int_{\Omega_0 \setminus \overline{B(y,r_i)}} |d(u_i + \varphi)|^Q \, d\mu = \int_{\Omega_0 \setminus \overline{B(y,r_i)}} |d(u_i + \varphi)|^Q \, d\mu.
\]

Therefore the singular function \(g(\cdot,y)\) on \(\Omega\) can be extended to be a singular function on all of \(\Omega_0\) with singularity at \(y\).

Now let \(M \in \partial_{cM}\Omega\) and let \((y_n)\) be an associated fundamental sequence in \(\Omega\). As this sequence can have no accumulation point in \(\Omega\), and as \(\Omega_0\) is compact, we see that this sequence has accumulation points in \((\Omega_0 \setminus \Omega) \cup \partial_{cM}\Omega_0\). If it has an accumulation point in \(\Omega_0 \setminus \Omega\), then \(M\) is the normalized singular function on \(\Omega_0\) with singularity at that accumulation point; hence the sequence \((y_n)\) must converge to this unique accumulation point. Otherwise all of the accumulation points of the sequence lie in the set \(\partial\Omega_0\), and hence \(M \in \partial_{cM}\Omega_0\). It therefore follows that \(\partial_{cM}\Omega \subset (\Omega_0 \setminus \Omega) \cup \partial_{cM}\Omega_0\).

To get equality note that every point in \(\Omega_0 \setminus \Omega\) corresponds to a unique Martin kernel function in \(\partial_{cM}\Omega\). Moreover, the restriction of every function in \(\partial_{cM}\Omega_0\) to \(\Omega\) lies in \(\partial_{cM}\Omega\). To see this, note that every point \(y_0 \in \Omega_0\) is a limit of a sequence of points from \(\Omega\); hence every singular function in \(\Omega_0\) with singularity at \(y\) can be approximated to any desired accuracy by a singular function in \(\Omega\) with a singularity at \(y\); this is because by the above discussion singular functions in \(\Omega\) with singularity in \(\Omega\) are extendable to be a singular function in \(\Omega_0\) with singularity in \(\Omega\), and the limit of such a sequence of singular functions is a singular function in \(\Omega_0\) with singularity at \(y\), and only one such singular function exists. Now a diagonalization argument yields that \(\partial_{cM}\Omega_0 \subset \partial_{cM}\Omega\). □

References


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