

BEURLING'S MINIMUM PRINCIPLE AND THE MINIMAL THINNESS

HIROAKI AIKAWA

ABSTRACT. Every domain with the Green function has the Martin boundary and every positive harmonic function on the domain is represented as the integral over the Martin boundary. The classical Fatou theorem concerning nontangential limits of harmonic functions is extended to this general context by using the notion of the minimal thinness. The identification of the Martin boundaries for specific domains is an interesting problem and has attracted many mathematicians. From smooth domains to nonsmooth domains the study of the Martin boundary has been expanded. It is now known that the Martin boundary of a uniform domain is homeomorphic to the Euclidean boundary.

On the other hand the study of the minimal thinness has been exploited a little. The minimal thinness of an NTA domain was characterized by the author in the early 90's. The quasiadditivity of capacity, the Hardy inequality and Beurling's minimum principle played important roles. Maz'ya was the first mathematician who recognized the significance of Beurling's minimum principle.

This article illustrates the backgrounds of the characterization of the minimal thinness and gives some new results. In fact, the minimal thinness of a uniform domain (with the capacity density condition) is characterized.

1. INTRODUCTION

Every domain with the Green function has the Martin boundary and every positive harmonic function on the domain is represented as the integral over the Martin boundary ([Martin \(1941\)](#)). The classical Fatou theorem concerning nontangential limits of harmonic functions ([Fatou \(1906\)](#)) is extended to this general context by using the notion of the minimal thinness ([Naïm \(1957\)](#)). The identification of the Martin boundaries for specific domains is an interesting problem and has attracted many mathematicians. From smooth domains to nonsmooth

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domains the study of the Martin boundary has been developed and a number of papers have been published. It is now known that the Martin boundary of a uniform domain is homeomorphic to the Euclidean boundary (Aikawa (2001)).

On the other hand the study of the minimal thinness has been exploited a little. See e.g. Lelong-Ferrand (1949), Essén (1988), Essén (1993), Ancona (1983), Ancona (1985), Aikawa (1985), Aikawa (1991), Aikawa (1993), Zhang (1987), Gardiner (1993), Miyamoto, et al. (2005), and Miyamoto & Yanagishita (2005). The minimal thinness of an NTA domain was characterized by the author in the early 90's (Aikawa (1991), and Aikawa (1993)). The quasiadditivity of capacity, the Hardy inequality and Beurling's minimum principle played important roles (Beurling (1965), Sjögren (1977), Ancona (1986), Aikawa (1991), Aikawa (1993)).

Maz'ya was the first mathematician who recognized the significance of Beurling's minimum principle (Maz'ya (1972)). The name "Beurling's theorem" is due to him. Unfortunately, his work was buried in the literature; Dahlberg proved the same results independently (Dahlberg (1976)).

The author is grateful to Professor Vladimir Maz'ya for drawing his attention to the paper Maz'ya (1972) on Beurling's minimum principle. Professor Maz'ya informed directly the author for the paper on the occasion of the international conference on potential theory in Amersfort (ICPT91). This information was very fruitful for the author.

Starting with a heuristic introduction of the Martin boundary, this article illustrates the backgrounds of the minimal thinness and gives some new results, i.e., characterizations of the minimal thinness in a uniform domain.

2. MARTIN BOUNDARY

Throughout this paper let D be a proper domain in \mathbb{R}^n with $n \geq 2$. Let $\delta_D(x) = \text{dist}(x, \partial D)$. We write $B(x, r)$ and $S(x, r)$ for the open ball and the sphere of center at x and radius r , respectively. By the symbol M we denote an absolute positive constant whose value is unimportant and may change from one occurrence and the next. If necessary, we use M_0, M_1, \dots , to specify them. If two positive quantities f and g satisfies $M^{-1} \leq f/g \leq M$ with some constant $M \geq 1$, then we say f and g are comparable and write $f \approx g$.

2.1. General Martin boundary theory. Let us begin with a heuristic introduction of the Martin boundary (Martin (1941)). Let $G_D(x, y)$ be the normalized Green function for D , i.e.,

- (i) For each $x \in D$ fixed, $G_D(x, \cdot) = 0$ on ∂D except for a polar set.
- (ii) For each $y \in D$ fixed, $\Delta G_D(\cdot, y) = -\delta_y$, where Δ and δ_y stand for the distributional Laplacian and the Dirac measure at y , respectively.

For simplicity we write $G(x, y)$ for $G_D(x, y)$.

If D is sufficiently smooth, then we have the Poisson integral formula: every harmonic function h on D with continuous boundary values is represented as

$$(1) \quad h(x) = \int_{\partial D} \frac{\partial G(x, y)}{\partial n_y} h(y) d\sigma(y),$$

where n_y and σ stand for the inward normal and the surface measure on ∂D . The kernel $\frac{\partial G(x, y)}{\partial n_y}$ is called the Poisson kernel.

Let us try to generalize (1) as far as possible. First, let us extend the representation to an arbitrary positive harmonic function, so that the Poisson kernel itself can be represented. To this end, we extend the measure $h(y)d\sigma(y)$ to a general measure on ∂D . We have the following *Herglotz theorem*: every nonnegative harmonic function h on D has a measure μ_h on ∂D such that

$$(2) \quad h(x) = \int_{\partial D} \frac{\partial G(x, y)}{\partial n_y} d\mu_h(y) \quad \text{for } x \in D.$$

Next, let us extend (2) to more general domains, which may not have normals. For this purpose we need to understand the normal derivative $\partial G(x, y)/\partial n_y$ in an appropriate way. Recall that $\partial/\partial n_y$ is decomposed into two operations:

- (i) divide by the distance $\delta_D(z)$ for $z \in D$;
- (ii) take the limit as $z \rightarrow y$, i.e.

$$\lim_{z \rightarrow y} \frac{G(x, z)}{\delta_D(z)}.$$

We fix $x_0 \in D$. Then, it is known that

$$G(x_0, z) \approx \delta_D(z) \quad \text{for } y \in D \text{ close to the boundary } \partial D,$$

provided D is a smooth domain, say $C^{1,\alpha}$ -domain ($0 < \alpha < 1$), or more generally a Liapunov-Dini domain ([Widman \(1967\)](#)).

Thus the limit of the ratio of the Green functions $\lim_{z \rightarrow y} \frac{G(x, z)}{G(x_0, z)}$ would be a substitute of the normal derivative. This limit can be considered, whenever the domain has the Green function. The regularity of the

boundary has no significance. This limit will be called the Martin kernel $K(x, y)$ and (2) will be generalized to the Martin integral representation:

$$h(x) = \int K(x, y) d\mu_h(y) \quad \text{for } x \in D.$$

Let us give more rigorous explanation. Consider the ratio of the Green functions

$$K(x, y) = K_y(x) = \frac{G(x, y)}{G(x_0, y)} \quad \text{for } y \in D \setminus \{x_0\}.$$

Observe that K_y is a positive harmonic function on $D \setminus \{y\}$. Suppose $\{y_j\}_j \subset D$ is a sequence without accumulation points in D . On any relatively compact subdomain, $\{K_{y_j}\}_j$ becomes a sequence of positive harmonic functions with $K_{y_j}(x_0) = 1$ for large j . Hence the Harnack principle implies that they are locally uniformly bounded, so that there is a subsequence converging to a positive harmonic function h^* . For simplicity, we assume that K_{y_j} itself converges to h^* . We regard that $\{y_j\}_j$ defines the boundary point ξ and write

$$h^* = K(\cdot, \xi).$$

We do this for all sequences $\{y_j\}_j$ and obtain a compactification \widehat{D} of D . This compactification is called the *Martin compactification* of D and the boundary $\widehat{D} \setminus D = \Delta$ is called the *Martin boundary*. Note that $K(x, \cdot)$ is continuous on $\widehat{D} \setminus \{x_0\}$ for each $x \in D$ and that $\{K(x, \cdot) : x \in D\}$ separates the boundary point in Δ . In fact, these two properties characterize the Martin compactification, which is known as Q -compactification. See [Constantinescu & Cornea \(1963\)](#).

For a nonnegative superharmonic function u on D and $E \subset D$ we denote by \widehat{R}_u^E the *regularized reduced function* of u on E (see e.g. [Helms \(1975\)](#)). We remark that if E is a compact subset of D , then $\widehat{R}_u^E = u$ in the interior of E ; and $v = \widehat{R}_u^E$ is the Dirichlet solution to $\Delta v = 0$ on $D \setminus E$ with boundary data $v = u$ on ∂E and $v = 0$ on ∂D .

The Martin compactification is closely related to the integral representation of positive harmonic functions h on D . Using the exhaustion of D and the regularized reduced functions, we easily observe that there is a measure μ on Δ such that

$$h(x) = \int_{\Delta} K(x, \xi) d\mu(\xi) \quad \text{for } x \in D.$$

See [Helms \(1975, Chapter 12\)](#). Unfortunately, the measure μ is not necessarily unique. In fact, if there is a point $\xi_0 \in \Delta$ such that

$$K(x, \xi_0) = \sum \alpha_j K(x, \xi_j)$$

with $\xi_j \in \Delta \setminus \{\xi_0\}$ and positive numbers α_j such that $\sum \alpha_j = 1$, then the measures μ and

$$\mu + \mu(\{\xi_0\})(\delta_{\xi_0} - \sum \alpha_j \delta_{\xi_j})$$

define the same harmonic function h ; these two measures do not coincide unless $\mu(\{\xi_0\}) = 0$. Thus we have to eliminate such points ξ_0 .

We say that a positive harmonic function h on D is *minimal* if any other positive harmonic function less than h coincides with a constant multiple of h . We let

$$\Delta_1 = \{y \in \Delta : K(\cdot, \xi) \text{ is a minimal harmonic function}\}.$$

The set Δ_1 is called the *minimal Martin boundary*. The complement $\Delta \setminus \Delta_1$ is the non minimal Martin boundary and is denoted by Δ_0 . Now we can state the Martin representation theorem.

Theorem A. *For a positive harmonic function h there exists a unique measure μ_h on Δ_1 such that*

$$h(x) = \int_{\Delta_1} K(x, \xi) d\mu_h(\xi) \quad \text{for } x \in D.$$

Combining the Riesz decomposition theorem, we obtain

Theorem B (Riesz-Martin representation theorem). *For a nonnegative superharmonic function u on D there exists a unique measure μ_u on $D \cup \Delta_1$ such that*

$$u(x) = \int_{D \cup \Delta_1} K(x, \xi) d\mu_u(\xi) \quad \text{for } x \in D.$$

For simplicity, we write $K\mu$ for $\int_{D \cup \Delta_1} K(\cdot, y) d\mu(y)$, if μ is a measure on $D \cup \Delta_1$.

2.2. Identification of the Martin boundary. So far, we have argued the abstract Martin theory. The identification of the Martin boundaries of specific domains is an interesting different problem and has attracted number of mathematicians. As one may expect, if D enjoys geometrical smoothness, then $\Delta = \Delta_1 = \partial D$.

Definition 1 (Jerison & Kenig (1982)). We say that a domain is an NTA domain if there exist positive constants M and r_1 such that

- (i) Corkscrew condition. For any $\xi \in \partial D$ and small positive r there exists a point $A_\xi(r) \in D \cap S(\xi, r)$ such that $\delta_D(A_\xi(r)) \approx r$. We refer to $A_\xi(r)$ as a *nontangential point at ξ* .
- (ii) The complement of D satisfies the corkscrew condition.

- (iii) Harnack chain condition. If $\varepsilon > 0$ and $x, y \in D$, $\delta_D(x) \geq \varepsilon$, $\delta_D(y) \geq \varepsilon$ and $|x - y| \leq M\varepsilon$, then there exists a Harnack chain from x and y whose length is independent of ε .

Uniform domains are more general. Roughly speaking, D is a uniform domain if D enjoys the above (i) and (iii).

Definition 2 (Gehring & Osgood (1979)). We say that D is a uniform domain if there exist constants M and M' such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable curve $\gamma \subset D$ for which

$$\begin{aligned} \ell(\gamma) &\leq M|x_1 - x_2|, \\ \min\{\ell(\gamma(x_1, y)), \ell(\gamma(x_2, y))\} &\leq M'\delta_D(y) \quad \text{for all } y \in \gamma. \end{aligned}$$

Here, $\ell(\gamma)$ and $\gamma(x_j, y)$ denote the length of γ and the subarc of γ connecting x_j and y , respectively.

Remark 1. Let D be a uniform domain and let $\xi \in \partial D$. For small $r > 0$ we have a nontangential point $A_\xi(r) \in D \cap S(\xi, r)$ at ξ .

Note that a uniform domain need not satisfy the exterior condition (ii), so that it may be irregular for the Dirichlet problem. Even in this nasty situation we have the following.

Theorem C (Aikawa (2001)). *Let D be a uniform domain. Then the Martin boundary of D is homeomorphic to its Euclidean boundary ∂D ; and each boundary point is minimal. This may be symbolically written as*

$$\Delta = \Delta_1 = \partial D.$$

3. MINIMAL THINNESS

3.1. Minimal thinness and the Fatou-Näim-Doob theorem. Let us consider the boundary behavior of nonnegative superharmonic functions. For this purpose, the notion of the minimal thinness is important.

Definition 3. Let $\xi \in \Delta_1$ and $E \subset D$. We say that E is minimally thin at ξ if

$$\widehat{R}_{K_\xi}^E \neq K_\xi.$$

This is equivalent to the fact that $\widehat{R}_{K_\xi}^E$ is a Green potential. We say that a function f on D has the minimal fine limit l at ξ if there is a set E minimally thin at ξ such that

$$\lim_{x \rightarrow \xi, x \in D \setminus E} \frac{u(x)}{K_\xi(x)} = 0.$$

We write

$$\text{mf lim}_{x \rightarrow \xi} \frac{u(x)}{K_\xi(x)} = l$$

if f has the minimal fine limit l at ξ .

Using the notion of minimal thinness, we extend the classical Fatou theorem. The extension is referred to as the Fatou -Naïm-Doob Theorem. See [Naïm \(1957\)](#).

Theorem D (Fatou-Naïm-Doob Theorem). *Let u and v be positive harmonic functions on D and let $v = K\mu_v$ with the Martin representation measure μ_v . Then the ratio u/v has the minimal fine limit at μ_v -a.e. boundary points in Δ_1 .*

The above theorem is rather abstract. The identification of the minimal thinness for specific domains is an interesting problem. However, there have been a few papers dealing with this problem, and there a lot remain mysterious for the minimal thinness; whereas the Martin boundaries of many of nonsmooth domains have

3.2. Wiener type criteria for the minimal thinness. For the upper half space [Lelong-Ferrand \(1949\)](#) gave a Wiener-type criterion for the minimal thinness. In the rest of this section, we let D be the upper half space $\mathbb{R}_+^n = \{x : x_n > 0\}$. Following [Essén & Jackson \(1980, Definition 2.2\)](#), we shall define the Green energy. For simplicity we let $n \geq 3$. Observe that the Martin kernel at $y \in \partial D$ has the following form:

$$K(x, y) = \begin{cases} \frac{x_n}{|x - y|^n} & \text{if } y \in \partial D \\ x_n & \text{if } y = \infty \end{cases}$$

up to a positive continuous function of y . We define the Green energy, a kind of capacity, as follows. Let E be a compact set. Consider the regularized reduced function $\widehat{R}_{x_n}^E$, which can be represented as the Green potential $G\mu_E$ of a measure μ_E on E , i.e.,

$$\widehat{R}_{x_n}^E = \int_E G(x, y) d\mu_E(y).$$

We let

$$\gamma(E) = \iint G(x, y) d\mu_E(x) d\mu_E(y),$$

and call it the Green energy of E . In a standard way, $\gamma(E)$ is extended to open sets, and then to general sets (See [Definition 4](#) below). We write $\gamma(E)$ for the extension as well. Observe that $\gamma(E)$ is homogeneous of degree n , i.e., $\gamma(rE) = r^n \gamma(E)$ for $r > 0$. This is because the kernel

$\frac{G(x, y)}{x_n y_n}$ is homogeneous of degree $-n$. In view of the homogeneity, we

let $I_i = \{x : 2^{-i-1} \leq |x| < 2^{-i}\}$ and consider the series $\sum_{i=1}^{\infty} 2^{in} \gamma(E \cap I_i)$.

Then, we have the following characterization.

Theorem E (Lelong-Ferrand (1949)). *Let $E \subset \mathbb{R}_+^n$. Then the following are equivalent:*

- (i) E is minimally thin at θ .
- (ii) $\sum_{i=1}^{\infty} 2^{in} \gamma(E \cap I_i) < \infty$.

The above theorem gives a complete characterization of minimal thinness for the upper half space. This can be generalized to a uniform domain. For this purpose we extend the notion of Green energy.

Definition 4. Let u be a nonnegative superharmonic function on D . For a compact subset K of D we let \widehat{R}_u^K be the regularized reduced function of u with respect to K . Observe that \widehat{R}_u^K is a Green potential of a measure λ_u^K on K . The energy

$$\gamma_u(K) = \iint G(x, y) d\lambda_u^K(x) d\lambda_u^K(y)$$

is called the Green energy of K relative to u . For an open subset V of D , we let

$$\gamma_u(V) = \sup\{\gamma_u(K) : K \text{ is compact, } K \subset V\},$$

and then, for a general subset E of D ,

$$\gamma_u(E) = \inf\{\gamma_u(V) : V \text{ is open, } E \subset V\}.$$

The quantity $\gamma_u(E)$ is also called the Green energy relative to u .

Remark 2. If $u \equiv 1$, then $\gamma_u(E)$ is the usual Green capacity $C_G(E)$ (see Landkof (1972, pp.174–177)). If $D = \{x = (x_1, \dots, x_n) : x_n > 0\}$ and $u(x) = x_n$, then $\gamma_u(E)$ is the Green energy defined by Essén & Jackson (1980, Definition 2.2).

We let $u(x) = g(x) = \min\{G(x, x_0), 1\}$ in Definition 4. Then

$$(3) \quad \widehat{R}_g^E(x_0) = \gamma_g(E) \quad \text{for } E \subset \{x \in D : g(x) < 1\}.$$

Theorem 1. *Let D be a uniform domain. Suppose that $E \subset D$ and $\xi \in \partial D$. Let $A_\xi(r) \in D \cap S(\xi, r)$ be a nontangential point for small $r > 0$ (Remark 1). Then the following are equivalent:*

- (i) E is minimally thin at ξ .

(ii) For some $i_0 \geq 1$

$$(4) \quad \sum_{i=i_0}^{\infty} 2^{i(n-2)} g(A_\xi(2^{-i}))^{-2} \gamma_g(E \cap B(\xi, 2^{1-i}) \setminus B(\xi, 2^{-i})) < \infty.$$

Since $g(x) \approx \delta_D(x)$ for a smooth domain D , say a $C^{1,\alpha}$ -domain ($0 < \alpha < 1$), or more generally a Liapunov-Dini domain (Widman (1967)), this theorem is a generalization of Theorem E. We have the following corollary.

Corollary 1. *Let D be a uniform domain in \mathbb{R}^n with $n \geq 3$. Suppose that $E \subset D$ and $\xi \in \partial D$. Then*

- (i) *If E is (ordinary) thin at ξ , then E is minimally thin at ξ .*
- (ii) *If E is included in a nontangential cone with vertex at ξ , then E is thin at ξ if and only if E is minimally thin at ξ .*

3.3. Proof of Theorem 1. Let $\Theta(x, y) = G(x, y)/(g(x)g(y))$. In view of Theorem C, we obtain that $\Theta(x, y)$ has a continuous extension on $\overline{D} \times \overline{D}$. By the same symbol we denote the continuous extension. The kernel Θ is referred to as Naïm's Θ kernel for D (Naïm (1957)). By definition Θ is symmetric.

Lemma 1. *Let $\xi \in \partial D$. For $r > 0$ small we let $\theta_\xi(r) = \Theta(A_\xi(r), \xi)$ with $A_\xi(r) \in D \cap S(\xi, r)$ (Remark 1). Then*

$$(5) \quad \theta_\xi(r) \approx g(A_\xi(r))^{-2} r^{2-n}.$$

Moreover, if $x \in S(\xi, r) \cap D$, then $\Theta(x, \xi) \approx \theta_\xi(r)$. Furthermore, if $r \approx R$, then $\theta_\xi(r) \approx \theta_\xi(R)$.

Proof. This is an easy consequence of the scale-invariant boundary Harnack principle. More directly, Aikawa (2001, Lemma 3) gives the estimates as follows. Note that the Green function for $D \cap B(\xi, Mr)$ is comparable with the original Green function for D near ξ , provided D satisfies the capacity density condition. We have

$$\frac{G(x, y)}{G(x', y)} \approx \frac{G(x, y')}{G(x', y')} \quad \text{for } x, x' \in D \setminus B(\xi, r) \text{ and } y, y' \in D \cap B(\xi, r/6).$$

By elementary calculations we have

$$\Theta(x, y) \frac{g(x)g(y)}{G(x', y)G(x, y')} \approx \frac{1}{G(x', y')},$$

where we recall $\Theta(x, y)$ is the continuous extension of $G(x, y)/(g(x)g(y))$. Letting $x' = A_\xi(r)$ and $y' = A_\xi(r/6)$, we obtain that $G(x', y') \approx r^{2-n}$, so that

$$\Theta(x, y) \frac{g(x)g(y)}{G(A_\xi(r), y)G(x, A_\xi(r/6))} \approx r^{n-2}.$$

Hence

$$\Theta(x, y) \approx \frac{r^{n-2}G(A_\xi(r), y)G(x, A_\xi(r/6))}{g(x)g(y)}.$$

Letting $x = A_\xi(r)$, we obtain

$$\begin{aligned} \Theta(A_\xi(r), y) &\approx \frac{r^{n-2}G(A_\xi(r), y)G(A_\xi(r), A_\xi(r/6))}{g(A_\xi(r))g(y)} \approx \frac{G(A_\xi(r), y)}{g(A_\xi(r))g(y)} \\ &= g(A_\xi(r))^{-1} \frac{G(A_\xi(r), y)}{g(y)}. \end{aligned}$$

The last term tends to

$$g(A_\xi(r))^{-1}K(A_\xi(r), \xi) \approx g(A_\xi(r))^{-2}r^{2-n}$$

as $y \rightarrow \xi$. Thus, (5) follows. The remaining can be easily proved by the Harnack inequality. \square

Proof of Theorem 1. In the same way as in [Aikawa \(1985\)](#), we can prove that E is minimally thin at ξ if and only if

$$(6) \quad \sum_{i=i_0}^{\infty} \widehat{R}_{K_\xi}^{E_i}(x_0) < \infty \quad \text{for some } i_0 \geq 1,$$

where $E_i = \{x \in E : 2^{-i} \leq |x - \xi| < 2^{1-i}\}$. By Lemma 1 we see that $K_\xi \approx \theta_\xi(2^{-i})g$ on $\{x \in D : 2^{-i} \leq |x - \xi| < 2^{1-i}\}$. Hence (6) is equivalent to

$$(7) \quad \sum_i \theta_\xi(2^{-i})\gamma_g(E_i) < \infty.$$

The proof is complete. \square

3.4. Whitney decomposition and the minimal thinness. The Green energy in Theorem 1 involves the Green function, and hence the above Wiener-type criterion is not so clear as the classical Wiener criterion.

Is it possible to characterize the minimal thinness by a usual (or Newtonian) capacity? This is a question raised by Hayman. Several results were given by Aikawa, Essén, Jackson, Rippon, Miyamoto, Yanagishita, Yoshida and others. It turned out that the key is the Whitney decomposition, a decomposition finer than the usual dyadic spherical decomposition. Here, we say that a family of closed cubes $\{Q_j\}$ with sides parallel to the coordinate axes is the *Whitney decomposition* of an open set D with $\partial D \neq \emptyset$ if

- (i) $\bigcup_j Q_j = D$.
- (ii) The interiors of Q_j are mutually disjoint.

(iii) Each Q_j satisfies

$$\text{diam}(Q_j) \leq \text{dist}(Q_j, \partial D) \leq 4 \text{diam}(Q_j).$$

See e.g. [Stein \(1970, Chapter VI\)](#) for details.

One of the reasons for considering the Whitney decomposition may be as follows: In a Whitney cube Q_j the Green function and the Newtonian kernel are comparable for $n \geq 3$, i.e.,

$$G(x, y) \approx \frac{|x - y|^{2-n}}{\text{diam}(Q_j)^2} \quad \text{for } x, y \in Q_j,$$

and the Green energy of is estimated as

$$\gamma(E) \approx \text{diam}(Q_j)^2 \text{Cap}(E) \quad \text{for } E \subset Q_j,$$

where Cap stands for the Newtonian capacity. If $n = 2$, then \mathbb{R}^n is recurrent and the fundamental harmonic function $-\log|x - y|$ takes the both sign. Hence the logarithmic capacity, the counterpart of the Newtonian capacity, is defined in a special fashion. See [Armitage & Gardiner \(2001\)](#), particularly p.150, for the logarithmic capacity. They illustrate the different appearances between the cases $n = 2$ and $n \geq 3$. By using the same symbol Cap for the logarithmic capacity, we can summarize: if $E \subset Q_j$, then

$$\gamma(E) \approx \begin{cases} \frac{\text{diam}(Q_j)^2}{\log(4 \text{diam}(Q_j) / \text{Cap}(E))} & \text{for } n = 2, \\ \text{diam}(Q_j)^2 \text{Cap}(E) & \text{for } n \geq 3, \end{cases}$$

provided D is sufficiently smooth.

We shall give further details after introducing Beurling's minimum principle. It will turn out that Beurling's minimum principle and the minimal thinness are inextricably related.

4. BEURLING'S MINIMUM PRINCIPLE

[Beurling \(1965\)](#) proved the following theorem. For a moment let D be a simply connected planar domain with Green function G_D and let $\xi \in \partial D$. Let $K(\cdot, \xi)$ be the Martin kernel at ξ in D . Suppose $S = \{z_j\}_1^\infty \subset D$ is a sequence of points in D converging to ξ as $j \rightarrow \infty$.

Definition 5. We say that S is an *equivalence sequence* for ξ if

$$h(z_j) \geq \lambda K(z_j, \xi) \text{ for } j = 1, 2, \dots, \implies h(z) \geq \lambda K(z, \xi) \text{ for all } z \in D,$$

whenever $\lambda > 0$ and h is a positive harmonic function on D .

Theorem F (Beurling's minimum principle). *Let D , ξ and S be as above. Then S is an equivalence sequence for ξ if and only if it contains a subsequence $\{z_{j_k}\}_k^\infty$ with the following two properties:*

- (i) $\sup_k G_D(z_{j_k}, z_{j_l}) < \infty$ whenever $k \neq l$.
- (ii) $\sum_{k=1}^{\infty} G_D(z, z_{j_k}) K(z_{j_k}, \xi) = \infty$ for $z \in D$.

This beautiful theorem has been attracted many mathematicians: [Armitage & Kuran \(1971\)](#), [Maz'ya \(1972\)](#), [Sjögren \(1976\)](#), [Sjögren \(1977\)](#), [Gardiner \(1993\)](#), [Essén \(1993\)](#), [Aikawa \(1991\)](#), [Aikawa \(1993\)](#), [Miyamoto et al. \(2005\)](#).

We recall that if D is the unit disk with center at 0, then the Green function is

$$G_D(z, w) = \log \left| \frac{1 - \bar{z}w}{z - w} \right| \quad \text{for } z, w \in D,$$

where \bar{z} is the complex conjugate of z . This explicit form gives

$$G_D(z, w) \approx \frac{\delta_D(z)\delta_D(w)}{|z - w|^2}$$

whenever

$$(8) \quad |z - w| \geq \frac{1}{2} \min\{\delta_D(z), \delta_D(w)\}.$$

In fact, for $z, w \in D$

$$G_D(z, w) = \log \left(1 + \frac{(2\delta_D(z) - \delta_D(z)^2)(2\delta_D(w) - \delta_D(w)^2)}{|z - w|^2} \right) \leq \frac{4\delta_D(z)\delta_D(w)}{|z - w|^2},$$

where the last inequality follows from the elementary inequality $\log(1 + t) \leq t$ for $t > 0$. If, moreover, z and w satisfy (8), then

$$\frac{4\delta_D(z)\delta_D(w)}{|z - w|^2} \leq 32,$$

so that the concavity of $\log(1 + t)$ gives

$$G_D(z, w) \geq \frac{\log 33}{32} \cdot \frac{4\delta_D(z)\delta_D(w)}{|z - w|^2}.$$

Analogous estimates extend to the higher dimensional case. Let D be a $C^{1,\alpha}$ -domain or more generally a Liapunov-Dini domain in \mathbb{R}^n with $n \geq 2$. Then

$$G(x, y) \approx \frac{\delta_D(x)\delta_D(y)}{|x - y|^n}$$

provided $|x - y| \geq \frac{1}{2} \min\{\delta_D(x), \delta_D(y)\}$. See [Widman \(1967\)](#). Here, the constant $\frac{1}{2}$ has no significance; it can be any positive number less than 1. It is easy to see that $G(x, y) \leq c$ if and only if $|x - y| \geq M \min\{\delta_D(x), \delta_D(y)\}$, where M depends only on c . So, we give the following definition.

Definition 6. We say that a sequence $\{x_j\} \subset D$ is *separated* if $|x_j - x_k| \geq M\delta_D(x_j)$ for $j \neq k$ with some positive constant M independent of j and k .

It is also easy to see that the Poisson kernel satisfies

$$P(x, \xi) \approx \frac{\delta_D(x)}{|x - \xi|^n} \quad \text{for } \xi \in \partial D \text{ and } x \in D,$$

whenever D is a $C^{1,\alpha}$ -domain or more generally a Liapunov-Dini domain. Since the Martin kernel is the multiple of the Poisson kernel and a positive continuous function of $\xi \in \partial D$, we can generalize Theorem F to the following theorem.

Theorem G (Maz'ya (1972), Dahlberg (1976)). *Let D be a $C^{1,\alpha}$ -domain or more generally a Liapunov-Dini domain in \mathbb{R}^n with $n \geq 2$. Suppose $\xi \in \partial D$ and a sequence S converges to ξ . Then S is an equivalence sequence for ξ if and only if it contains a separated subsequence $\{x_j\}_j$ such that*

$$\sum_j \left(\frac{\delta_D(x_j)}{|x_j - \xi|} \right)^n = \infty.$$

The above theorem was proved by hard calculus with the aid of the estimates of the Green function and its derivatives (Widman (1967)). The author (Aikawa (1991) and Aikawa (1993)) observed that it can be derived from the *quasiadditivity of capacity*, which will be illustrated in the succeeding sections.

5. QUASIADDITIVITY OF CAPACITY (GENERAL THEORY)

Following Aikawa (1993), we state the quasiadditivity of capacity in a general setting. The quasiadditivity will play a crucial role for refined Wiener-type criteria.

Let X be a locally compact Hausdorff space and let k be a nonnegative lower semicontinuous function on $X \times X$. For a measure μ on X we write

$$k(x, \mu) = \int_X k(x, y) d\mu(y) \quad \text{for } x \in X.$$

Define the capacity C_k with respect to k by

$$C_k(E) = \inf \{ \|\mu\| : k(\cdot, \mu) \geq 1 \text{ on } E \},$$

where $\|\mu\|$ stands for the total mass of μ . It is well known that C_k is countably subadditive, i.e.

$$C_k(E) \leq \sum_j C_k(E_j) \quad \text{if } E = \bigcup_j E_j.$$

This section is devoted to a reverse inequality, up to a multiplicative constant, holds for some decomposition of E .

Definition 7. Let $\{Q_j\}$ and $\{Q_j^*\}$ be families of Borel subsets of X such that

- (i) $Q_j \subset Q_j^*$,
- (ii) $X = \bigcup_j Q_j$,
- (iii) Q_j^* do not overlap so often, i.e., $\sum \chi_{Q_j^*} \leq N$.

Then we say that $\{Q_j, Q_j^*\}$ is a *quasidisjoint decomposition* of X . If Q_j^* can be understood from the context, we suppress Q_j^* and simply write $\{Q_j\}$.

We note that Q_j and Q_j^* need not be disjoint.

Definition 8. Let σ be a (Borel) measure on X . We say that σ is *comparable to C_k with respect to $\{Q_j\}$* if

- (9) $\sigma(Q_j) \approx C_k(Q_j)$ for every Q_j ,
- (10) $\sigma(E) \leq MC_k(E)$ for every Borel set $E \subset X$.

Definition 9. We say that the kernel k has *the Harnack property with respect to $\{Q_j, Q_j^*\}$* if

$$k(x, y) \approx k(x', y) \quad \text{for } x, x' \in Q_j \text{ and } y \in X \setminus Q_j^*$$

with the constant of comparison independent of Q_j .

Let us state [Aikawa \(1993, Theorem 1\)](#) with its proof.

Theorem 2. *Let $\{Q_j, Q_j^*\}$ be a quasidisjoint decomposition of X . Suppose that the kernel k has the Harnack property with respect to $\{Q_j, Q_j^*\}$. If there is a measure σ comparable to C_k with respect to $\{Q_j, Q_j^*\}$, then for every $E \subset X$*

$$(11) \quad C_k(E) \approx \sum_j C_k(E \cap Q_j).$$

We shall say that C_k is quasiadditive with respect to $\{Q_j, Q_j^\}$ if (11) holds.*

Proof. Let $E \subset X$. We may assume that $C_k(E) < \infty$. By definition we can find a measure μ such that $k(\cdot, \mu) \geq 1$ on E and $\|\mu\| \leq 2C_k(E)$. For each Q_j we have the following two cases:

- (i) $k(x, \mu|_{Q_j^*}) \geq \frac{1}{2}$ for all $x \in E \cap Q_j$.
- (ii) $k(x, \mu|_{Q_j^*}) \geq \frac{1}{2}$ for some $x \in E \cap Q_j$.

If (i) holds, then $C_k(E \cap Q_j) \leq 2\|\mu|_{Q_j^*}\|$ by definition. Since Q_j^* do not overlap so often, we obtain

$$(12) \quad \sum' C_k(E \cap Q_j) \leq 2 \sum' \|\mu_j\| \leq M\|\mu\| \leq MC_k(E),$$

where \sum' denotes the summation over all Q_j for which (i) holds. If (ii) holds, then the Harnack property of k yields that $k(\cdot, \mu) \geq k(\cdot, \mu|_{X \setminus Q_j^*}) \geq M$ on Q_j , so that

$$k(\cdot, \mu) \geq M \text{ on } \bigcup'' Q_j,$$

where \bigcup'' denotes the union over all Q_j for which (ii) holds. Hence

$$C_k\left(\bigcup'' Q_j\right) \leq M\|\mu\|.$$

Since σ is comparable to C_k , it follows from the countable additivity of σ that

$$\begin{aligned} \sum'' C_k(E \cap Q_j) &\leq \sum'' C_k(Q_j) \leq M \sum'' \sigma(Q_j) \\ &\leq M\sigma\left(\bigcup'' Q_j\right) \leq MC_k\left(\bigcup'' Q_j\right) \leq M\|\mu\| \leq MC_k(E). \end{aligned}$$

This, together with (12), completes the proof. \square

6. QUASIADDITIVITY OF THE GREEN ENERGY

It is, in general, difficult to find a measure comparable to a given capacity. In [Aikawa \(1991\)](#), we found such a measure for the Riesz capacity with the aid of a certain weighted norm inequality. Let us observe that Hardy's inequality is a better tool.

Let us begin with the capacity density condition. We recall that Cap stands for the logarithmic capacity if $n = 2$, and the Newtonian capacity if $n \geq 3$.

Definition 10. We say that D satisfies the capacity density condition if there exist constants $M > 1$ and $r_0 > 0$ such that

$$\text{Cap}(B(\xi, r) \setminus D) \geq \begin{cases} M^{-1}r & \text{if } n = 2, \\ M^{-1}r^{n-2} & \text{if } n \geq 3, \end{cases}$$

whenever $\xi \in \partial D$ and $0 < r < r_0$.

[Ancona \(1986\)](#) gave the most general Hardy inequality for domains with the capacity density condition. Here, we remark that a domain satisfies the capacity density condition if and only if it is uniformly Δ -regular in the sense of [Ancona \(1986\)](#).

Lemma A (Hardy's inequality). *Assume that D enjoys the capacity density condition. Then, there is a positive constant M depending only on D such that*

$$\int_D \left| \frac{\psi(x)}{\delta_D(x)} \right|^2 dx \leq M \int_D |\nabla \psi(x)|^2 dx \quad \text{for all } \psi \in W_0^{1,2}(D),$$

where $W_0^{1,2}(D)$ stands for the usual Sobolev space, namely the completion of $C_0^\infty(D)$ with norm $\left(\int_D (|\psi|^2 + |\nabla \psi|^2) dx \right)^{1/2}$.

Let u be a positive superharmonic function on D . Define the measure σ_u on D by

$$\sigma_u(E) = \int_E \left(\frac{u(x)}{\delta_D(x)} \right)^2 dx \quad \text{for } E \subset D.$$

Let $k(x, y) = G(x, y)/(u(x)u(y))$. Then, it is not so difficult to see that $\gamma_u(E) = C_k(E)$ (see [Fuglede \(1965\)](#)). Let $\{Q_j\}$ be the Whitney decomposition of D . For each Whitney cube Q_j we let x_j and Q_j^* be the center and the double of Q_j , respectively. Suppose that u satisfies the Harnack property with respect to $\{Q_j, Q_j^*\}$. It is easy to see that for $x, y \in Q_j$,

$$k(x, y) \approx \begin{cases} u(x_j)^{-2} \log \frac{4 \operatorname{diam}(Q_j)}{|x-y|} & \text{if } n = 2, \\ u(x_j)^{-2} |x-y|^{2-n} & \text{if } n \geq 3. \end{cases}$$

Hence we have the following lemma.

Lemma 2. *Let $\{Q_j\}$ be the Whitney decomposition of D with doubles $\{Q_j^*\}$. Suppose that u is a positive superharmonic function on D satisfying the Harnack property with respect to $\{Q_j, Q_j^*\}$. If $E \subset Q_j$, then*

$$\gamma_u(E) \approx \begin{cases} \frac{u(x_j)^2}{\log(4 \operatorname{diam}(Q_j)/\operatorname{Cap}(E))} & \text{for } n = 2, \\ u(x_j)^2 \operatorname{Cap}(E) & \text{for } n \geq 3. \end{cases}$$

In particular,

$$\gamma_u(Q_j) \approx \sigma_u(Q_j) \approx u(x_j)^2 \operatorname{diam}(Q_j)^{n-2} \quad \text{for } n \geq 2.$$

Moreover, Hardy's inequality yields the following lemma.

Lemma 3. *Let D satisfy the capacity density condition and let $\{Q_j\}$ be the Whitney decomposition of D with doubles $\{Q_j^*\}$. Suppose that*

u is a positive superharmonic function on D satisfying the Harnack property with respect to $\{Q_j, Q_j^*\}$. Then

$$\sigma_u(E) \leq M\gamma_u(E) \quad \text{for Borel sets } E \subset D.$$

Proof. Let K be a compact subset of E and write $v_K = \widehat{R}_u^K = G\lambda_u^K$. Then

$$\gamma_u(K) = \iint G(x, y) d\lambda_u^K(x) d\lambda_u^K(y) = \int_D |\nabla v_K|^2 dx$$

See e.g. [Constantinescu & Cornea \(1963, Satz 7.2\)](#). Since $v_K = u$ on K except for a polar set, the same equality holds a.e. on K . Hence it follows from Lemma [A](#) (Hardy's inequality) that

$$\begin{aligned} \gamma_u(E) &\geq \gamma_u(K) = \int_D |\nabla v_K|^2 dx \\ &\geq M \int_D \left(\frac{v_K(x)}{\delta_D(x)} \right)^2 dx \geq M \int_K \left(\frac{u(x)}{\delta_D(x)} \right)^2 dx = M\sigma_u(K). \end{aligned}$$

Since $K \subset E$ is an arbitrary compact subset of E , we have the required inequality. \square

Theorem 3. *Let D enjoy the capacity density condition and let $\{Q_j\}$ be the Whitney decomposition of D with doubles Q_j^* . Suppose that a positive superharmonic function u satisfies*

$$(13) \quad \sup_{Q_j^*} u \leq M_0 \inf_{Q_j^*} u$$

with M_0 independent of Q_j . Then σ_u is comparable to γ_u with respect to $\{Q_j\}$, and hence γ_u is quasiadditive with respect to $\{Q_j\}$, i.e. $\gamma_u(E) \approx \sum_j \gamma_u(E \cap Q_j)$ for $E \subset D$.

Proof. Lemma [2](#) and Lemma [3](#) show [\(9\)](#) and [\(10\)](#), respectively. Hence, Theorem [2](#) proves this theorem. \square

Let h be a positive harmonic function. Then, the Harnack inequality shows that $u(x) = \min\{h(x)^a, b\}$ with $0 < a \leq 1$ and $b > 0$ satisfies [\(13\)](#). Hence we have the following corollary.

Corollary 2. *Let D and $\{Q_j\}$ be as in Theorem [3](#). Let $x_0 \in D$ and let $g(x) = \min\{G(x, x_0), 1\}$. Then γ_g is quasiadditive with respect to $\{Q_j\}$, i.e.,*

$$\gamma_g(E) \approx \sum_j \gamma_g(E \cap Q_j) \approx \begin{cases} \sum_j \frac{g(x_j)^2}{\log(4 \operatorname{diam}(Q_j) / \operatorname{Cap}(E \cap Q_j))} & \text{if } n = 2, \\ \sum_j g(x_j)^2 \operatorname{Cap}(E \cap Q_j) & \text{if } n \geq 3. \end{cases}$$

7. REFINED WIENER-TYPE CRITERIA FOR THE MINIMAL THINNESS

In this section we let D be a uniform domain with the capacity density condition. Let $\{Q_j\}_j$ be the Whitney decomposition of D . For each Whitney cube Q_j we let x_j be the center. We recall that $A_\xi(r) \in S(\xi, r) \cap D$ is a nontangential point for $\xi \in \partial D$ (Remark 1). The quasiadditivity of the Green energy gives the following Wiener-type criteria finer than Theorem 1.

Theorem 4. *Let D be a uniform domain with the capacity density condition. Suppose $\xi \in \partial D$ and $E \subset D$. Then E is minimally thin at ξ if and only if*

$$\sum_j \left(\frac{g(x_j)}{g(A_\xi(|x_j - \xi|))} \right)^2 \frac{1}{\log(4 \operatorname{diam}(Q_j) / \operatorname{Cap}(E \cap Q_j))} < \infty \quad \text{if } n = 2,$$

$$\sum_j \left(\frac{g(x_j)}{g(A_\xi(|x_j - \xi|))} \right)^2 |x_j - \xi|^{2-n} \operatorname{Cap}(E \cap Q_j) < \infty \quad \text{if } n \geq 3.$$

Proof. Invoking Corollary 2, we can rewrite (4) as

$$\sum_j \theta_\xi(|x_j - \xi|) \gamma_g(E \cap Q_j) < \infty,$$

where we recall x_j is the center of a Whitney cube Q_j . Thus the Wiener-type condition (7) is refined as in Theorem 4. The theorem is proved. \square

Corollary 3. *Let D be a uniform domain with the capacity density condition. Suppose $\xi \in \partial D$ and $E \subset D$. If E is measurable and minimally thin at ξ , then*

$$(14) \quad \int_E \left(\frac{g(x)}{g(A_\xi(|x - \xi|))} \right)^2 \frac{|x - \xi|^{2-n}}{\delta_D(x)^2} dx < \infty.$$

Proof. The refined Wiener criterion and Lemma 2 imply that

$$\sum_j g(A_\xi(|x_j - \xi|))^{-2} |x_j - \xi|^{2-n} \frac{g(x_j)^2}{\operatorname{diam}(Q_j)^2} \int_{E \cap Q_j} dx < \infty.$$

This shows the corollary. \square

Remark 3. Essén (1988) introduced first the refined Wiener criterion for the half-space. He used the weak L^1 -estimate due to Sjögren (1977). We note that the weak L^1 -estimate need not hold for a uniform domain.

Definition 11. Suppose $\xi \in \partial D$ and $E \subset D$. We say that E is *minimally thin at ξ for harmonic functions* if there is a finite measure

μ concentrated on ∂D such that $\mu(\{\xi\}) = 0$ and $K_\xi \leq K\mu$ on E . We say that E determines the point measure at $\xi \in \partial D$ if

$$K\mu \geq K_\xi \text{ on } E \implies \mu(\{\xi\}) > 0$$

for every finite measure μ concentrated on ∂D (Beurling (1965), Dahlberg (1976), Maz'ya (1972) and Sjögren (1977)).

Remark 4. We note that E is minimally thin at ξ for harmonic functions if and only if E does not determine the point measure at ξ . A sequence S is equivalent for ξ in the sense of Beurling (see Definition 5) if and only if S is not minimally thin at ξ for harmonic functions. Obviously if E is minimally thin at ξ for harmonic functions, then it is minimally thin; but the converse is not necessarily true.

Remark 5. Hayman & Kennedy (1976, p.481 and Theorem 7.37) defined sets “rarefied for harmonic functions”. We note that the term “rarefied sets” was introduced by Essén & Jackson (1980). A set is “rarefied” in the terminology of Lelong-Ferrand (1949) if and only if it is “semirarefied” in the terminology of Essén and Jackson. A set rarefied for harmonic functions corresponds to a rarefied set defined by Essén and Jackson. See Essén (1993) for further information.

Theorem 5. *Let D be a uniform domain with the capacity density condition. Suppose $\xi \in \partial D$ and $E \subset D$. Then E is minimally thin at ξ for harmonic functions if and only if*

$$\sum_{E \cap Q_j \neq \emptyset} \left(\frac{g(x_j)}{g(A_\xi(|x_j - \xi|))} \right)^2 \left(\frac{\text{diam}(Q_j)}{|x_j - \xi|} \right)^{n-2} < \infty.$$

Proof. It is not so difficult to see that E is minimally thin at ξ for harmonic functions if and only if $\tilde{E} = \cup_{E \cap Q_j \neq \emptyset} Q_j$ is minimally thin at ξ . Hence Lemma 1 and Theorem 5 readily imply the theorem. \square

Remark 6. It is easy to see that $\{x_i\}$ is separated if and only if the number of points x_i included in a Whitney cube Q_j is bounded by a positive constant independent of Q_j . Thus Theorem 5 coincides with Ancona's criterion for a sequence to determine a point measure (Ancona (1988, Theorem 7.4)).

Corollary 4. *Let D be a uniform domain with the capacity density condition. Suppose that $\xi \in \partial D$ and $0 < \rho < 1$. If E is minimally thin at ξ for harmonic functions, then $E_\rho = \bigcup_{x \in E} B(x, \rho \delta_D(x))$ satisfies*

$$\int_{E_\rho} \left(\frac{g(x)}{g(A_\xi(|x - \xi|))} \right)^2 \frac{|x - \xi|^{2-n}}{\delta_D(x)^2} dx < \infty.$$

If D is a $C^{1,\alpha}$ -domain or more generally a Liapunov-Dini domain, then $g(x) \approx \delta_D(x)$ (Widman (1967)), so that the above theorems and corollaries are generalizations of the results of Beurling (1965), Maz'ya (1972), Dahlberg (1976), Sjögren (1977) and Essén (1993, Section 2). For the completeness we state them as the following corollaries.

Corollary 5. *Let D be a $C^{1,\alpha}$ -domain or more generally a Liapunov-Dini domain. Suppose $\xi \in \partial D$ and $E \subset D$. Then E is minimally thin at ξ if and only if*

$$\sum_j \frac{\text{diam}(Q_j)^2}{|x_j - \xi|^2 \log(4 \text{diam}(Q_j) / \text{Cap}(E \cap Q_j))} < \infty \quad \text{if } n = 2,$$

$$\sum_j \frac{\text{diam}(Q_j)^2 \text{Cap}(E \cap Q_j)}{|x_j - \xi|^n} < \infty \quad \text{if } n \geq 3.$$

Corollary 6. *Let D be a $C^{1,\alpha}$ -domain or more generally a Liapunov-Dini domain. Suppose $\xi \in \partial D$ and $E \subset D$. If E is measurable and minimally thin at ξ , then*

$$\int_E \frac{dx}{|x - \xi|^n} < \infty.$$

Corollary 7. *Let D be a $C^{1,\alpha}$ -domain or more generally a Liapunov-Dini domain. Suppose $\xi \in \partial D$ and $E \subset D$. Then E is minimally thin at ξ for harmonic functions if and only if*

$$\sum_{E \cap Q_j \neq \emptyset} \left(\frac{\text{diam}(Q_j)}{|x_j - \xi|} \right)^n < \infty.$$

Corollary 8. *Let D be a $C^{1,\alpha}$ -domain or more generally a Liapunov-Dini domain. Let $\xi \in \partial D$ and let $0 < \rho < 1$. If E is minimally thin at ξ for harmonic functions, then $E_\rho = \bigcup_{x \in E} B(x, \rho \delta_D(x))$ satisfies*

$$\int_{E_\rho} \frac{dx}{|x - \xi|^n} < \infty.$$

8. FURTHER REMARKS

We conclude the paper by raising two open problems:

- (i) In our argument we have used the geometric assumption of the domain to characterize the minimal thinness. However, there might be a direct extension of Beurling's minimum principle to the higher dimensional case; such an extension allows us

to give complete understanding of the minimal thinness in the framework of the general Martin boundary theory.

- (ii) [Essén & Jackson \(1980\)](#) gave the covering properties of minimally thin sets in a half space. Such covering properties of minimally thin sets in a uniform domain remain open.

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DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO 060-0810,
JAPAN

E-mail address: aik@math.sci.hokudai.ac.jp