BESSEL CAPACITY, HAUSDORFF CONTENT AND THE TANGENTIAL BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

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Abstract. We compare the Bessel capacity with the Hausdorff content. For $E \subset \mathbb{R}^n$ we let $\bar{E}_{\gamma,c} = \bigcup_{x \in E} B(x,c\delta_{E}(x)^{\gamma})$ with $c > 0$ and $0 < \gamma \leq 1$. If $E$ is an open set and $0 < \gamma < 1$, then $\bar{E}_{\gamma,c}$ is larger than $E$. It is shown that the Bessel capacity of $\bar{E}_{\gamma,c}$ is estimated above by the Hausdorff content of $E$. This estimation is applied to the tangential boundary behavior of harmonic functions in the upper half space.

1. Introduction

Let $K(r) \equiv 0$ be a nonnegative nonincreasing lower semicontinuous (l. s. c.) function for $r > 0$. For $x \in \mathbb{R}^n$ we define $K(x) = K(|x|)$, and assume that $K(x)$ is locally integrable on $\mathbb{R}^n$. For $E \subset \mathbb{R}^n$ we define the capacity $C_K$ by

$$C_K(E) = \inf\{ \|\mu\| : K * \mu \geq 1 \text{ on } E \},$$

where $\|\mu\|$ denotes the total mass of a measure $\mu$. Let $k_\alpha(r) = r^{\alpha-n}$ for $0 < \alpha < n$. This is the Riesz kernel of order $\alpha$. If $K(r) = k_\alpha(r)$, then we write $C_\alpha$ for $C_K$ and call it the Riesz capacity of order $\alpha$.

Let $h(r)$ be a positive nondecreasing function for $r > 0$ and $h(0) = 0$. Such a function is called a measure function. We define the content $M_h$ by

$$M_h(E) = \inf\{ \sum h(r_j) : E \subset \bigcup B(x_j, r_j) \},$$

where $B(x, r)$ stands for the open ball with center at $x$ and radius $r$. If $h(r) = r^\beta$, then we write $M_\beta$ for $M_h$ and call it $\beta$-content. There is a close connection between $C_\alpha$ and $M_\beta$. The following theorem is well-known (cf. [4, §IV] and [6, Theorems 5.13 and 5.14]).
Theorem A.

(i) If \( M_{n-\alpha}(E) = 0 \), then \( C_{\alpha}(E) = 0 \).

(ii) Let \( n - \alpha < \beta \leq n \). Then \( C_{\alpha}(E) = 0 \) implies \( M_{\beta}(E) = 0 \).

(iii) There is a set \( E \) such that \( C_{\alpha}(E) = 0 \) and \( M_{n-\alpha}(E) > 0 \).

It is easy to see that \( C_{\alpha} \) and \( M_{n-\alpha} \) are both homogeneous of degree \( n - \alpha \). From this fact, we can easily obtain the above (i). However, in view of (iii), \( M_{n-\alpha}(E) = 0 \) is not characterized by \( C_{\alpha}(E) = 0 \). We have only partial comparison (ii).

One of the main purposes of this paper is to compare \( C_{\alpha} \) with a certain quantity, which may be regarded as an \((n - \alpha)\)-dimensional quantity. Hereafter we shall use the following notation. By the symbol \( A \) we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use \( A_1, A_2, \ldots \), to specify them. We shall say that two positive quantities \( f \) and \( g \) are comparable, written \( f \approx g \), if and only if there exists a constant \( A \) such that \( A^{-1}g \leq f \leq Ag \). By \(|E|\) we denote the Lebesgue measure of \( E \).

For \( c > 0 \) and \( 0 < \gamma \leq 1 \) we define

\[
\tilde{E}_{\gamma,c} = \bigcup_{x \in E} B(x, c\delta_E(x)^\gamma),
\]

where \( \delta_E(x) = \text{dist}(x, E^c) \). If \( E \) is an open set and \( 0 < \gamma < 1 \), then \( \tilde{E}_{\gamma,c} \) is a proper extension of \( E \). Moreover, if \( E = B(0, r) \) and \( r > 0 \) is small, then \( \tilde{E}_{\gamma,c} \) is a ball with radius comparable to \( cr^{\gamma} \), so that

\[
M_{\beta}(\tilde{E}_{\gamma,c}) \approx r^{\gamma\beta} \approx M_{\beta}(E)^\gamma.
\]

So, one may regard \( M_{\beta}(\tilde{E}_{\gamma,c}) \) as a \( \beta\gamma \)-dimensional quantity. If \( \beta = n \), then \( M_{\beta}(E) \) is comparable with the Lebesgue measure \(|E|\). Let \( g_{\alpha} \) be the Bessel kernel. The Riesz and the Bessel kernels have the same asymptotics as \( r \to 0 \). However, \( g_{\alpha}(r) \) decreases rapidly as \( r \to \infty \) and hence \( g_{\alpha} \) is integrable on \( \mathbb{R}^n \). The capacity \( C_{g_{\alpha}}(E) \) is called the Bessel capacity of index \((\alpha, 1)\) and is denoted by \( B_{\alpha,1}(E) \). It is well known that

\[
C_{\alpha}(E) \approx B_{\alpha,1}(E) \text{ for } E \subset U,
\]

where \( U \) is a bounded set. Thus the Riesz capacity \( C_{\alpha} \) and the Bessel capacity \( B_{\alpha,1} \) have the same null sets. In the previous paper [3] we have proved

**Theorem B.** Let \( 0 < \alpha < n \), \( c = 1 \) and \( \gamma = (n - \alpha)/n \). Then

\[
|\tilde{E}_{\gamma,c}| \leq AB_{\alpha,1}(E),
\]

where \( A > 0 \) depends only on \( n \) and \( \alpha \).

Here we generalize Theorem B to
Theorem 1. Let $0 < n - \alpha < \beta \leq n$, $\gamma = (n - \alpha)/\beta$ and $c > 0$. Then
\[ M_\beta(\bar{E}_{\gamma,c}) \leq AB_{\alpha,1}(E), \]
where $A > 0$ depends only on $n$, $\alpha$, $\beta$ and $c$.

Actually, in [3], general kernels and capacities were treated. Our argument here for Theorem 1 is very different from that of [3] and heavily depends on the Bessel kernel. The case when $\beta = n$ was dealt with in [3]. We see that $M_\beta(E)$ and the Lebesgue measure $|E|$ are comparable in this case. The main idea in [3] was to compare a test measure for the capacity with the Lebesgue measure on a ball whose volume is equal to its capacity. In case $\beta < n$, a difficulty arises from the lack of a measure corresponding to the Lebesgue measure. We shall employ the Frostman lemma and the Besicovitch covering lemma (see Lemmas A and B below). We shall convert the measure given by the Frostman lemma so that the converted measure becomes a test measure for the dual definition of $B_{\alpha,1}$ (see Lemma C below).

We can consider a counterpart of Theorem 1 for $L^p$-capacity theory. Let $1 < p < \infty$. We define
\[ C_{K,p}(E) = \inf\{\|f\|_p^p : K * f \geq 1 \text{ on } E\}. \]
If $K = k_\alpha$, then we write $R_{\alpha,p}(E)$ for $C_{K,p}(E)$ and call it the Riesz capacity of index $(\alpha, p)$. If $K = g_\alpha$, then we write $B_{\alpha,p}(E)$ for $C_{K,p}(E)$ and call it the Bessel capacity of index $(\alpha, p)$. In case $\alpha p < n$, the Riesz capacity $R_{\alpha,p}$ is homogeneous of degree $n - \alpha p$; the Riesz capacity $R_{\alpha,p}(E)$ and the Bessel capacity $B_{\alpha,p}(E)$ are comparable for $E \subset U$, where $U$ is a bounded set.

Theorem 2. Let $1 < p < \infty$, $0 < n - \alpha p < \beta \leq n$, $\gamma = (n - \alpha p)/\beta$ and $c > 0$. Then
\[ M_\beta(\bar{E}_{\gamma,c}) \leq AB_{\alpha,p}(E), \]
where $A > 0$ depends only on $n$, $\alpha$, $p$, $\beta$ and $c$.

The proof of Theorem 2 will use the same converted measure as in the proof of Theorem 1, the dual definition of $B_{\alpha,p}$ and the Hedberg–Wolff lemma (see Lemmas D and E). We shall later generalize these theorems, in connection with Nagel-Stein approach regions ([11]). We shall introduce a notion of “thin sets” and combine it with the generalized version of Theorems 1 and 2 to obtain the tangential boundary behavior of harmonic functions given as the Poisson integral of Bessel potentials.

The plan of this paper is as follows. We shall prove Theorems 1 and 2 in Sections 2 and 3, respectively. A theorem similar to Theorem 2 for the case $\alpha p = n$ will be given also in Section 3. In Section 4 we shall introduce the Nagel-Stein approach region and generalize Theorems 1 and 2. The boundary behavior of harmonic functions will be considered in Section 5. Finally, a norm estimate of tangential maximal functions of Poisson integrals will be given in Section 6. We shall observe that our arguments yield different proofs of Ahern-Nagel [2, Theorem 6.2 and Corollary 6.3].

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2. Proof of Theorem 1

Let us recall the fundamental lemma due to Frostman (see e.g. [4, Theorem 1 on p. 7] and [6, Lemma 5.4]).

**Lemma A.** Let $h$ be a measure function. Suppose $F$ is a compact set such that $M_h(F) > 0$. Then there is a measure $\mu$ supported on $F$ such that

$$\|\mu\| \approx M_h(F),$$

$$\mu(B(x,r)) \leq h(r) \text{ for all } x \in \mathbb{R}^n \text{ and } r > 0.$$ 

We also need the Besicovitch covering lemma (see e.g. [14, Theorem 1.3.5]).

**Lemma B.** Let $E$ be a set in $\mathbb{R}^n$ and suppose that $r(x)$ is a positive bounded function on $E$. Then we can select $\{x_j\} \subset E$ with the following properties:

(i) $E \subset \bigcup_j B(x_j, r(x_j))$.

(ii) The multiplicity of $\{B(x_j, r(x_j))\}$ is bounded by a positive constant $N$ depending only on the dimension. In other words, $\sum \chi_{B(x_j, r(x_j))} \leq N$.

We note the dual definition of $C_K$.

**Lemma C.** Let $E$ be an analytic set. Then

$$C_K(E) = \sup\{\|\mu\| : \mu \text{ is concentrated on } E, K * \mu \leq 1 \text{ on } \mathbb{R}^n\}.$$

For each integer $\nu$ we let $G_\nu$ be the family of cubes

$$Q = \{(x_1, \ldots, x_n) : \frac{k_i}{2^\nu} \leq x_i < \frac{k_i + 1}{2^\nu}, i = 1, \ldots, n\},$$

where $k_1, \ldots, k_n$ are integers. We let $G = \{G_\nu\}_{\nu = -\infty}^\infty$. For a cube $Q$ of side length $\ell$ we put $\tau_h(Q) = h(\ell)$ and define

$$m_h(E) = \inf\{\sum_{j=1}^{\infty} \tau_h(Q_j) : E \subset \bigcup_{j=1}^{\infty} Q_j, Q_j \in G\}.$$

Then it is easy to see that

$$M_h(E) \approx m_h(E) \text{ for any set } E$$

([4, (1.3) on p. 7]). We observe that $m_h$ has the increasing property.

**Lemma 1.** Let $\lim_{r \to \infty} h(r) = \infty$. If $E_j \uparrow E$, then $\lim_{j \to \infty} m_h(E_j) = m_h(E)$. In particular, if $E$ is an $F_\sigma$-set, then

$$m_h(E) = \sup_{F \subset E} m_h(F).$$

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Proof. It is clear that \( \lim_{j \to \infty} m_h(E_j) \leq m_h(E) \). Hence, it is sufficient to show the opposite inequality, under the assumption that \( \lim_{j \to \infty} m_h(E_j) < \infty \). Let \( \varepsilon > 0 \). By definition we find cubes \( Q_{j,i} \in G \) such that

\[
E_j \subset \bigcup_{i=1}^{\infty} Q_{j,i},
\]

\[
\sum_{i=1}^{\infty} \tau_h(Q_{j,i}) < m_h(E_j) + \varepsilon 2^{-j}.
\]

Since \( \lim_{j \to \infty} m_h(E_j) < \infty \) and \( \lim_{r \to \infty} h(r) = \infty \), it follows that the side lengths of \( Q_{j,i} \) are bounded. Hence we can select maximal cubes \( Q_1, Q_2, \ldots, Q_\nu, \ldots \) whose union covers \( E = \bigcup_{j=1}^{\infty} E_j \). Now, in the same way as in [12, Theorem 52], we can show

\[
\sum_{\nu=1}^{\infty} \tau_h(Q_\nu) \leq \lim_{j \to \infty} m_h(E_j) + 2\varepsilon,
\]

and hence \( m_h(E) \leq \lim_{j \to \infty} m_h(E_j) + 2\varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, the lemma follows.

As a corollary to (2.1) and Lemma 1 we have the following:

**Corollary 1.** Let \( \lim_{r \to \infty} h(r) = \infty \). If \( E \) is an \( F_{\sigma} \)-set, then

\[
M_h(E) \approx \sup_{\substack{F \subset E \text{ is compact}}} M_h(F).
\]

**Remark.** The assumption that \( \lim_{r \to \infty} h(r) = \infty \) is essential in Lemma 1. In fact, suppose that \( \lim_{r \to \infty} h(r) = a < \infty \). Then, by definition, \( m_h(E) \leq a \) for any bounded set \( E \). On the other hand it is easy to see that \( m_h(\mathbb{R}^n) = \infty \) if \( \lim\inf_{r \to 0} h(r)/r > 0 \). Thus the increasing property does not hold in general. This example is suggested by K. Hatano. We observe that [4, (3.2) on p.9] actually requires some additional assumption like \( \lim_{r \to \infty} h(r) = \infty \) or the boundedness of \( E \).

From Lemmas A, C and 1 we show the following lemma.

**Lemma 2.** Let \( 0 < n - \alpha < \beta \leq n \). Then

\[
M_\beta(E) \leq AB_\alpha,1(E),
\]

where \( A > 0 \) depends only on \( n, \alpha \) and \( \beta \).

**Proof.** Since \( B_{\alpha,1} \) is an outer capacity, i.e.,

\[
B_{\alpha,1}(E) = \inf_{E \subset U, \text{U is open}} B_{\alpha,1}(U),
\]


we may assume that \( E \) is an open set. Let \( F \) be a compact subset of \( E \). By Lemma A there is a measure \( \mu \) on \( F \) such that

\[
\| \mu \| \approx M_\beta(F),
\]

\[
\mu(B(x, r)) \leq r^\beta \text{ for all } x \in \mathbb{R}^n \text{ and } r > 0.
\]

Observe from (2.3) that

\[
g_\alpha \ast \mu(x) = \int_0^\infty g_\alpha(r)d\mu(B(x, r)) = \int_0^\infty \mu(B(x, r))d(-g_\alpha(r)) \\
\leq \int_0^\infty r^\beta d(-g_\alpha(r)) = A_1 < \infty.
\]

Hence Lemma C and (2.2) yield

\[
B_{\alpha, 1}(E) \geq A_1^{-1}\| \mu \| \approx M_\beta(F).
\]

Taking the supremum over all \( F \), we obtain the required inequality from Corollary 1. The lemma follows.

**Proof of Theorem 1.** By (2.1) and Lemma 1 we may assume that \( E \) is a bounded set. Since \( B_{\alpha, 1} \) is an outer capacity, we may furthermore assume that \( E \) is an open set. By Lemma 2 we have only to show that

\[
M_\beta(\widetilde{E}_{\gamma, c} \setminus E) \leq AB_{\alpha, 1}(E).
\]

In view of Corollary 1 it is sufficient to show that

\[
M_\beta(F) \leq AB_{\alpha, 1}(E)
\]

for any compact subset \( F \) of \( \widetilde{E}_{\gamma, c} \setminus E \), since \( \widetilde{E}_{\gamma, c} \setminus E \) is an \( F_\sigma \)-set. By Lemma A we can find a measure \( \mu \) on \( F \) satisfying (2.2) and (2.3).

By definition, for each \( x \in \widetilde{E}_{\gamma, c} \setminus E \), there is \( x^* \in E \) such that \( x \in B(x^*, c\delta_E(x^*)^\gamma) \). We let

\[
r(x) = \sup_{x^* \in E, x \in B(x^*, c\delta_E(x^*)^\gamma)} \delta_E(x^*).
\]

We observe that \( r(x) \) is a positive bounded function on \( \widetilde{E}_{\gamma, c} \setminus E \). We invoke Lemma B and find \( \{x_j\} \subset F \) such that

\[
F \subset \bigcup B(x_j, 2cr_j^\gamma) \text{ with } r_j = r(x_j),
\]

\[
\text{the multiplicity of } \{B(x_j, 2cr_j^\gamma)\} \text{ is bounded by } N.
\]
By definition we can find $x^*_j \in E$ such that

\begin{align}
(2.7) & \quad r_j/2 < \delta_E(x^*_j) \leq r_j, \\
(2.8) & \quad |x_j - x^*_j| < cr_j^\gamma.
\end{align}

We put $\mu_j = \mu|_{B(x_j, 2cr_j^\gamma)}$ and observe from (2.5) and (2.6) that

\begin{equation}
(2.9) \quad \mu \leq \sum \mu_j \leq N \mu.
\end{equation}

From $\mu_j$ we construct a measure $\lambda_j$ as follows: for Borel sets $S$

\begin{align*}
\lambda_j(S) &= \mu_j(4(S - x^*_j) + x_j) & \text{if } cr_j^\gamma \leq r_j, \\
\lambda_j(S) &= \mu_j(4cr_j^\gamma - 1(S - x^*_j) + x_j) & \text{if } cr_j^\gamma > r_j.
\end{align*}

It is easy to see that

\begin{align}
(2.10) & \quad \lambda_j \text{ is concentrated on } B(x^*_j, \frac{1}{2} \min\{cr_j^\gamma, r_j\}), \\
(2.11) & \quad \|\lambda_j\| = \|\mu_j\|, \\
(2.12) & \quad \lambda_j(B(x, \rho)) = \mu_j(B(x, \rho)) = \|\mu_j\|
\end{align}

for $\rho \geq \max\{|x - x_j| + 2cr_j^\gamma, |x - x^*_j| + \frac{1}{2} \min\{cr_j^\gamma, r_j\}\}$.

Moreover, in view of (2.3)

\begin{equation}
(2.13) \quad \|\lambda_j\| = \|\mu_j\| \leq (2cr_j^\gamma)^\beta;
\end{equation}

for all $x \in \mathbb{R}^n$ and $r > 0$

\begin{align}
(2.14) & \quad \lambda_j(B(x, r)) \leq (4r)^\beta & \text{if } cr_j^\gamma \leq r_j, \\
(2.15) & \quad \lambda_j(B(x, r)) \leq (4cr_j^\gamma - 1r)^\beta & \text{if } cr_j^\gamma > r_j.
\end{align}

It follows from (2.7) that $B(x^*_j, r_j/2) \subset E$ and so from (2.10) that the measure $\lambda_j$ is concentrated on $E$. Let $\lambda = \sum \lambda_j$. We claim

\begin{equation}
(2.16) \quad g_\alpha \ast \lambda \leq A_2 \text{ on } \mathbb{R}^n.
\end{equation}

If we have (2.16), then the proof is easy. Since $\lambda$ is concentrated on $E$, it follows from Lemma C and (2.11) that

$$B_{\alpha,1}(E) \geq A_2^{-1}\|\lambda\| = A_2^{-1}\sum \|\mu_j\| \geq A_2^{-1}\|\mu\|.$$
This, together with (2.2), yields (2.4).

Let us prove (2.16). Hereafter we fix \( x \in \mathbb{R}^n \). First we claim

\[
\text{(2.17)} \quad g_\alpha \ast \lambda_j(x) \leq A
\]

with \( A \) independent of \( j \) and \( x \). Suppose \( cr_j^\gamma \leq r_j \). Then by (2.14)

\[
g_\alpha \ast \lambda_j(x) = \int_0^\infty \lambda_j(B(x, r))d(-g_\alpha(r)) \leq \int_0^\infty (4r)^\beta d(-g_\alpha(r)) = A < \infty.
\]

Thus (2.17) follows. Suppose \( cr_j^\gamma > r_j \). Then by (2.13) and (2.15)

\[
g_\alpha \ast \lambda_j(x) = \int_0^\infty \lambda_j(B(x, r))d(-g_\alpha(r)) \leq \int_0^{r_j/2} (4cr_j^{\gamma-1}r)^\beta d(-g_\alpha(r)) + (2cr_j^\gamma) \int_{r_j/2}^\infty d(-g_\alpha(r)) \leq Ar_j^{(\gamma-1)\beta} r_j^{\beta+\alpha-n} + Ar_j^\gamma r_j^{\alpha-n} = A < \infty.
\]

Thus (2.17) follows in this case, too.

Let us write

\[
\lambda' = \sum' \lambda_j, \quad \lambda'' = \sum'' \lambda_j,
\]

where \( \sum' \) (resp. \( \sum'' \)) denotes the summation over \( j \) for which \( x \in B(x_j, 2cr_j^\gamma) \) (resp. \( x \notin B(x_j, 2cr_j^\gamma) \)). In view of (2.6), the number of \( j \) appearing in \( \sum' \) is at most \( N \). Hence by (2.17)

\[
\text{(2.18)} \quad g_\alpha \ast \lambda'(x) \leq A.
\]

Next, we consider \( g_\alpha \ast \lambda''(x) \). Let us estimate \( \lambda''(B(x, r)) = \sum'' \lambda_j(B(x, r)) \). In the summation \( \sum'' \), we may consider only \( j \) such that \( \lambda_j(B(x, r)) > 0 \). By (2.10) this implies that \( |x-x_j^*| \leq r + cr_j^\gamma/2 \). In view of the definition of \( \sum'' \), we have \( |x-x_j| \geq 2cr_j^\gamma \). Using these inequalities and (2.8), we obtain

\[
r + cr_j^\gamma/2 \geq |x-x_j^*| \geq |x-x_j| - |x_j-x_j^*| \geq 2cr_j^\gamma - cr_j^\gamma = cr_j^\gamma,
\]

so that \( r \geq cr_j^\gamma/2, |x-x_j^*| \leq 2r, |x_j-x_j^*| \leq 2r \) and \( |x-x_j| \leq 4r \). Hence

\[
\max\{|x-x_j| + 2cr_j^\gamma, |x-x_j^*| + \frac{1}{2} \min\{cr_j^\gamma, r_j\}\} \leq \max\{8r, 3r\} = 8r.
\]

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Therefore, (2.12) implies that 

\[ \lambda_j(B(x, 8r)) = \mu_j(B(x, 8r)) \]

so that

\[ \lambda''(B(x, r)) = \sum '' \lambda_j(B(x, r)) \]
\[ \leq \sum '' \lambda_j(B(x, 8r)) = \sum '' \mu_j(B(x, 8r)) \]
\[ \leq \sum \mu_j(B(x, 8r)) \leq N \mu(B(x, 8r)), \]

where the last inequality follows from (2.9). Hence by (2.3)

(2.19) \[ \lambda''(B(x, r)) \leq N(8r)\beta \] for all \( r > 0 \).

Thus

\[ g_\alpha * \lambda''(x) = \int_0^\infty \lambda''(B(x, r))d(-g_\alpha(r)) \leq A \int_0^\infty r^{\beta}d(-g_\alpha(r)) = A < \infty. \]

This, together with (2.18), yields (2.16). The proof is complete.

3. Proof of Theorem 2

Let \( \frac{1}{p} + \frac{1}{q} = 1 \). We have the dual definition of \( C_{K,p} \) ([8, Theorem 14]).

Lemma D. Let \( E \) be an analytic set. Then

\[ C_{K,p}(E) = \sup\{\|\mu\|^p : \mu \text{ is concentrated on } E, \|K * \mu\|_q \leq 1\}. \]

Let \( \alpha p \leq n \). We put

\[ W^\mu_{\alpha,p}(x) = \int_0^1 \left( \frac{\mu(B(x, r))}{r^{n-\alpha p}} \right)^{q-1} \frac{dr}{r}. \]

Hedberg and Wolff [7] proved the following lemma (see also [1] and [14, Theorem 4.7.5]).

Lemma E. Let \( \alpha p \leq n \). Then

\[ \|g_\alpha * \mu\|_q^q \approx \int W^\mu_{\alpha,p}(x)d\mu(x). \]

In the same way as in the proof of Lemma 2, we obtain the following lemma from Lemmas A, D and E.
Lemma 3. Let $1 < p < \infty$ and $0 \leq n - \alpha p < \beta \leq n$. Then

$$M_\beta(E) \leq AB_{\alpha,p}(E),$$

where $A > 0$ depends only on $n$, $\alpha$, $p$ and $\beta$.

Proof. Since $B_{\alpha,p}$ is an outer capacity, we may assume that $E$ is an open set. Let $F$ be a compact subset of $E$. By Lemma A there is a measure $\mu$ on $F$ satisfying (2.2) and (2.3). Observe from (2.3) that

$$W_{\alpha,p}^\mu(x) \leq \int_0^1 \left( \frac{r^{\beta}}{r^{n-\alpha p}} \right)^{q-1} \frac{dr}{r} = A < \infty,$$

since $n - \alpha p < \beta$. Hence Lemma E yields $\|g_\alpha \ast \mu\|_q \leq A\|\mu\|$, or equivalently

$$\left\| g_\alpha \ast \frac{\mu}{A\|\mu\|^{1/q}} \right\|_q \leq 1.$$

Hence Lemma D and (2.2) yield

$$B_{\alpha,p}(E) \geq \left( \frac{\|\mu\|}{A\|\mu\|^{1/q}} \right)^p = A\|\mu\| \approx M_\beta(F).$$

Taking the supremum over all $F$, we obtain the required inequality from Corollary 1.

Proof of Theorem 2. We may assume that $E$ is a bounded open set. In view of Lemma 3 and Corollary 1 it is sufficient to show that

$$(3.1) \quad M_\beta(F) \leq AB_{\alpha,p}(E)$$

for any compact set $F \subset \tilde{E}_{r_j,cr_j} \setminus E$. In the same way as in the proof of Theorem 1 we can find a measure $\mu$ on $F$ satisfying (2.2) and (2.3). We find balls $B(x_j,2cr_j)$ satisfying (2.5) and (2.6). Let $\mu_j = \mu|_{B(x_j,2cr_j)}$ and let $\lambda_j$, $\lambda$, $\lambda'$ and $\lambda''$ be as in the proof of Theorem 1. Observe that (2.9)–(2.15) and (2.19) hold. In particular $\lambda$ is concentrated on $E$ and

$$(3.2) \quad \|\lambda\| \approx \|\mu\| \approx M_\beta(F).$$

If $cr_j^\gamma \leq r_j$, then by (2.14)

$$W_{\alpha,p}^{\lambda_j}(x) \leq A \int_0^1 \left( \frac{4r^{\gamma}}{r^{n-\alpha p}} \right)^{q-1} \frac{dr}{r} = A < \infty.$$
If \( cr_j^\gamma > r_j \), then by (2.13) and (2.15)
\[
W_{\alpha,p}^\lambda(x) \leq A \int_0^1 \left( \frac{\min\{4cr_j^{\gamma-1}r, 2cr_j^\gamma\}}{r^{n-\alpha p}} \right)^{q-1} \frac{dr}{r} \leq A < \infty.
\]
Thus \( W_{\alpha,p}^\lambda(x) \leq A \) in any case, and hence from (2.6) we have \( W_{\alpha,p}^{\lambda'}(x) \leq A \). From (2.19) we have
\[
W_{\alpha,p}^{\lambda''}(x) \leq A \int_0^1 \left( \frac{(8r)^\beta}{r^{n-\alpha p}} \right)^{q-1} \frac{dr}{r} = A < \infty.
\]
Thus \( W_{\alpha,p}^\lambda(x) \leq A \). Hence Lemma E yields \( \|g_\alpha * \lambda\|_q \leq A\|\lambda\|_q \), or equivalently
\[
\left\| g_\alpha * \frac{\lambda}{A\|\lambda\|^{1/q}} \right\|_q \leq 1.
\]
Since \( \lambda \) is concentrated on \( E \), it follows from Lemma D and (3.2) that
\[
B_{\alpha,p}(E) \geq \left( \frac{\|\lambda\|}{A\|\lambda\|^{1/q}} \right)^p = A\|\lambda\| \approx M_\beta(F).
\]
Thus (3.1) follows. The theorem is proved.

Observe that if \( r > 0 \) is small, then
\[
B_{\alpha,p}(B(0,r)) \approx \begin{cases} r^{n-\alpha p} & \text{if } \alpha p < n, \\ \left( \log \frac{1}{r} \right)^{1-p} & \text{if } \alpha p = n. \end{cases}
\]
Therefore, it may be natural to consider a logarithmic expansion in case \( \alpha p = n \).

**Theorem 2’**. Let \( 1 < p < \infty \), \( \alpha p = n \), \( 0 < \beta \leq n \) and \( c > 0 \). We put
\[
\varphi(r) = \varphi_{\beta,p}(r) = \begin{cases} \left( \log \frac{1}{r} \right)^{(1-p)/\beta} & , \ 0 < r < 1/2, \\ 2(\log 2)^{(1-p)/\beta r} & , \ r \geq 1/2. \end{cases}
\]
and
\[
\tilde{E}_{\varphi,c} = \bigcup_{x \in E} B(x, c\varphi(\delta_E(x))).
\]
Then
\[
M_\beta(\tilde{E}_{\varphi,c}) \leq AB_{\alpha,p}(E),
\]
where \( A > 0 \) depends only on \( n, \alpha, p, \beta \) and \( c \).

**Proof.** We can prove the theorem in a way similar to Theorem 2. But for the completeness we give a proof. We observe that \( \varphi(r) \) is a positive continuous increasing function. We may assume that \( E \) is a bounded open set. In view of Lemma 3 and Corollary 1 it is sufficient to show that

\[
M_{\beta}(F) \leq AB_{\alpha,p}(E)
\]

for any compact subset \( F \subset \tilde{E}_{\varphi,c} \setminus E \). In the same way as in the proof of Theorem 1 we can find a measure \( \mu \) on \( F \) satisfying (2.2) and (2.3). Let

\[
\rho(x) = \sup_{x^{*} \in E} \delta_{E}(x^{*})
\]

and observe that \( \rho(x) \) is a positive bounded function on \( \tilde{E}_{\varphi,c} \setminus E \). By Lemma B we find \( \{x_{j}\} \subset F \) such that

\[
F \subset \bigcup B(x_{j}, 2c\varphi(r_{j})) \text{ with } r_{j} = \rho(x_{j}),
\]

(3.5) the multiplicity of \( \{B(x_{j}, 2c\varphi(r_{j}))\} \) is bounded by \( N \).

By definition we can find \( x_{j}^{*} \in E \) such that

\[
r_{j}/2 < \delta_{E}(x_{j}^{*}) \leq r_{j} \text{ and } |x_{j} - x_{j}^{*}| < c\varphi(r_{j}).
\]

(3.7) We put \( \mu_{j} = \mu|_{B(x_{j}, 2c\varphi(r_{j}))} \) and observe from (3.5) and (3.6) that

\[
\mu \leq \sum \mu_{j} \leq N\mu.
\]

From \( \mu_{j} \) we construct a measure \( \lambda_{j} \) as follows: for Borel sets \( S \)

\[
\lambda_{j}(S) = \mu_{j}(4(S - x_{j}^{*}) + x_{j}) \quad \text{if } c\varphi(r_{j}) \leq r_{j},
\]

\[
\lambda_{j}(S) = \mu_{j}(4c\varphi(r_{j})r_{j}^{-1}(S - x_{j}^{*}) + x_{j}) \quad \text{if } c\varphi(r_{j}) > r_{j}.
\]

It is easy to see that

\[
\lambda_{j} \text{ is concentrated on } B(x_{j}^{*}, \frac{1}{2} \min\{c\varphi(r_{j}), r_{j}\}),
\]

\[
\|\lambda_{j}\| = \|\mu_{j}\| \leq (2c\varphi(r_{j}))^{\beta},
\]

\[
\lambda_{j}(B(x, \rho)) = \mu_{j}(B(x, \rho)) = \|\mu_{j}\|
\]

for \( \rho \geq \max\{|x - x_{j}| + 2c\varphi(r_{j}), |x - x_{j}^{*}| + \frac{1}{2} \min\{c\varphi(r_{j}), r_{j}\}\} \).
and for all \( x \in \mathbb{R}^n \) and \( r > 0 \)

\[
\begin{align*}
\lambda_j(B(x, r)) &\leq (4r)^\beta & &\text{if } c\varphi(r_j) \leq r_j, \\
\lambda_j(B(x, r)) &\leq (4c\varphi(r_j)r_j^{-1}r)^\beta & &\text{if } c\varphi(r_j) > r_j.
\end{align*}
\]

Let \( \lambda = \sum \lambda_j \). It follows from (3.7) that \( B(x_j^*, r_j/2) \subset E \) so that the measure \( \lambda_j \) is concentrated on \( E \), and so is \( \lambda \). We claim

(3.8) \[ W_{\alpha,p}^{\lambda_j}(x) \leq A \]

with \( A \) independent of \( j \) and \( x \). If \( c\varphi(r_j) \leq r_j \), then

\[
W_{\alpha,p}^{\lambda_j}(x) \leq A \int_0^1 \frac{(4r)^\beta}{r} dr = A < \infty,
\]

so that (3.8) follows. If \( c\varphi(r_j) > r_j \), then

\[
W_{\alpha,p}^{\lambda_j}(x) \leq A \int_0^1 \min\{ (4c\varphi(r_j)r_j^{-1}r)^\beta, (2c\varphi(r_j))^{\beta(q-1)} \} \frac{dr}{r}
\]

\[
\leq \begin{cases} A\varphi(r_j)^{\beta(q-1)} \left( \frac{1}{\beta(q-1)} + \log \frac{1}{r_j} \right) & \text{if } 0 < r_j < 1, \\
A\varphi(r_j)^{\beta(q-1)} \frac{1}{\beta(q-1)} r_j^{-\beta(q-1)} & \text{if } r_j \geq 1,
\end{cases}
\]

so that in view of the definition of \( \varphi \) we have (3.8) in this case, too. Let us write

\[
\lambda' = \sum' \lambda_j, \quad \lambda'' = \sum'' \lambda_j,
\]

where \( \sum' \) (resp. \( \sum'' \)) denotes the summation over \( j \) for which \( x \in B(x_j, 2c\varphi(r_j)) \) (resp. \( x \notin B(x_j, 2c\varphi(r_j)) \)). In view of (3.6) the number of \( j \) appearing in \( \sum' \) is at most \( N \). Hence (3.8) implies that

(3.9) \[ W_{\alpha,p}^{\lambda'}(x) \leq A. \]

In the same way as in the proof of Theorem 1 we estimate \( \lambda''(B(x, r)) \). Observe that if \( x \notin B(x_j, 2c\varphi(r_j)) \) and \( \lambda_j(B(x, r)) > 0 \), then \( |x - x_j| + 2c\varphi(r_j) < 8r \), so that \( \lambda_j(B(x, 8r)) = \mu_j(B(x, 8r)) \) and (2.19) holds. Therefore

\[
W_{\alpha,p}^{\lambda''}(x) \leq A \int_0^1 (8r)^{\beta(q-1)} \frac{dr}{r} = A < \infty.
\]

This, together with (3.9), yields

\[ W_{\alpha,p}^{\lambda} \leq A \text{ on } \mathbb{R}^n. \]

Hence Lemmas D and E and (2.2) imply

\[ B_{\alpha,p}(E) \geq A\|\lambda\| \approx \|\mu\| \approx M_\beta(F). \]

Thus (3.4) follows. The theorem is proved.
4. Generalization

Let $\Omega$ be a set in $\mathbb{R}^{n+1}$ with $\Omega \cap \partial \mathbb{R}^{n+1} = \{0\}$. For simplicity we assume that $\Omega \supset \{(0, y) : y > 0\}$. Put $\Omega(y) = \{x : (x, y) \in \Omega\}$. We say that $\Omega$ satisfies the Nagel-Stein condition (abbreviated to (NS)), if

(i) $|\Omega(y)| \leq Ay^n$ with $A = A(\Omega)$;
(ii) there is $a_0 > 0$ such that $(x_1, y_1) \in \Omega$ and $|x - x_1| < a_0(y - y_1) \implies (x, y) \in \Omega$.

It is easy to see that $\Omega(y)$ is an increasing set function of $y$, i.e., if $y_1 < y_2$, then $\Omega(y_1) \subset \Omega(y_2)$. For $E$ we put

$$e_{E; \gamma,c; \Omega} = \bigcup_{x \in E} (x + \Omega(c\delta_E(x)^\gamma)).$$

We have a generalization of Theorems 1, 2 and 2'.

**Theorem 3.** Let $1 \leq p < \infty$, $0 < \alpha < n$, $0 \leq n - \alpha p < \beta \leq n$, $\gamma = (n - \alpha p)/\beta$, $c > 0$ and let $\varphi(r) = \varphi_{\beta,p}(r)$ be as in (3.3) if $\alpha p = n$. Let $\Omega$ satisfy (NS). Then

$$M^\beta(e_{\gamma,c; \Omega}) \leq AB_{\alpha,p}(E) \quad \text{if } \alpha p < n,$$

$$M^\beta(e_{\varphi,c; \Omega}) \leq AB_{\alpha,p}(E) \quad \text{if } \alpha p = n,$$

where $A > 0$ depends only on $n$, $\alpha$, $p$, $\beta$, $c$ and $\Omega$.

We shall prove this theorem as a corollary to Theorems 1, 2 and 2' and the following lemma.

**Lemma 4.** Let $0 < \beta \leq n$ and let $\Omega$ satisfy (NS). If $V$ is an open subset of $\mathbb{R}^n$, then

$$M^\beta \left( \bigcup_{x \in V} (x + \Omega(\delta_V(x))) \right) \leq AM^\beta(V),$$

where $\delta_V(x) = \text{dist}(x, V^c)$ and $A > 0$ depends only on $\beta$, $\Omega$ and $n$.

We assume Lemma 4, then the proof of Theorem 3 is easy.

**Proof of Theorem 3.** We prove the theorem only in the case $\alpha p < n$, since the case $\alpha p = n$ is similarly proved. First we claim that

$$(4.1) \quad e_{E; \gamma,c; \Omega} \subset \bigcup_{x \in e_{E; \gamma,c}} (x + \Omega(\delta_{E; \gamma,c}(x))).$$

Suppose $x \in E$. By definition $B(x, c\delta_E(x)^\gamma) \subset e_{E; \gamma,c}$, so that $c\delta_E(x)^\gamma \leq \delta_{E; \gamma,c}(x)$. Hence

$$e_{E; \gamma,c; \Omega} = \bigcup_{x \in E} (x + \Omega(c\delta_E(x)^\gamma)) \subset \bigcup_{x \in E} (x + \Omega(\delta_{E; \gamma,c}(x))) \subset \bigcup_{x \in e_{E; \gamma,c}} (x + \Omega(\delta_{E; \gamma,c}(x))).$$
Thus (4.1) follows. Combining (4.1), Lemma 4 with \( V = \tilde{E}_{\gamma,c} \) and Theorems 1 and 2, we obtain

\[
M_\beta(\tilde{E}_{\gamma,c};\Omega) \leq M_\beta \left( \bigcup_{x \in \tilde{E}_{\gamma,c}} (x + \Omega(\delta_{\tilde{E}_{\gamma,c}}(x))) \right) \leq A M_\beta(\tilde{E}_{\gamma,c}) \leq A B_{\alpha,p}(E).
\]

Thus the theorem is proved.

For a proof of Lemma 4 we consider the Whitney decomposition of \( V \), i.e. \( Q_k \) are closed cubes with sides parallel to the axes with the following properties:

(i) \( \bigcup Q_k = V \);
(ii) the interiors of \( Q_k \) are mutually disjoint;
(iii) \[
\text{(4.2) } \text{diam}(Q_k) \leq \text{dist}(Q_k, V^c) \leq 4 \text{diam}(Q_k)
\]

([13, Theorem 1 on p.167]). Let \( \tilde{Q}_k \) be the cube which has the same center as \( Q_k \) but is expanded by the factor \( 9/8 \). Then

\[
\text{(4.3) } \text{the multiplicity of } \tilde{Q}_k \text{ is bounded by } N_1,
\]

where \( N_1 \) depends only on the dimension \( n \) ([13, Proposition 3 on p.169]). In view of (4.2) we can choose a constant \( c_0, 0 < c_0 < 1 \), with the property that

\[
\text{(4.4) } B(x, c_0 \delta_V(x)) \cap Q_k \neq \emptyset \implies B(x, c_0 \delta_V(x)) \subset \tilde{Q}_k.
\]

Using these facts, we can prove the following lemma.

**Lemma 5.** Suppose \( V \) is an open subset of \( \mathbb{R}^n \). Then there is a covering \( \mathcal{B} = \{B(x_j, r_j)\} \) of \( V \) such that

\[
\text{(4.5) } r_j \geq \delta_V(x_j),
\]

\[
\text{(4.6) } \sum_j r_j^\beta \leq A M_\beta(V),
\]

where \( A > 0 \) depends only on the dimension \( n \) and \( \beta \).

**Proof.** Since \( V \) is an open set, it follows that \( M_\beta(V) > 0 \). By definition we can find a covering \( \{B(\xi_j, \rho_j)\} \) of \( V \) such that

\[
\text{(4.7) } \sum_j \rho_j^\beta \leq 2 M_\beta(V).
\]

From this covering we construct a covering \( \mathcal{B} \) with the required properties.
Let $\bigcup_k Q_k$ be the Whitney decomposition of $V$ and let $\tilde{Q}_k$ be the expanded cube as before the lemma. We let

\[ K_1 = \{ k : \text{there is } B(\xi_j, \rho_j) \text{ meeting } Q_k \text{ such that } \rho_j \geq c_0 \delta_V(\xi_j) \}, \]

\[ K_2 = \{ k : \text{if } B(\xi_j, \rho_j) \text{ meets } Q_k, \text{ then } \rho_j < c_0 \delta_V(\xi_j) \}, \]

where $c_0$ is the constant appearing in (4.4).

First suppose $k \in K_1$. We can find $j = j(k)$ such that $B(\xi_j, \rho_j) \cap Q_k \neq \emptyset$ and $\rho_j \geq c_0 \delta_V(\xi_j)$. Let $\xi \in B(\xi_j, \rho_j) \cap Q_k$. We have from (4.2)

\[ \text{diam}(Q_k) \leq \text{dist}(Q_k, V^c) \leq \delta_V(\xi) \leq \delta_V(\xi_j) + \rho_j \leq (1 + c_0^{-1})\rho_j. \]

Hence $Q_k \subset B(\xi_j, (2 + c_0^{-1})\rho_j)$, so that

\[ (4.8) \quad \bigcup_{k \in K_1} Q_k \subset \bigcup_{k \in K_1} B(\xi_j(k), (2 + c_0^{-1})\rho_j(k)), \]

\[ (4.9) \quad (2 + c_0^{-1})\rho_j(k) \geq (2 + c_0^{-1})c_0 \delta_V(\xi_j(k)) \geq \delta_V(\xi_j(k)). \]

Second suppose $k \in K_2$. Since $\rho_j < c_0 \delta_V(\xi_j)$ for $B(\xi_j, \rho_j) \cap Q_k \neq \emptyset$, we obtain from (4.4) that

\[ Q_k \subset \bigcup_{B(\xi_j, \rho_j) \cap Q_k \neq \emptyset} B(\xi_j, \rho_j) \subset \tilde{Q}_k. \]

From the first inclusion we have

\[
|Q_k| \leq A \sum_{B(\xi_j, \rho_j) \cap Q_k \neq \emptyset} \rho_j^n = A|Q_k| \sum_{B(\xi_j, \rho_j) \cap Q_k \neq \emptyset} \left( \frac{\rho_j}{\text{diam}(Q_k)} \right)^n
\leq A|Q_k| \sum_{B(\xi_j, \rho_j) \cap Q_k \neq \emptyset} \left( \frac{\rho_j}{\text{diam}(Q_k)} \right)^\beta,
\]

so that the second inclusion yields

\[ \text{diam}(Q_k)^\beta \leq A \sum_{B(\xi_j, \rho_j) \cap Q_k \neq \emptyset} \rho_j^\beta \leq A \sum_{B(\xi_j, \rho_j) \subset \tilde{Q}_k} \rho_j^\beta. \]

Hence

\[ (4.10) \quad \sum_{k \in K_2} \text{diam}(Q_k)^\beta \leq A \sum_{k \in K_2} \sum_{B(\xi_j, \rho_j) \subset \tilde{Q}_k} \rho_j^\beta \leq AN_1 \sum_j \rho_j^\beta, \]

where the last inequality follows from (4.3). Note that $Q_k \subset B(x_{Q_k}, \text{diam}(Q_k))$ with $x_{Q_k}$ being the center of $Q_k$. We have from (4.2)

\[ (4.11) \quad \delta_V(x_{Q_k}) \leq \text{dist}(Q_k, V^c) + \text{diam}(Q_k) \leq 5 \text{diam}(Q_k). \]
We observe from (4.7), (4.8) and (4.10) that
\[ B = \left\{ B(\xi_j(k), (2 + c_0^{-1})\rho_j(k)) : k \in K_1 \right\} \cup \left\{ B(x_{Q_k}, 5 \text{diam}(Q_k)) : k \in K_2 \right\} \]
is a covering of \( V \) and
\[
\sum_{k \in K_1} ((2 + c_0^{-1})\rho_j(k))^\beta \leq (2 + c_0^{-1})^\beta \sum_j \rho_j^\beta \leq 2(2 + c_0^{-1})^\beta M_\beta(V),
\]
\[
\sum_{k \in K_2} (5 \text{diam}(Q_k))^\beta \leq A \sum_j \rho_j^\beta \leq AM_\beta(V).
\]
Thus (4.6) follows. We obtain from (4.9) and (4.11) that our covering \( B \) satisfies (4.5).

The lemma is proved.

**Proof of Lemma 4.** First we claim
\[
(4.12) \Omega(y) \subset x + \Omega(y + \frac{2}{a_0} |x|),
\]
where \( a_0 \) is the constant appearing in (NS). We may assume that \( x \neq 0 \). Suppose \( \xi \in \Omega(y) \). Then \( (\xi, y) \in \Omega \) and
\[
|\xi - x| = |x| < 2|x| = a_0(y + \frac{2}{a_0} |x| - y).
\]
Hence (NS) implies that \( \xi - x \in \Omega(y + 2|x|/a_0) \), or equivalently \( \xi \in x + \Omega(y + 2|x|/a_0) \).

The claim is proved.

By Lemma 5 we find a covering \( B = \{ B(x_j, r_j) \} \) of \( V \) satisfying (4.5) and (4.6). Suppose \( x \in B(x_j, r_j) \). Then \( |x - x_j| < r_j \) and \( \delta_V(x) \leq 2r_j \) by (4.5), so that
\[
\Omega(\delta_V(x)) \subset x_j - x + \Omega(\delta_V(x) + \frac{2}{a_0} |x - x_j|) \subset x_j - x + \Omega(A_3r_j)
\]
with \( A_3 = 2 + 2/a_0 \) by (4.12). Hence \( x + \Omega(\delta_V(x)) \subset x_j + \Omega(A_3r_j) \), so that
\[
\bigcup_{x \in B(x_j, r_j)} (x + \Omega(\delta_V(x))) \subset x_j + \Omega(A_3r_j).
\]
By [11, Lemma 1 (d)] we find points \( u_{j,\nu} \) (\( \nu = 1, ..., M \)) such that
\[
\Omega(A_3r_j) \subset \bigcup_{\nu=1}^M B(u_{j,\nu}, 3A_3r_j),
\]
where the number \( M \) depends only on \( \Omega \). Therefore
\[
\bigcup_{x \in V} (x + \Omega(\delta_V(x))) \subset \bigcup_{j} \bigcup_{\nu=1}^M B(x_j + u_{j,\nu}, 3A_3r_j).
\]
Hence by (4.6)
\[
M_\beta \left( \bigcup_{x \in V} (x + \Omega(\delta_V(x))) \right) \leq \sum_{j} \sum_{\nu=1}^M (3A_3r_j)^\beta \leq AM_\beta(V).
\]
The lemma is proved.
5. Boundary behavior of harmonic functions

In what follows we are interested in the boundary behavior of harmonic functions in \( \mathbb{R}^{n+1}_+ \). In [3] we introduced the notion of thinness at the boundary. For a set \( E \subset \mathbb{R}^{n+1}_+ \) we put \( E_t = \{ (x,y) \in E : 0 < y < t \} \) and \( E^* = \bigcup_{(x,y) \in E} B(x,y) \). We recall that \( B(x,y) \) is the \( n \)-dimensional ball with center at \( x \) and radius \( y \), so that \( E^* \) is a set on the boundary \( \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+ \). We shall combine the above notation and write simply \( E^*_t \) for \((E_t)^*\), i.e.,

\[
E^*_t = \bigcup_{0 < y < t} B(x,y).
\]

**Definition.** Let \( E \subset \mathbb{R}^{n+1}_+ \). We say that \( E \) is \( B_{\alpha,p} \)-thin at \( \partial \mathbb{R}^{n+1}_+ \) if

\[
\lim_{t \to 0} B_{\alpha,p}(E^*_t) = 0.
\]

For a function \( f \) on \( \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+ \) we denote by \( PI(f) \) its Poisson integral, i.e.

\[
PI(f)(x,y) = \int_{\mathbb{R}^n} \frac{A_n y}{(|x-z| + y)^{(n+1)/2}} f(z) dz,
\]

where \( A_n > 0 \) is such that \( PI(1) = 1 \). In [3] we have proved

**Theorem C.** Let \( 1 \leq p < \infty \) and \( \alpha p \leq n \). Let \( \Omega \subset \mathbb{R}^{n+1}_+ \) and suppose \( \Omega \cap \partial \mathbb{R}^{n+1}_+ = \{0\} \). Suppose \( f \in L^p(\mathbb{R}^n) \). Then there is a set \( E \subset \mathbb{R}^{n+1}_+ \) such that \( E \) is \( B_{\alpha,p} \)-thin at \( \partial \mathbb{R}^{n+1}_+ \) and that

\[
\lim_{t \to 0} B_{\alpha,p}(E^*_t) = 0.
\]

for \( B_{\alpha,p} \)-a.e. \( x \in \partial \mathbb{R}^{n+1}_+ \), i.e. there is a set \( F \subset \partial \mathbb{R}^{n+1}_+ \) such that \( B_{\alpha,p}(F) = 0 \) and (5.1) holds at every \( x \in \partial \mathbb{R}^{n+1}_+ \setminus F \).

Using Theorem 3, we can show

**Theorem 4.** Let \( 1 \leq p < \infty \), \( 0 < \alpha < n \), \( 0 \leq n - \alpha p < \beta \leq n \), \( \gamma = (n - \alpha p)/\beta \), \( c > 0 \) and let \( \varphi(r) = \varphi_{\beta,p}(r) \) be as in (3.3) if \( \alpha p = n \). Suppose \( \Omega \) satisfies (NS). Let

\[
\Omega_{\gamma,c} = \{(x,y) : x \in \Omega(cy^\gamma)\} \quad \text{and} \quad \Omega_{\varphi,c} = \{(x,y) : x \in \Omega(c\varphi(y))\}.
\]

If \( E \) is \( B_{\alpha,p} \)-thin at \( \partial \mathbb{R}^{n+1}_+ \), then

\[
M_{\beta} \left( \bigcap_{t>0} \{ x : (x + \Omega_{\gamma,c}) \cap E_t \neq \emptyset \} \right) = 0 \quad \text{if} \ \alpha p < n,
\]

\[
M_{\beta} \left( \bigcap_{t>0} \{ x : (x + \Omega_{\varphi,c}) \cap E_t \neq \emptyset \} \right) = 0 \quad \text{if} \ \alpha p = n.
\]
In other words, there is a set \( F \subset \partial \mathbb{R}^{n+1}_+ \) of \( \beta \)-dimensional Hausdorff measure zero such that for \( x \in \partial \mathbb{R}^{n+1}_+ \setminus F \), \( \Omega_{\gamma,c} \) and \( \Omega_{\varphi,c} \) lie eventually outside \( E \), i.e., there is \( t = t_x > 0 \) such that \( E_t \cap (x + \Omega_{\gamma,c}) = \emptyset \) and \( E_t \cap (x + \Omega_{\varphi,c}) = \emptyset \).

**Proof.** We prove the theorem only in the case \( \alpha p < n \), since the case \( \alpha p = n \) is similarly proved. We can easily show that

\[
\{ x \in \mathbb{R}^n : (x + \Omega_{\gamma,c}) \cap E \neq \emptyset \} \subset \bigcup_{x \in E^*} \{ x - \Omega(c\delta_{E^*}(x)^\gamma)) \},
\]

where \( \delta_{E^*}(x) = \text{dist}(x, E^*) \) ([3, Lemma 2]). We apply Theorem 3 with \( E \) replaced by \( E^* \). Then

\[
M_\beta(\{ x \in \mathbb{R}^n : (x + \Omega_{\gamma,c}) \cap E \neq \emptyset \}) \leq M_\beta \left( \bigcup_{x \in E^*} \{ x - \Omega(c\delta_{E^*}(x)^\gamma)) \} \right) \leq AB_{\alpha,p}(E^*).
\]

Apply this inequality with \( E \) replaced by \( E_t \). Then the definition of thinness implies that

\[
M_\beta(\{ x \in \mathbb{R}^n : (x + \Omega_{\gamma,c}) \cap E_t \neq \emptyset \}) \leq AB_{\alpha,p}(E_t^*) \to 0 \quad \text{as} \quad t \to 0.
\]

Thus the theorem follows.

As a corollary to Theorems C and 4 we have

**Theorem 5.** Let \( 1 \leq p < \infty \), \( 0 < \alpha < n \), \( 0 \leq n - \alpha p < \beta \leq n \), \( \gamma = (n - \alpha p)/\beta \), \( c > 0 \) and let \( \varphi(r) = \varphi_{\beta,p}(r) \) be as in (3.3) if \( \alpha p = n \). Suppose \( \Omega \) satisfies (NS) and let \( \Omega_{\gamma,c} \) and \( \Omega_{\varphi,c} \) be as in Theorem 4. If \( f \in L^p(\mathbb{R}^n) \), then there is a set \( F \subset \partial \mathbb{R}^{n+1}_+ \) of \( \beta \)-dimensional Hausdorff measure zero such that

\[
\lim_{P \to x} \int_{P \times x + \Omega_{\gamma,c}} |f| = g_\alpha \ast f(x) \quad \text{for all} \quad c > 0 \quad \text{if} \quad \alpha p < n,
\]

\[
\lim_{P \to x} \int_{P \times x + \Omega_{\varphi,c}} |f| = g_\alpha \ast f(x) \quad \text{for all} \quad c > 0 \quad \text{if} \quad \alpha p = n
\]

at every \( x \in \partial \mathbb{R}^{n+1}_+ \setminus F \).

Let \( \Omega \) be the nontangential cone \( \{ (x,y) : |x| < y \} \). Then the approach regions in Theorem 5 are represented as \( \Omega_{\gamma,c} = \{ (x,y) : |x| < cy^\gamma \} \) and \( \Omega_{\varphi,c} = \{ (x,y) : |x| < c\varphi(y) \} \). Hence our Theorem 5 particularly yields the following corollary.

**Corollary 2.** Let \( 1 \leq p < \infty \), \( 0 < \alpha < n \), \( 0 \leq n - \alpha p < \beta \leq n \), \( \gamma = (n - \alpha p)/\beta \), \( c > 0 \) and let \( \varphi(r) = \varphi_{\beta,p}(r) \) be as in (3.3) if \( \alpha p = n \). If \( f \in L^p(\mathbb{R}^n) \), then there is a set \( F \subset \partial \mathbb{R}^{n+1}_+ \) such that \( M_\beta(F) = 0 \) and

\[
\lim_{P \to x} \int_{P \times x + \Omega_{\gamma,c}} |f| = g_\alpha \ast f(x) \quad \text{for all} \quad c > 0 \quad \text{if} \quad \alpha p < n,
\]

\[
\lim_{P \to x} \int_{P \times x + \Omega_{\varphi,c}} |f| = g_\alpha \ast f(x) \quad \text{for all} \quad c > 0 \quad \text{if} \quad \alpha p = n
\]
at every \( x \in \partial \mathbb{R}^{n+1}_+ \setminus F \).

Remark. Ahern and Nagel [2, Corollary 6.3] showed the above corollary for \( \alpha p < n \) by using a different method. Mizuta [9] studied the tangential boundary behavior of harmonic functions with gradient in \( L^p \). If \( p \geq 2 \), then his result improves Corollary 2. Ahern and Nagel [2, Corollary 7.3] also gave the same result.

6. Integration with respect to Hausdorff content

For a function \( F \) on \( \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+ \) we denote by \( NF(x) \) the nontangential maximal function of the Poisson integral of \( F \), i.e.

\[
NF(x) = \sup_{x+\Gamma} |PI(F)|,
\]

where \( \Gamma = \{(x, y) : |x| < y\} \) is the nontangential cone with vertex at the origin. Similarly, we define a tangential maximal function by

\[
\mathcal{M}_{\gamma,c}F(x) = \sup_{x+\Omega_{\gamma,c}} |PI(F)| \quad \text{and} \quad \mathcal{M}_{\varphi,c}F(x) = \sup_{x+\Omega_{\varphi,c}} |PI(F)|,
\]

where \( \Omega_{\gamma,c} \) and \( \Omega_{\varphi,c} \) are as in Theorem 4. We define the integral of \( u \geq 0 \) with respect to the Hausdorff content \( M_\beta \) by

\[
\int u^p dM_\beta = \int_0^\infty M_\beta(\{x : u(x) > t\}) dt^p.
\]

If \( \beta = n \), then the above integral is comparable to the usual Lebesgue integral.

**Theorem 6.** Let \( 1 < p < \infty, 0 < \alpha < n, 0 \leq n - \alpha p < \beta \leq n, \gamma = (n - \alpha p)/\beta, c > 0 \) and let \( \varphi(r) = \varphi_{\beta,p}(r) \) be as in (3.3) if \( \alpha p < n \). Suppose \( \Omega \) satisfies (NS). If \( f \in L^p(\mathbb{R}^n) \), then

\[
\int \mathcal{M}_{\gamma,c}(g_\alpha * f)^p dM_\beta \leq A\|f\|_p^p, \quad \text{if } \alpha p < n,
\]

\[
\int \mathcal{M}_{\varphi,c}(g_\alpha * f)^p dM_\beta \leq A\|f\|_p^p, \quad \text{if } \alpha p = n,
\]

where \( A > 0 \) depends only on \( n, \alpha, p, c, \beta \) and \( \Omega \).

**Proof.** We prove the theorem only in the case \( \alpha p < n \), since the case \( \alpha p = n \) is similarly proved. Let \( t > 0, E = \{(x, y) : |PI(g_\alpha * f)(x, y)| > t\} \) and \( E^* \) be as in Section 5. It is easy to see that \( E^* = \{x : N(g_\alpha * f)(x) > t\} \) and \( \{x : \mathcal{M}_{\gamma,c}(g_\alpha * f)(x) > t\} = \{x \in \mathbb{R}^n : (x + \Omega_{\gamma,c}) \cap E \neq \emptyset\} \). Hence, by (5.2) and Hansson’s theorem ([5] and [10, 3.7]),

\[
\int \mathcal{M}_{\gamma,c}(g_\alpha * f)^p dM_\beta = \int_0^\infty M_\beta(\{x : \mathcal{M}_{\gamma,c}(g_\alpha * f)(x) > t\}) dt^p
\]

\[
\leq A \int_0^\infty B_{\alpha,p}(\{x : N(g_\alpha * f)(x) > t\}) dt^p
\]

\[
\leq A \int_0^\infty B_{\alpha,p}(\{x : g_\alpha * Nf(x) > t\}) dt^p
\]

\[
\leq A\|Nf\|_p^p \leq A\|f\|_p^p,
\]

\[
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\]
where the second inequality follows from the obvious inequality $N(g_\alpha \ast f) \leq g_\alpha \ast Nf$ (cf. [10, p.344]). The theorem is proved.

**Remark.** If $\beta = n$, then Theorem 6 is included in [10, Theorem 3.8]. If $\beta < n$, then Theorem 6 improves [10, Theorem 3.12]. Ahern and Nagel [2, Theorem 6.2] showed Theorem 6 for $\alpha p < n$ by using a different method.

**References**


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