BESSEL CAPACITY, HAUSDORFF CONTENT AND THE TANGENTIAL BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

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Abstract. We compare the Bessel capacity with the Hausdorff content. For $E \subset \mathbb{R}^n$ we let $\widetilde{E}_{\gamma,c} = \bigcup_{x\in E} B(x, c\delta_E(x)^\gamma)$ with $c > 0$ and $0 < \gamma \le 1$. If E is an open set and $0 < \gamma < 1$, then $\widetilde{E}_{\gamma,c}$ is larger than E. It is shown that the Bessel capacity of $\widetilde{E}_{\gamma,c}$ is estimated above by the Hausdorff content of E. This estimation is applied to the tangential boundary behavior of harmonic functions in the upper half space.

1. Introduction

Let $K(r) \not\equiv 0$ be a nonnegative nonincreasing lower semicontinuous (l. s. c.) function for $r > 0$. For $x \in \mathbb{R}^n$ we define $K(x) = K(|x|)$, and assume that $K(x)$ is locally integrable on $\mathbb{R}^n$. For $E \subset \mathbb{R}^n$ we define the capacity C_K by

$$C_K(E) = \inf\{\|\mu\| : K * \mu \ge 1 \text{ on } E\},$$

where $\|\mu\|$ denotes the total mass of a measure μ. Let $k_\alpha(r) = r^{\alpha-n}$ for $0 < \alpha < n$. This is the Riesz kernel of order α. If $K(r) = k_\alpha(r)$, then we write C_α for C_K and call it the Riesz capacity of order α.

Let $h(r)$ be a positive nondecreasing function for $r > 0$ and $h(0) = 0$. Such a function is called a measure function. We define the content M_h by

$$M_h(E) = \inf\{\sum h(r_j) : E \subset \bigcup B(x_j, r_j)\},$$

where $B(x, r)$ stands for the open ball with center at x and radius r. If $h(r) = r^\beta$, then we write M_β for M_h and call it β-content. There is a close connection between C_α and M_β. The following theorem is well-known (cf. [4, §IV] and [6, Theorems 5.13 and 5.14]).

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Theorem A.

(i) *If* $M_{n-\alpha}(E)=0$, *then* $C_\alpha(E)=0$.
(ii) *Let* $n-\alpha<\beta\le n$. *Then* $C_\alpha(E)=0$ *implies* $M_\beta(E)=0$.
(iii) *There is a set* E *such that* $C_\alpha(E)=0$ *and* $M_{n-\alpha}(E)>0$.

It is easy to see that C_α and $M_{n-\alpha}$ are both homogeneous of degree $n-\alpha$. From this fact, we can easily obtain the above (i). However, in view of (iii), $M_{n-\alpha}(E)=0$ is not characterized by $C_\alpha(E)=0$. We have only partial comparison (ii).

One of the main purposes of this paper is to compare C_α with a certain quantity, which may be regarded as an $(n-\alpha)$-dimensional quantity. Hereafter we shall use the following notation. By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_1, A_2, \dots,$ to specify them. We shall say that two positive quantities f and g are comparable, written $f\approx g$, if and only if there exists a constant A such that $A^{-1}g\le f\le Ag$. By $|E|$ we denote the Lebesgue measure of E.

For $c>0$ and $0<\gamma\le 1$ we define

$$\widetilde{E}_{\gamma,c}=\bigcup_{x\in E}B(x,c\delta_E(x)^\gamma),$$

where $\delta_E(x)=\operatorname{dist}(x,E^c)$. If E is an open set and $0<\gamma<1$, then $\widetilde{E}_{\gamma,c}$ is a proper extension of E. Moreover, if $E=B(0,r)$ and $r>0$ is small, then $\widetilde{E}_{\gamma,c}$ is a ball with radius comparable to cr^γ, so that

$$M_\beta(\widetilde{E}_{\gamma,c})\approx r^{\gamma\beta}\approx M_\beta(E)^\gamma.$$

So, one may regard $M_\beta(\widetilde{E}_{\gamma,c})$ as a $\beta\gamma$-dimensional quantity. If $\beta=n$, then $M_\beta(E)$ is comparable with the Lebesgue measure $|E|$. Let g_α be the Bessel kernel. The Riesz and the Bessel kernels have the same asymptotics as $r\to 0$. However, $g_\alpha(r)$ decreases rapidly as $r\to\infty$ and hence g_α is integrable on $\mathbb{R}^n$. The capacity $C_{g_\alpha}(E)$ is called the Bessel capacity of index $(\alpha,1)$ and is denoted by $B_{\alpha,1}(E)$. It is well known that

$$C_\alpha(E)\approx B_{\alpha,1}(E)\text{ for }E\subset U,$$

where U is a bounded set. Thus the Riesz capacity C_α and the Bessel capacity $B_{\alpha,1}$ have the same null sets. In the previous paper [3] we have proved

Theorem B. *Let* $0<\alpha<n$, $c=1$ *and* $\gamma=(n-\alpha)/n$. *Then*

$$|\widetilde{E}_{\gamma,c}|\le AB_{\alpha,1}(E),$$

where $A>0$ *depends only on* n *and* α.

Here we generalize Theorem B to

Theorem 1. *Let $0 < n - \alpha < \beta \le n$, $\gamma = (n-\alpha)/\beta$ and $c > 0$. Then*

$$M_\beta(\widetilde{E}_{\gamma,c}) \le AB_{\alpha,1}(E),$$

where $A > 0$ depends only on n, α, β and c.

Actually, in [3], general kernels and capacities were treated. Our argument here for Theorem 1 is very different from that of [3] and heavily depends on the Bessel kernel. The case when $\beta = n$ was dealt with in [3]. We see that $M_\beta(E)$ and the Lebesgue measure $|E|$ are comparable in this case. The main idea in [3] was to compare a test measure for the capacity with the Lebesgue measure on a ball whose volume is equal to its capacity. In case $\beta < n$, a difficulty arises from the lack of a measure corresponding to the Lebesgue measure. We shall employ the Frostman lemma and the Besicovitch covering lemma (see Lemmas A and B below). We shall convert the measure given by the Frostman lemma so that the converted measure becomes a test measure for the dual definition of $B_{\alpha,1}$ (see Lemma C below).

We can consider a counterpart of Theorem 1 for L^p-capacity theory. Let $1 < p < \infty$. We define

$$C_{K,p}(E) = \inf\{\|f\|_p^p : K * f \ge 1 \text{ on } E\}.$$

If $K = k_\alpha$, then we write $R_{\alpha,p}(E)$ for $C_{K,p}(E)$ and call it the Riesz capacity of index (α, p). If $K = g_\alpha$, then we write $B_{\alpha,p}(E)$ for $C_{K,p}(E)$ and call it the Bessel capacity of index (α, p). In case $\alpha p < n$, the Riesz capacity $R_{\alpha,p}$ is homogeneous of degree $n - \alpha p$; the Riesz capacity $R_{\alpha,p}(E)$ and the Bessel capacity $B_{\alpha,p}(E)$ are comparable for $E \subset U$, where U is a bounded set.

Theorem 2. *Let $1 < p < \infty$, $0 < n - \alpha p < \beta \le n$, $\gamma = (n - \alpha p)/\beta$ and $c > 0$. Then*

$$M_\beta(\widetilde{E}_{\gamma,c}) \le AB_{\alpha,p}(E),$$

where $A > 0$ depends only on n, α, p, β and c.

The proof of Theorem 2 will use the same converted measure as in the proof of Theorem 1, the dual definition of $B_{\alpha,p}$ and the Hedberg–Wolff lemma (see Lemmas D and E). We shall later generalize these theorems, in connection with Nagel-Stein approach regions ([11]). We shall introduce a notion of "thin sets" and combine it with the generalized version of Theorems 1 and 2 to obtain the tangential boundary behavior of harmonic functions given as the Poisson integral of Bessel potentials.

The plan of this paper is as follows. We shall prove Theorems 1 and 2 in Sections 2 and 3, respectively. A theorem similar to Theorem 2 for the case $\alpha p = n$ will be given also in Section 3. In Section 4 we shall introduce the Nagel-Stein approach region and generalize Theorems 1 and 2. The boundary behavior of harmonic functions will be considered in Section 5. Finally, a norm estimate of tangential maximal functions of Poisson integrals will be given in Section 6. We shall observe that our arguments yield different proofs of Ahern-Nagel [2, Theorem 6.2 and Corollary 6.3].

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2. Proof of Theorem 1

Let us recall the fundamental lemma due to Frostman (see e.g. [4, Theorem 1 on p. 7] and [6, Lemma 5.4]).

Lemma A. *Let h be a measure function. Suppose F is a compact set such that $M_h(F) > 0$. Then there is a measure μ supported on F such that*

$$\|\mu\| \approx M_h(F),$$
$$\mu(B(x,r)) \le h(r) \text{ for all } x \in \mathbb{R}^n \text{ and } r > 0.$$

We also need the Besicovitch covering lemma (see e.g. [14, Theorem 1.3.5]).

Lemma B. *Let E be a set in $\mathbb{R}^n$ and suppose that $r(x)$ is a positive bounded function on E. Then we can select $\{x_j\} \subset E$ with the following properties:*

(i) $E \subset \bigcup_j B(x_j, r(x_j))$.
(ii) *The multiplicity of $\{B(x_j, r(x_j))\}$ is bounded by a positive constant N depending only on the dimension. In other words, $\sum \chi_{B(x_j,r(x_j))} \le N$.*

We note the dual definition of C_K.

Lemma C. *Let E be an analytic set. Then*

$$C_K(E) = \sup\{\|\mu\| : \mu \text{ is concentrated on } E, K * \mu \le 1 \text{ on } \mathbb{R}^n\}.$$

For each integer ν we let G_ν be the family of cubes

$$Q = \{(x_1, \dots, x_n) : \frac{k_i}{2^\nu} \le x_i < \frac{k_i + 1}{2^\nu},\ i = 1, \dots, n\},$$

where $k_1, \dots, k_n$ are integers. We let $G = \{G_\nu\}_{\nu=-\infty}^{\infty}$. For a cube Q of side length ℓ we put $\tau_h(Q) = h(\ell)$ and define

$$m_h(E) = \inf\{\sum_{j=1}^{\infty} \tau_h(Q_j) : E \subset \bigcup_{j=1}^{\infty} Q_j,\ Q_j \in G\}.$$

Then it is easy to see that

$$M_h(E) \approx m_h(E) \quad \text{for any set } E \tag{2.1}$$

([4, (1.3) on p. 7]). We observe that m_h has the increasing property.

Lemma 1. *Let $\lim_{r\to\infty} h(r) = \infty$. If $E_j \uparrow E$, then $\lim_{j\to\infty} m_h(E_j) = m_h(E)$. In particular, if E is an F_σ-set, then*

$$m_h(E) = \sup_{\substack{F \subset E \\ F \text{ is compact}}} m_h(F).$$

Proof. It is clear that $\lim_{j\to\infty} m_h(E_j) \le m_h(E)$. Hence, it is sufficient to show the opposite inequality, under the assumption that $\lim_{j\to\infty} m_h(E_j) < \infty$. Let $\varepsilon > 0$. By definition we find cubes $Q_{j,i} \in G$ such that

$$E_j \subset \bigcup_{i=1}^{\infty} Q_{j,i},$$
$$\sum_{i=1}^{\infty} \tau_h(Q_{j,i}) < m_h(E_j) + \varepsilon 2^{-j}.$$

Since $\lim_{j\to\infty} m_h(E_j) < \infty$ and $\lim_{r\to\infty} h(r) = \infty$, it follows that the side lengths of $Q_{j,i}$ are bounded. Hence we can select maximal cubes $Q_1, Q_2, \dots, Q_\nu, \dots$ whose union covers $E = \bigcup_{j=1}^{\infty} E_j$. Now, in the same way as in [12, Theorem 52], we can show

$$\sum_{\nu=1}^{\infty} \tau_h(Q_\nu) \le \lim_{j\to\infty} m_h(E_j) + 2\varepsilon,$$

and hence $m_h(E) \le \lim_{j\to\infty} m_h(E_j) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the lemma follows.

As a corollary to (2.1) and Lemma 1 we have the following:

Corollary 1. *Let* $\lim_{r\to\infty} h(r) = \infty$. *If* E *is an* F_σ*-set, then*

$$M_h(E) \approx \sup_{\substack{F\subset E \\ F \text{ is compact}}} M_h(F).$$

Remark. The assumption that $\lim_{r\to\infty} h(r) = \infty$ is essential in Lemma 1. In fact, suppose that $\lim_{r\to\infty} h(r) = a < \infty$. Then, by definition, $m_h(E) \le a$ for any bounded set E. On the other hand it is easy to see that $m_h(\mathbb{R}^n) = \infty$ if $\liminf_{r\to 0} h(r)/r > 0$. Thus the increasing property does not hold in general. This example is suggested by K. Hatano. We observe that [4, (3.2) on p.9] actually requires some additional assumption like $\lim_{r\to\infty} h(r) = \infty$ or the boundedness of E.

From Lemmas A, C and 1 we show the following lemma.

Lemma 2. *Let* $0 < n - \alpha < \beta \le n$. *Then*

$$M_\beta(E) \le A B_{\alpha,1}(E),$$

where $A > 0$ *depends only on* n, α *and* β.

Proof. Since $B_{\alpha,1}$ is an outer capacity, i.e.,

$$B_{\alpha,1}(E) = \inf_{\substack{E\subset U \\ U \text{ is open}}} B_{\alpha,1}(U),$$

we may assume that E is an open set. Let F be a compact subset of E. By Lemma A there is a measure μ on F such that

$$\|\mu\| \approx M_\beta(F), \tag{2.2}$$

$$\mu(B(x,r)) \le r^\beta \text{ for all } x \in \mathbb{R}^n \text{ and } r > 0. \tag{2.3}$$

Observe from (2.3) that

$$g_\alpha * \mu(x) = \int_0^\infty g_\alpha(r) d\mu(B(x,r)) = \int_0^\infty \mu(B(x,r)) d(-g_\alpha(r)) \le \int_0^\infty r^\beta d(-g_\alpha(r)) = A_1 < \infty.$$

Hence Lemma C and (2.2) yield

$$B_{\alpha,1}(E) \ge A_1^{-1}\|\mu\| \approx M_\beta(F).$$

Taking the supremum over all F, we obtain the required inequality from Corollary 1. The lemma follows.

Proof of Theorem 1. By (2.1) and Lemma 1 we may assume that E is a bounded set. Since $B_{\alpha,1}$ is an outer capacity, we may furthermore assume that E is an open set. By Lemma 2 we have only to show that

$$M_\beta(\widetilde{E}_{\gamma,c} \setminus E) \le AB_{\alpha,1}(E).$$

In view of Corollary 1 it is sufficient to show that

$$M_\beta(F) \le AB_{\alpha,1}(E) \tag{2.4}$$

for any compact subset F of $\widetilde{E}_{\gamma,c} \setminus E$, since $\widetilde{E}_{\gamma,c} \setminus E$ is an F_σ-set. By Lemma A we can find a measure μ on F satisfying (2.2) and (2.3).

By definition, for each $x \in \widetilde{E}_{\gamma,c} \setminus E$, there is $x^* \in E$ such that $x \in B(x^*, c\delta_E(x^*)^\gamma)$. We let

$$r(x) = \sup_{\substack{x^* \in E \\ x \in B(x^*, c\delta_E(x^*)^\gamma)}} \delta_E(x^*).$$

We observe that $r(x)$ is a positive bounded function on $\widetilde{E}_{\gamma,c} \setminus E$. We invoke Lemma B and find $\{x_j\} \subset F$ such that

$$F \subset \bigcup B(x_j, 2cr_j^\gamma) \text{ with } r_j = r(x_j), \tag{2.5}$$

$$\text{the multiplicity of } \{B(x_j, 2cr_j^\gamma)\} \text{ is bounded by } N. \tag{2.6}$$

By definition we can find $x_j^* \in E$ such that

$$r_j/2 < \delta_E(x_j^*) \le r_j, \tag{2.7}$$
$$|x_j - x_j^*| < cr_j^\gamma. \tag{2.8}$$

We put $\mu_j = \mu|_{B(x_j, 2cr_j^\gamma)}$ and observe from (2.5) and (2.6) that

$$\mu \le \sum \mu_j \le N\mu. \tag{2.9}$$

From μ_j we construct a measure λ_j as follows: for Borel sets S

$$\lambda_j(S) = \mu_j(4(S - x_j^*) + x_j) \qquad \text{if } cr_j^\gamma \le r_j,$$
$$\lambda_j(S) = \mu_j(4cr_j^{\gamma-1}(S - x_j^*) + x_j) \qquad \text{if } cr_j^\gamma > r_j.$$

It is easy to see that

$$\lambda_j \text{ is concentrated on } B(x_j^*, \frac{1}{2}\min\{cr_j^\gamma, r_j\}), \tag{2.10}$$
$$\|\lambda_j\| = \|\mu_j\|, \tag{2.11}$$
$$\lambda_j(B(x,\rho)) = \mu_j(B(x,\rho)) = \|\mu_j\| \tag{2.12}$$
$$\text{for } \rho \ge \max\{|x - x_j| + 2cr_j^\gamma, |x - x_j^*| + \frac{1}{2}\min\{cr_j^\gamma, r_j\}\}.$$

Moreover, in view of (2.3)

$$\|\lambda_j\| = \|\mu_j\| \le (2cr_j^\gamma)^\beta; \tag{2.13}$$

for all $x \in \mathbb{R}^n$ and $r > 0$

$$\lambda_j(B(x,r)) \le (4r)^\beta \qquad \text{if } cr_j^\gamma \le r_j, \tag{2.14}$$
$$\lambda_j(B(x,r)) \le (4cr_j^{\gamma-1}r)^\beta \qquad \text{if } cr_j^\gamma > r_j. \tag{2.15}$$

It follows from (2.7) that $B(x_j^*, r_j/2) \subset E$ and so from (2.10) that the measure λ_j is concentrated on E. Let $\lambda = \sum \lambda_j$. We claim

$$g_\alpha * \lambda \le A_2 \text{ on } \mathbb{R}^n. \tag{2.16}$$

If we have (2.16), then the proof is easy. Since λ is concentrated on E, it follows from Lemma C and (2.11) that

$$B_{\alpha,1}(E) \ge A_2^{-1}\|\lambda\| = A_2^{-1}\sum \|\mu_j\| \ge A_2^{-1}\|\mu\|.$$

This, together with (2.2), yields (2.4).

Let us prove (2.16). Hereafter we fix $x \in \mathbb{R}^n$. First we claim

$$g_\alpha * \lambda_j(x) \le A \tag{2.17}$$

with A independent of j and x. Suppose $cr_j^\gamma \le r_j$. Then by (2.14)

$$g_\alpha * \lambda_j(x) = \int_0^\infty \lambda_j(B(x,r))d(-g_\alpha(r)) \le \int_0^\infty (4r)^\beta d(-g_\alpha(r)) = A < \infty.$$

Thus (2.17) follows. Suppose $cr_j^\gamma > r_j$. Then by (2.13) and (2.15)

$$\begin{aligned} g_\alpha * \lambda_j(x) &= \int_0^\infty \lambda_j(B(x,r))d(-g_\alpha(r)) \\ &\le \int_0^\infty \min\{(2cr_j^\gamma)^\beta, (4cr_j^{\gamma-1}r)^\beta\}d(-g_\alpha(r)) \\ &= \int_0^{r_j/2} (4cr_j^{\gamma-1}r)^\beta d(-g_\alpha(r)) + (2cr_j^\gamma)^\beta \int_{r_j/2}^\infty d(-g_\alpha(r)) \\ &\le Ar_j^{(\gamma-1)\beta} r_j^{\beta+\alpha-n} + Ar_j^{\gamma\beta} r_j^{\alpha-n} = A < \infty. \end{aligned}$$

Thus (2.17) follows in this case, too.

Let us write

$$\lambda' = \sum{}' \lambda_j, \quad \lambda'' = \sum{}'' \lambda_j,$$

where $\sum'$ (resp. $\sum''$) denotes the summation over j for which $x \in B(x_j, 2cr_j^\gamma)$ (resp. $x \notin B(x_j, 2cr_j^\gamma)$). In view of (2.6), the number of j appearing in $\sum'$ is at most N. Hence by (2.17)

$$g_\alpha * \lambda'(x) \le A. \tag{2.18}$$

Next, we consider $g_\alpha * \lambda''(x)$. Let us estimate $\lambda''(B(x,r)) = \sum'' \lambda_j(B(x,r))$. In the summation $\sum''$, we may consider only j such that $\lambda_j(B(x,r)) > 0$. By (2.10) this implies that $|x - x_j^*| \le r + cr_j^\gamma/2$. In view of the definition of $\sum''$, we have $|x - x_j| \ge 2cr_j^\gamma$. Using these inequalities and (2.8), we obtain

$$r + cr_j^\gamma/2 \ge |x - x_j^*| \ge |x - x_j| - |x_j - x_j^*| \ge 2cr_j^\gamma - cr_j^\gamma = cr_j^\gamma,$$

so that $r \ge cr_j^\gamma/2$, $|x - x_j^*| \le 2r$, $|x_j - x_j^*| \le 2r$ and $|x - x_j| \le 4r$. Hence

$$\max\{|x - x_j| + 2cr_j^\gamma, |x - x_j^*| + \frac{1}{2}\min\{cr_j^\gamma, r_j\}\} \le \max\{8r, 3r\} = 8r.$$

Therefore, (2.12) implies that $\lambda_j(B(x,8r)) = \mu_j(B(x,8r))$, so that

$$\begin{aligned}\lambda''(B(x,r)) &= \sum{}''\lambda_j(B(x,r)) \\ &\le \sum{}''\lambda_j(B(x,8r)) = \sum{}''\mu_j(B(x,8r)) \\ &\le \sum \mu_j(B(x,8r)) \le N\mu(B(x,8r)),\end{aligned}$$

where the last inequality follows from (2.9). Hence by (2.3)

$$\lambda''(B(x,r)) \le N(8r)^\beta \text{ for all } r > 0. \tag{2.19}$$

Thus

$$g_\alpha * \lambda''(x) = \int_0^\infty \lambda''(B(x,r))d(-g_\alpha(r)) \le A\int_0^\infty r^\beta d(-g_\alpha(r)) = A < \infty.$$

This, together with (2.18), yields (2.16). The proof is complete.

3. Proof of Theorem 2

Let $\frac{1}{p} + \frac{1}{q} = 1$. We have the dual definition of $C_{K,p}$ ([8, Theorem 14]).

Lemma D. *Let E be an analytic set. Then*

$$C_{K,p}(E) = \sup\{\|\mu\|^p : \mu \text{ is concentrated on } E, \|K*\mu\|_q \le 1\}.$$

Let $\alpha p \le n$. We put

$$W^\mu_{\alpha,p}(x) = \int_0^1 \left(\frac{\mu(B(x,r))}{r^{n-\alpha p}}\right)^{q-1}\frac{dr}{r}.$$

Hedberg and Wolff [7] proved the following lemma (see also [1] and [14, Theorem 4.7.5]).

Lemma E. *Let $\alpha p \le n$. Then*

$$\|g_\alpha * \mu\|_q^q \approx \int W^\mu_{\alpha,p}(x)d\mu(x).$$

In the same way as in the proof of Lemma 2, we obtain the following lemma from Lemmas A, D and E.

Lemma 3. *Let $1 < p < \infty$ and $0 \le n - \alpha p < \beta \le n$. Then*

$$M_\beta(E) \le AB_{\alpha,p}(E),$$

where $A > 0$ depends only on n, α, p and β.

Proof. Since $B_{\alpha,p}$ is an outer capacity, we may assume that E is an open set. Let F be a compact subset of E. By Lemma A there is a measure μ on F satisfying (2.2) and (2.3). Observe from (2.3) that

$$W^\mu_{\alpha,p}(x) \le \int_0^1 \left(\frac{r^\beta}{r^{n-\alpha p}}\right)^{q-1} \frac{dr}{r} = A < \infty,$$

since $n - \alpha p < \beta$. Hence Lemma E yields $\|g_\alpha * \mu\|_q^q \le A\|\mu\|$, or equivalently

$$\left\| g_\alpha * \frac{\mu}{A\|\mu\|^{1/q}} \right\|_q \le 1.$$

Hence Lemma D and (2.2) yield

$$B_{\alpha,p}(E) \ge \left(\frac{\|\mu\|}{A\|\mu\|^{1/q}}\right)^p = A\|\mu\| \approx M_\beta(F).$$

Taking the supremum over all F, we obtain the required inequality from Corollary 1.

Proof of Theorem 2. We may assume that E is a bounded open set. In view of Lemma 3 and Corollary 1 it is sufficient to show that

$$M_\beta(F) \le AB_{\alpha,p}(E) \tag{3.1}$$

for any compact set $F \subset \widetilde{E}_{\gamma,c} \setminus E$. In the same way as in the proof of Theorem 1 we can find a measure μ on F satisfying (2.2) and (2.3). We find balls $B(x_j, 2cr_j^\gamma)$ satisfying (2.5) and (2.6). Let $\mu_j = \mu|_{B(x_j,2cr_j^\gamma)}$ and let λ_j, λ, λ' and λ'' be as in the proof of Theorem 1. Observe that (2.9)–(2.15) and (2.19) hold. In particular λ is concentrated on E and

$$\|\lambda\| \approx \|\mu\| \approx M_\beta(F). \tag{3.2}$$

If $cr_j^\gamma \le r_j$, then by (2.14)

$$W^{\lambda_j}_{\alpha,p}(x) \le A \int_0^1 \left(\frac{(4r)^\beta}{r^{n-\alpha p}}\right)^{q-1} \frac{dr}{r} = A < \infty.$$

If $cr_j^\gamma > r_j$, then by (2.13) and (2.15)

$$W^{\lambda_j}_{\alpha,p}(x) \le A \int_0^1 \left(\frac{(\min\{4cr_j^{\gamma-1}r, 2cr_j^\gamma\})^\beta}{r^{n-\alpha p}} \right)^{q-1} \frac{dr}{r} \le A < \infty.$$

Thus $W^{\lambda_j}_{\alpha,p}(x) \le A$ in any case, and hence from (2.6) we have $W^{\lambda'}_{\alpha,p}(x) \le A$. From (2.19) we have

$$W^{\lambda''}_{\alpha,p}(x) \le A \int_0^1 \left(\frac{(8r)^\beta}{r^{n-\alpha p}} \right)^{q-1} \frac{dr}{r} = A < \infty.$$

Thus $W^{\lambda}_{\alpha,p}(x) \le A$. Hence Lemma E yields $\|g_\alpha * \lambda\|_q^q \le A\|\lambda\|$, or equivalently

$$\left\| g_\alpha * \frac{\lambda}{A\|\lambda\|^{1/q}} \right\|_q \le 1.$$

Since λ is concentrated on E, it follows from Lemma D and (3.2) that

$$B_{\alpha,p}(E) \ge \left(\frac{\|\lambda\|}{A\|\lambda\|^{1/q}} \right)^p = A\|\lambda\| \approx M_\beta(F).$$

Thus (3.1) follows. The theorem is proved.

Observe that if $r > 0$ is small, then

$$B_{\alpha,p}(B(0,r)) \approx \begin{cases} r^{n-\alpha p} & \text{if } \alpha p < n, \\ \left(\log \frac{1}{r} \right)^{1-p} & \text{if } \alpha p = n. \end{cases}$$

Therefore, it may be natural to consider a logarithmic expansion in case $\alpha p = n$.

Theorem 2'. *Let $1 < p < \infty$, $\alpha p = n$, $0 < \beta \le n$ and $c > 0$. We put*

$$\varphi(r) = \varphi_{\beta,p}(r) = \begin{cases} \left(\log \frac{1}{r} \right)^{(1-p)/\beta}, & 0 < r < 1/2, \\ 2(\log 2)^{(1-p)/\beta} r, & r \ge 1/2 \end{cases} \tag{3.3}$$

and

$$\widetilde{E}_{\varphi,c} = \bigcup_{x \in E} B(x, c\varphi(\delta_E(x))).$$

Then

$$M_\beta(\widetilde{E}_{\varphi,c}) \le AB_{\alpha,p}(E),$$

where $A > 0$ depends only on n, α, p, β and c.

Proof. We can prove the theorem in a way similar to Theorem 2. But for the completeness we give a proof. We observe that $\varphi(r)$ is a positive continuous increasing function. We may assume that E is a bounded open set. In view of Lemma 3 and Corollary 1 it is sufficient to show that

$$M_\beta(F) \le AB_{\alpha,p}(E) \tag{3.4}$$

for any compact subset $F \subset \widetilde{E}_{\varphi,c} \setminus E$. In the same way as in the proof of Theorem 1 we can find a measure μ on F satisfying (2.2) and (2.3). Let

$$\rho(x) = \sup_{\substack{x^* \in E \\ x \in B(x^*, c\varphi(\delta_E(x^*)))}} \delta_E(x^*)$$

and observe that $\rho(x)$ is a positive bounded function on $\widetilde{E}_{\varphi,c} \setminus E$. By Lemma B we find $\{x_j\} \subset F$ such that

$$F \subset \bigcup B(x_j, 2c\varphi(r_j)) \text{ with } r_j = \rho(x_j), \tag{3.5}$$

$$\text{the multiplicity of } \{B(x_j, 2c\varphi(r_j))\} \text{ is bounded by } N. \tag{3.6}$$

By definition we can find $x_j^* \in E$ such that

$$r_j/2 < \delta_E(x_j^*) \le r_j \text{ and } |x_j - x_j^*| < c\varphi(r_j). \tag{3.7}$$

We put $\mu_j = \mu|_{B(x_j, 2c\varphi(r_j))}$ and observe from (3.5) and (3.6) that

$$\mu \le \sum \mu_j \le N\mu.$$

From μ_j we construct a measure λ_j as follows: for Borel sets S

$$\begin{aligned} \lambda_j(S) &= \mu_j(4(S - x_j^*) + x_j) && \text{if } c\varphi(r_j) \le r_j, \\ \lambda_j(S) &= \mu_j(4c\varphi(r_j)r_j^{-1}(S - x_j^*) + x_j) && \text{if } c\varphi(r_j) > r_j. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &\lambda_j \text{ is concentrated on } B(x_j^*, \tfrac{1}{2}\min\{c\varphi(r_j), r_j\}), \\ &\|\lambda_j\| = \|\mu_j\| \le (2c\varphi(r_j))^\beta, \\ &\lambda_j(B(x,\rho)) = \mu_j(B(x,\rho)) = \|\mu_j\| \\ &\qquad \text{for } \rho \ge \max\{|x - x_j| + 2c\varphi(r_j), |x - x_j^*| + \tfrac{1}{2}\min\{c\varphi(r_j), r_j\}\}, \end{aligned}$$

and for all $x \in \mathbb{R}^n$ and $r > 0$

$$\lambda_j(B(x,r)) \le (4r)^\beta \qquad \text{if } c\varphi(r_j) \le r_j,$$
$$\lambda_j(B(x,r)) \le (4c\varphi(r_j)r_j^{-1}r)^\beta \quad \text{if } c\varphi(r_j) > r_j.$$

Let $\lambda = \sum \lambda_j$. It follows from (3.7) that $B(x_j^*, r_j/2) \subset E$ so that the measure λ_j is concentrated on E, and so is λ. We claim

$$W^{\lambda_j}_{\alpha,p}(x) \le A \tag{3.8}$$

with A independent of j and x. If $c\varphi(r_j) \le r_j$, then

$$W^{\lambda_j}_{\alpha,p}(x) \le A \int_0^1 (4r)^{\beta(q-1)} \frac{dr}{r} = A < \infty,$$

so that (3.8) follows. If $c\varphi(r_j) > r_j$, then

$$\begin{aligned} W^{\lambda_j}_{\alpha,p}(x) &\le A \int_0^1 \min\{(4c\varphi(r_j)r_j^{-1}r)^\beta, (2c\varphi(r_j))^\beta\}^{q-1} \frac{dr}{r} \\ &\le A\varphi(r_j)^{\beta(q-1)} \int_0^1 \min\{\frac{r}{r_j}, 1\}^{\beta(q-1)} \frac{dr}{r} \\ &\le \begin{cases} A\varphi(r_j)^{\beta(q-1)}(\frac{1}{\beta(q-1)} + \log\frac{1}{r_j}) & \text{if } 0 < r_j < 1, \\ A\varphi(r_j)^{\beta(q-1)}\frac{1}{\beta(q-1)} r_j^{-\beta(q-1)} & \text{if } r_j \ge 1, \end{cases} \end{aligned}$$

so that in view of the definition of φ we have (3.8) in this case, too. Let us write

$$\lambda' = {\sum}'\lambda_j, \quad \lambda'' = {\sum}''\lambda_j,$$

where $\sum'$ (resp. $\sum''$) denotes the summation over j for which $x \in B(x_j, 2c\varphi(r_j))$ (resp. $x \notin B(x_j, 2c\varphi(r_j))$). In view of (3.6) the number of j appearing in $\sum'$ is at most N. Hence (3.8) implies that

$$W^{\lambda'}_{\alpha,p}(x) \le A. \tag{3.9}$$

In the same way as in the proof of Theorem 1 we estimate $\lambda''(B(x,r))$. Observe that if $x \notin B(x_j, 2c\varphi(r_j))$ and $\lambda_j(B(x,r)) > 0$, then $|x - x_j| + 2c\varphi(r_j) < 8r$, so that $\lambda_j(B(x,8r)) = \mu_j(B(x,8r))$ and (2.19) holds. Therefore

$$W^{\lambda''}_{\alpha,p}(x) \le A \int_0^1 (8r)^{\beta(q-1)} \frac{dr}{r} = A < \infty.$$

This, together with (3.9), yields

$$W^{\lambda}_{\alpha,p} \le A \text{ on } \mathbb{R}^n.$$

Hence Lemmas D and E and (2.2) imply

$$B_{\alpha,p}(E) \ge A\|\lambda\| \approx \|\mu\| \approx M_\beta(F).$$

Thus (3.4) follows. The theorem is proved.

4. Generalization

Let Ω be a set in $\mathbb{R}^{n+1}_+$ with $\overline{\Omega} \cap \partial\mathbb{R}^{n+1}_+ = \{0\}$. For simplicity we assume that $\Omega \supset \{(0, y) : y > 0\}$. Put $\Omega(y) = \{x : (x, y) \in \Omega\}$. We say that Ω satisfies the Nagel-Stein condition (abbreviated to (NS)), if

(i) $|\Omega(y)| \le Ay^n$ with $A = A(\Omega)$;
(ii) there is $a_0 > 0$ such that

$$(x_1, y_1) \in \Omega \text{ and } |x - x_1| < a_0(y - y_1) \implies (x, y) \in \Omega.$$

It is easy to see that $\Omega(y)$ is an increasing set function of y, i.e., if $y_1 < y_2$, then $\Omega(y_1) \subset \Omega(y_2)$. For E we put

$$\widetilde{E}_{\gamma,c;\Omega} = \bigcup_{x\in E} \left(x + \Omega(c\delta_E(x)^\gamma)\right).$$

We have a generalization of Theorems 1, 2 and 2'.

Theorem 3. *Let $1 \le p < \infty$, $0 < \alpha < n$, $0 \le n - \alpha p < \beta \le n$, $\gamma = (n - \alpha p)/\beta$, $c > 0$ and let $\varphi(r) = \varphi_{\beta,p}(r)$ be as in (3.3) if $\alpha p = n$. Let Ω satisfy (NS). Then*

$$M_\beta(\widetilde{E}_{\gamma,c;\Omega}) \le AB_{\alpha,p}(E) \qquad \text{if } \alpha p < n,$$
$$M_\beta(\widetilde{E}_{\varphi,c;\Omega}) \le AB_{\alpha,p}(E) \qquad \text{if } \alpha p = n,$$

where $A > 0$ depends only on n, α, p, β, c and Ω.

We shall prove this theorem as a corollary to Theorems 1, 2 and 2' and the following lemma.

Lemma 4. *Let $0 < \beta \le n$ and let Ω satisfy (NS). If V is an open subset of $\mathbb{R}^n$, then*

$$M_\beta\left(\bigcup_{x\in V} (x + \Omega(\delta_V(x)))\right) \le AM_\beta(V),$$

where $\delta_V(x) = \operatorname{dist}(x, V^c)$ and $A > 0$ depends only on β, Ω and n.

If we assume Lemma 4, then the proof of Theorem 3 is easy.

Proof of Theorem 3. We prove the theorem only in the case $\alpha p < n$, since the case $\alpha p = n$ is similarly proved. First we claim that

$$\widetilde{E}_{\gamma,c;\Omega} \subset \bigcup_{x\in\widetilde{E}_{\gamma,c}} (x + \Omega(\delta_{\widetilde{E}_{\gamma,c}}(x))). \tag{4.1}$$

Suppose $x \in E$. By definition $B(x, c\delta_E(x)^\gamma) \subset \widetilde{E}_{\gamma,c}$, so that $c\delta_E(x)^\gamma \le \delta_{\widetilde{E}_{\gamma,c}}(x)$. Hence

$$\widetilde{E}_{\gamma,c;\Omega} = \bigcup_{x\in E} (x + \Omega(c\delta_E(x)^\gamma)) \subset \bigcup_{x\in E} (x + \Omega(\delta_{\widetilde{E}_{\gamma,c}}(x))) \subset \bigcup_{x\in\widetilde{E}_{\gamma,c}} (x + \Omega(\delta_{\widetilde{E}_{\gamma,c}}(x))).$$

Thus (4.1) follows. Combining (4.1), Lemma 4 with $V = \widetilde{E}_{\gamma,c}$ and Theorems 1 and 2, we obtain

$$M_\beta(\widetilde{E}_{\gamma,c;\Omega}) \le M_\beta\left(\bigcup_{x\in\widetilde{E}_{\gamma,c}} (x+\Omega(\delta_{\widetilde{E}_{\gamma,c}}(x))\right) \le AM_\beta(\widetilde{E}_{\gamma,c}) \le AB_{\alpha,p}(E).$$

Thus the theorem is proved.

For a proof of Lemma 4 we consider the Whitney decomposition of V, i.e. Q_k are closed cubes with sides parallel to the axes with the following properties:

(i) $\bigcup Q_k = V$;
(ii) the interiors of Q_k are mutually disjoint;
(iii)

$$\operatorname{diam}(Q_k) \le \operatorname{dist}(Q_k, V^c) \le 4\operatorname{diam}(Q_k) \tag{4.2}$$

([13, Theorem 1 on p.167]). Let $\widetilde{Q}_k$ be the cube which has the same center as Q_k but is expanded by the factor 9/8. Then

$$\text{the multiplicity of } \widetilde{Q}_k \text{ is bounded by } N_1, \tag{4.3}$$

where N_1 depends only on the dimension n ([13, Proposition 3 on p.169]). In view of (4.2) we can choose a constant c_0, $0 < c_0 < 1$, with the property that

$$B(x, c_0\delta_V(x)) \cap Q_k \ne \emptyset \implies B(x, c_0\delta_V(x)) \subset \widetilde{Q}_k. \tag{4.4}$$

Using these facts, we can prove the following lemma.

Lemma 5. *Suppose V is an open subset of $\mathbb{R}^n$. Then there is a covering $\mathcal{B} = \{B(x_j, r_j)\}$ of V such that*

$$r_j \ge \delta_V(x_j), \tag{4.5}$$

$$\sum_j r_j^\beta \le AM_\beta(V), \tag{4.6}$$

where $A > 0$ depends only on the dimension n and β.

Proof. Since V is an open set, it follows that $M_\beta(V) > 0$. By definition we can find a covering $\{B(\xi_j, \rho_j)\}$ of V such that

$$\sum_j \rho_j^\beta \le 2M_\beta(V). \tag{4.7}$$

From this covering we construct a covering $\mathcal{B}$ with the required properties.

Let $\bigcup_k Q_k$ be the Whitney decomposition of V and let $\widetilde{Q}_k$ be the expanded cube as before the lemma. We let

$$\mathcal{K}_1 = \{k : \text{ there is } B(\xi_j,\rho_j) \text{ meeting } Q_k \text{ such that } \rho_j \ge c_0\delta_V(\xi_j)\},$$
$$\mathcal{K}_2 = \{k : \text{ if } B(\xi_j,\rho_j) \text{ meets } Q_k, \text{ then } \rho_j < c_0\delta_V(\xi_j)\},$$

where c_0 is the constant appearing in (4.4).

First suppose $k \in \mathcal{K}_1$. We can find $j = j(k)$ such that $B(\xi_j,\rho_j) \cap Q_k \neq \emptyset$ and $\rho_j \ge c_0\delta_V(\xi_j)$. Let $\xi \in B(\xi_j,\rho_j) \cap Q_k$. We have from (4.2)

$$\operatorname{diam}(Q_k) \le \operatorname{dist}(Q_k, V^c) \le \delta_V(\xi) \le \delta_V(\xi_j) + \rho_j \le (1 + c_0^{-1})\rho_j.$$

Hence $Q_k \subset B(\xi_j, (2 + c_0^{-1})\rho_j)$, so that

$$\bigcup_{k\in\mathcal{K}_1} Q_k \subset \bigcup_{k\in\mathcal{K}_1} B(\xi_{j(k)}, (2 + c_0^{-1})\rho_{j(k)}), \tag{4.8}$$

$$(2 + c_0^{-1})\rho_{j(k)} \ge (2 + c_0^{-1})c_0\delta_V(\xi_{j(k)}) \ge \delta_V(\xi_{j(k)}). \tag{4.9}$$

Second suppose $k \in \mathcal{K}_2$. Since $\rho_j < c_0\delta_V(\xi_j)$ for $B(\xi_j,\rho_j) \cap Q_k \neq \emptyset$, we obtain from (4.4) that

$$Q_k \subset \bigcup_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} B(\xi_j,\rho_j) \subset \widetilde{Q}_k.$$

From the first inclusion we have

$$\begin{aligned}|Q_k| \le A \sum_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} \rho_j^n &= A|Q_k| \sum_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} \left(\frac{\rho_j}{\operatorname{diam}(Q_k)}\right)^n \\ &\le A|Q_k| \sum_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} \left(\frac{\rho_j}{\operatorname{diam}(Q_k)}\right)^\beta,\end{aligned}$$

so that the second inclusion yields

$$\operatorname{diam}(Q_k)^\beta \le A \sum_{B(\xi_j,\rho_j)\cap Q_k\neq\emptyset} \rho_j^\beta \le A \sum_{B(\xi_j,\rho_j)\subset\widetilde{Q}_k} \rho_j^\beta.$$

Hence

$$\sum_{k\in\mathcal{K}_2} \operatorname{diam}(Q_k)^\beta \le A \sum_{k\in\mathcal{K}_2} \sum_{B(\xi_j,\rho_j)\subset\widetilde{Q}_k} \rho_j^\beta \le AN_1 \sum_j \rho_j^\beta, \tag{4.10}$$

where the last inequality follows from (4.3). Note that $Q_k \subset B(x_{Q_k}, \operatorname{diam}(Q_k))$ with x_{Q_k} being the center of Q_k. We have from (4.2)

$$\delta_V(x_{Q_k}) \le \operatorname{dist}(Q_k, V^c) + \operatorname{diam}(Q_k) \le 5\operatorname{diam}(Q_k). \tag{4.11}$$

We observe from (4.7), (4.8) and (4.10) that

$$\mathcal{B} = \{B(\xi_{j(k)}, (2 + c_0^{-1})\rho_{j(k)}) : k \in \mathcal{K}_1\} \cup \{B(x_{Q_k}, 5\,\mathrm{diam}(Q_k)) : k \in \mathcal{K}_2\}$$

is a covering of V and

$$\sum_{k\in\mathcal{K}_1} ((2 + c_0^{-1})\rho_{j(k)})^\beta \le (2 + c_0^{-1})^\beta \sum_j \rho_j^\beta \le 2(2 + c_0^{-1})^\beta M_\beta(V),$$

$$\sum_{k\in\mathcal{K}_2} (5\,\mathrm{diam}(Q_k))^\beta \le A \sum_j \rho_j^\beta \le AM_\beta(V).$$

Thus (4.6) follows. We obtain from (4.9) and (4.11) that our covering $\mathcal{B}$ satisfies (4.5). The lemma is proved.

Proof of Lemma 4. First we claim

$$\Omega(y) \subset x + \Omega(y + \frac{2}{a_0}|x|), \tag{4.12}$$

where a_0 is the constant appearing in (NS). We may assume that $x \ne 0$. Suppose $\xi \in \Omega(y)$. Then $(\xi, y) \in \Omega$ and

$$|(\xi - x) - \xi| = |x| < 2|x| = a_0(y + \frac{2}{a_0}|x| - y).$$

Hence (NS) implies that $\xi - x \in \Omega(y + 2|x|/a_0)$, or equivalently $\xi \in x + \Omega(y + 2|x|/a_0)$. The claim is proved.

By Lemma 5 we find a covering $\mathcal{B} = \{B(x_j, r_j)\}$ of V satisfying (4.5) and (4.6). Suppose $x \in B(x_j, r_j)$. Then $|x - x_j| < r_j$ and $\delta_V(x) \le 2r_j$ by (4.5), so that

$$\Omega(\delta_V(x)) \subset x_j - x + \Omega(\delta_V(x) + \frac{2}{a_0}|x - x_j|) \subset x_j - x + \Omega(A_3 r_j)$$

with $A_3 = 2 + 2/a_0$ by (4.12). Hence $x + \Omega(\delta_V(x)) \subset x_j + \Omega(A_3 r_j)$, so that

$$\bigcup_{x\in B(x_j,r_j)} (x + \Omega(\delta_V(x))) \subset x_j + \Omega(A_3 r_j).$$

By [11, Lemma 1 (d)] we find points $u_{j,\nu}$ $(\nu = 1, ..., M)$ such that

$$\Omega(A_3 r_j) \subset \bigcup_{\nu=1}^{M} B(u_{j,\nu}, 3A_3 r_j),$$

where the number M depends only on Ω. Therefore

$$\bigcup_{x\in V} (x + \Omega(\delta_V(x))) \subset \bigcup_j \bigcup_{\nu=1}^{M} B(x_j + u_{j,\nu}, 3A_3 r_j).$$

Hence by (4.6)

$$M_\beta\left(\bigcup_{x\in V} (x + \Omega(\delta_V(x)))\right) \le \sum_j \sum_{\nu=1}^{M} (3A_3 r_j)^\beta \le AM_\beta(V).$$

The lemma is proved.

5. Boundary behavior of harmonic functions

In what follows we are interested in the boundary behavior of harmonic functions in $\mathbb{R}^{n+1}_+$. In [3] we introduced the notion of thinness at the boundary. For a set $E \subset \mathbb{R}^{n+1}_+$ we put $E_t = \{(x,y) \in E : 0 < y < t\}$ and $E^* = \bigcup_{(x,y)\in E} B(x,y)$. We recall that $B(x,y)$ is the n-dimensional ball with center at x and radius y, so that E^* is a set on the boundary $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$. We shall combine the above notation and write simply E_t^* for $(E_t)^*$, i.e.,

$$E_t^* = \bigcup_{\substack{(x,y)\in E \\ 0<y<t}} B(x,y).$$

Definition. Let $E \subset \mathbb{R}^{n+1}_+$. We say that E is $B_{\alpha,p}$-thin at $\partial\mathbb{R}^{n+1}_+$ if

$$\lim_{t\to 0} B_{\alpha,p}(E_t^*) = 0.$$

For a function f on $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$ we denote by $PI(f)$ its Poisson integral, i.e.

$$PI(f)(x,y) = \int_{\mathbb{R}^n} \frac{A_n y}{(|x-z|^2 + y^2)^{(n+1)/2}} f(z)dz,$$

where $A_n > 0$ is such that $PI(1) = 1$. In [3] we have proved

Theorem C. *Let $1 \le p < \infty$ and $\alpha p \le n$. Let $\Omega \subset \mathbb{R}^{n+1}_+$ and suppose $\overline{\Omega} \cap \partial\mathbb{R}^{n+1}_+ = \{0\}$. Suppose $f \in L^p(\mathbb{R}^n)$. Then there is a set $E \subset \mathbb{R}^{n+1}_+$ such that E is $B_{\alpha,p}$-thin at $\partial\mathbb{R}^{n+1}_+$ and that*

$$\lim_{\substack{P\to x \\ P\in(x+\Omega)\setminus E}} PI(g_\alpha * f)(P) = g_\alpha * f(x) \tag{5.1}$$

for $B_{\alpha,p}$-a.e. $x \in \partial\mathbb{R}^{n+1}_+$, i.e. there is a set $F \subset \partial\mathbb{R}^{n+1}_+$ such that $B_{\alpha,p}(F) = 0$ and (5.1) holds at every $x \in \partial\mathbb{R}^{n+1}_+ \setminus F$.

Using Theorem 3, we can show

Theorem 4. *Let $1 \le p < \infty$, $0 < \alpha < n$, $0 \le n - \alpha p < \beta \le n$, $\gamma = (n-\alpha p)/\beta$, $c > 0$ and let $\varphi(r) = \varphi_{\beta,p}(r)$ be as in (3.3) if $\alpha p = n$. Suppose Ω satisfies (NS). Let*

$$\Omega_{\gamma,c} = \{(x,y) : x \in \Omega(cy^\gamma)\} \text{ and } \Omega_{\varphi,c} = \{(x,y) : x \in \Omega(c\varphi(y))\}.$$

If E is $B_{\alpha,p}$-thin at $\partial\mathbb{R}^{n+1}_+$, then

$$M_\beta\left(\bigcap_{t>0}\{x : (x+\Omega_{\gamma,c}) \cap E_t \ne \emptyset\}\right) = 0 \qquad \text{if } \alpha p < n,$$

$$M_\beta\left(\bigcap_{t>0}\{x : (x+\Omega_{\varphi,c}) \cap E_t \ne \emptyset\}\right) = 0 \qquad \text{if } \alpha p = n.$$

In other words, there is a set $F \subset \partial\mathbb{R}^{n+1}_+$ of β-dimensional Hausdorff measure zero such that for $x \in \partial\mathbb{R}^{n+1}_+ \setminus F$, $\Omega_{\gamma,c}$ and $\Omega_{\varphi,c}$ lie eventually outside E, i.e., there is $t = t_x > 0$ such that $E_t \cap (x + \Omega_{\gamma,c}) = \emptyset$ and $E_t \cap (x + \Omega_{\varphi,c}) = \emptyset$.

Proof. We prove the theorem only in the case $\alpha p < n$, since the case $\alpha p = n$ is similarly proved. We can easily show that

$$\{x \in \mathbb{R}^n : (x + \Omega_{\gamma,c}) \cap E \neq \emptyset\} \subset \bigcup_{x \in E^*} (x - \Omega(c\delta_{E^*}(x)^\gamma)),$$

where $\delta_{E^*}(x) = \operatorname{dist}(x, E^{*c})$ ([3, Lemma 2]). We apply Theorem 3 with E replaced by E^*. Then

(5.2)

$$M_\beta(\{x \in \mathbb{R}^n : (x + \Omega_{\gamma,c}) \cap E \neq \emptyset\}) \le M_\beta\left(\bigcup_{x \in E^*} (x - \Omega(c\delta_{E^*}(x)^\gamma))\right) \le AB_{\alpha,p}(E^*).$$

Apply this inequality with E replaced by E_t. Then the definition of thinness implies that

$$M_\beta(\{x \in \mathbb{R}^n : (x + \Omega_{\gamma,c}) \cap E_t \neq \emptyset\}) \le AB_{\alpha,p}(E_t^*) \to 0 \text{ as } t \to 0.$$

Thus the theorem follows.

As a corollary to Theorems C and 4 we have

Theorem 5. *Let $1 \le p < \infty$, $0 < \alpha < n$, $0 \le n - \alpha p < \beta \le n$, $\gamma = (n - \alpha p)/\beta$, $c > 0$ and let $\varphi(r) = \varphi_{\beta,p}(r)$ be as in (3.3) if $\alpha p = n$. Suppose Ω satisfies (NS) and let $\Omega_{\gamma,c}$ and $\Omega_{\varphi,c}$ be as in Theorem 4. If $f \in L^p(\mathbb{R}^n)$, then there is a set $F \subset \partial\mathbb{R}^{n+1}_+$ of β-dimensional Hausdorff measure zero such that*

$$\lim_{\substack{P \to x \\ P \in x + \Omega_{\gamma,c}}} PI(g_\alpha * f)(P) = g_\alpha * f(x) \text{ for all } c > 0 \qquad \text{if } \alpha p < n,$$

$$\lim_{\substack{P \to x \\ P \in x + \Omega_{\varphi,c}}} PI(g_\alpha * f)(P) = g_\alpha * f(x) \text{ for all } c > 0 \qquad \text{if } \alpha p = n$$

at every $x \in \partial\mathbb{R}^{n+1}_+ \setminus F$.

Let Ω be the nontangential cone $\{(x, y) : |x| < y\}$. Then the approach regions in Theorem 5 are represented as $\Omega_{\gamma,c} = \{(x, y) : |x| < cy^\gamma\}$ and $\Omega_{\varphi,c} = \{(x, y) : |x| < c\varphi(y)\}$. Hence our Theorem 5 particularly yields the following corollary.

Corollary 2. *Let $1 \le p < \infty$, $0 < \alpha < n$, $0 \le n - \alpha p < \beta \le n$, $\gamma = (n - \alpha p)/\beta$, $c > 0$ and let $\varphi(r) = \varphi_{\beta,p}(r)$ be as in (3.3) if $\alpha p = n$. If $f \in L^p(\mathbb{R}^n)$, then there is a set $F \subset \partial\mathbb{R}^{n+1}_+$ such that $M_\beta(F) = 0$ and*

$$\lim_{\substack{P \to x \\ P \in x + \Omega_{\gamma,c}}} PI(g_\alpha * f)(P) = g_\alpha * f(x) \text{ for all } c > 0 \qquad \text{if } \alpha p < n,$$

$$\lim_{\substack{P \to x \\ P \in x + \Omega_{\varphi,c}}} PI(g_\alpha * f)(P) = g_\alpha * f(x) \text{ for all } c > 0 \qquad \text{if } \alpha p = n,$$

at every $x \in \partial\mathbb{R}^{n+1}_+ \setminus F$.

Remark. Ahern and Nagel [2, Corollary 6.3] showed the above corollary for $\alpha p < n$ by using a different method. Mizuta [9] studied the tangential boundary behavior of harmonic functions with gradient in L^p. If $p \geq 2$, then his result improves Corollary 2. Ahern and Nagel [2, Corollary 7.3] also gave the same result.

6. Integration with respect to Hausdorff content

For a function F on $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$ we denote by $NF(x)$ the nontangential maximal function of the Poisson integral of F, i.e.

$$NF(x) = \sup_{x+\Gamma} |PI(F)|,$$

where $\Gamma = \{(x, y) : |x| < y\}$ is the nontangential cone with vertex at the origin. Similarly, we define a tangential maximal function by

$$\mathcal{M}_{\gamma,c}F(x) = \sup_{x+\Omega_{\gamma,c}} |PI(F)| \quad \text{and} \quad \mathcal{M}_{\varphi,c}F(x) = \sup_{x+\Omega_{\varphi,c}} |PI(F)|,$$

where $\Omega_{\gamma,c}$ and $\Omega_{\varphi,c}$ are as in Theorem 4. We define the integral of $u \geq 0$ with respect to the Hausdorff content M_β by

$$\int u^p dM_\beta = \int_0^\infty M_\beta(\{x : u(x) > t\}) dt^p.$$

If $\beta = n$, then the above integral is comparable to the usual Lebesgue integral.

Theorem 6. *Let* $1 < p < \infty$, $0 < \alpha < n$, $0 \leq n - \alpha p < \beta \leq n$, $\gamma = (n - \alpha p)/\beta$, $c > 0$ *and let* $\varphi(r) = \varphi_{\beta,p}(r)$ *be as in (3.3) if* $\alpha p = n$. *Suppose* Ω *satisfies (NS). If* $f \in L^p(\mathbb{R}^n)$, *then*

$$\int \mathcal{M}_{\gamma,c}(g_\alpha * f)^p dM_\beta \leq A\|f\|_p^p, \qquad \text{if } \alpha p < n,$$

$$\int \mathcal{M}_{\varphi,c}(g_\alpha * f)^p dM_\beta \leq A\|f\|_p^p, \qquad \text{if } \alpha p = n,$$

where $A > 0$ *depends only on* n, α, p, c, β *and* Ω.

Proof. We prove the theorem only in the case $\alpha p < n$, since the case $\alpha p = n$ is similarly proved. Let $t > 0$, $E = \{(x, y) : |PI(g_\alpha * f)(x, y)| > t\}$ and E^* be as in Section 5. It is easy to see that $E^* = \{x : N(g_\alpha * f)(x) > t\}$ and $\{x : \mathcal{M}_{\gamma,c}(g_\alpha * f)(x) > t\} = \{x \in \mathbb{R}^n : (x + \Omega_{\gamma,c}) \cap E \neq \emptyset\}$. Hence, by (5.2) and Hansson's theorem ([5] and [10, 3.7]),

$$\begin{aligned}\int \mathcal{M}_{\gamma,c}(g_\alpha * f)^p dM_\beta &= \int_0^\infty M_\beta(\{x : \mathcal{M}_{\gamma,c}(g_\alpha * f)(x) > t\}) dt^p \\ &\leq A \int_0^\infty B_{\alpha,p}(\{x : N(g_\alpha * f)(x) > t\}) dt^p \\ &\leq A \int_0^\infty B_{\alpha,p}(\{x : g_\alpha * Nf(x) > t\}) dt^p \\ &\leq A\|Nf\|_p^p \leq A\|f\|_p^p,\end{aligned}$$

where the second inequality follows from the obvious inequality $N(g_\alpha * f) \le g_\alpha * Nf$ (cf. [10, p.344]). The theorem is proved.

Remark. If $\beta = n$, then Theorem 6 is included in [10, Theorem 3.8]. If $\beta < n$, then Theorem 6 improves [10, Theorem 3.12]. Ahern and Nagel [2, Theorem 6.2] showed Theorem 6 for $\alpha p < n$ by using a different method.

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