Quasiadditivity and measure property of capacity and the tangential boundary behavior of harmonic functions

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Abstract. A certain quasiadditive property of capacity is shown. Namely, it is proved that if a set E is dispersely decomposed into subsets, then the capacity of E is comparable to the sum of the capacities of the subsets. From the quasiadditivity it is derived that the Lebesgue measure of a certain expanded set is estimated by the capacity of the original set. The estimation has an application to the boundary behavior of harmonic functions.

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1. Introduction

Throughout this article we denote by A a positive constant whose value is unimportant and may change from line to line. We say that two positive quantities f and g are comparable, written $f \approx g$, if there exists a constant A such that $A^{-1}g \le f \le Ag$. We say that a capacity C is quasiadditive if

$$C(E) \approx \sum C(E_j)$$

for some decomposition $E = \bigcup E_j$. In [3] and [4] the first author considered the quasiadditivity for certain capacities with respect to the Whitney decomposition. Here we give a different type of quasiadditivity.

Let $K(r) \not\equiv 0$ be a nonnegative nonincreasing lower semicontinuous (l. s. c.) function for $r > 0$. We assume that

$$\lim_{r\to 0} K(r) = \infty \text{ and } \lim_{r\to\infty} K(r) = 0.$$

For $x \in \mathbb{R}^N$ we define $K(x) = K(|x|)$, and assume that $K(x)$ is integrable over $\mathbb{R}^N$. We define the capacity C_K by

$$C_K(E) = \inf\{\|\mu\| : K * \mu \ge 1 \text{ on } E\}.$$

Define a positive function $\eta(r)$ by $|B(0,\eta(r))| = C_K(B(0,r))$ and put $\eta^*(r) = \max\{\eta(r), 2r\}$.

Theorem 1. *Suppose* $\{B(x_j,\eta_p^*(r_j))\}$ *is disjoint and* E *is an analytic subset of* $\bigcup B(x_j,r_j)$*. Then*

$$C_K(E) \approx \sum C_K(E\cap B(x_j,r_j)).$$

Theorem 1 has a counterpart in L^p-capacity and in energy capacity. Let $1 < p < \infty$. We define

$$C_{K,p}(E) = \inf\{\|f\|_p^p : K*f \ge 1 \text{ on } E,\ f\ge 0\}.$$

Define a positive function $\eta_p(r)$ by $|B(0,\eta_p(r))| = C_{K,p}(B(0,r))$ and put $\eta_p^*(r) = \max\{\eta_p(r), 2r\}$.

Theorem 2. *Suppose* $\{B(x_j,\eta_p^*(r_j))\}$ *is disjoint and* E *is an analytic subset of* $\bigcup B(x_j,r_j)$*. Then*

$$C_{K,p}(E) \approx \sum C_{K,p}(E\cap B(x_j,r_j)).$$

The energy capacity is defined by

$$e_K(E) = \sup\{\|\mu\|^2 :\ \mu \text{ is concentrated on } E,\ \int K*\mu d\mu \le 1\}.$$

We define $\eta_e(r)$ by $|B(0,\eta_e(r))| = e_K(B(0,r))$ and put $\eta_e^*(r) = \max\{\eta_e(r), 2r\}$.

Theorem 3. *Suppose* $\{B(x_j,\eta_e^*(r_j))\}$ *is disjoint and* E *is an analytic subset of* $\bigcup B(x_j,r_j)$*. Then*

$$e_K(E) \approx \sum e_K(E\cap B(x_j,r_j)).$$

From Theorems 1–3 we can deduce the following measure property of C_K, $C_{K,p}$ and e_K. For notational convenience we extend $C_{K,p}$, η_p and η_p^* for $p \ge 1$. Thus, $C_{K,1}$, η_1 and η_1^* mean C_K, η and η^*, respectively. For a proper subset E we put $\delta_E(x) = \operatorname{dist}(x, E^c)$. We put

$$\widetilde{E}_{K,p} = \bigcup_{x\in E} B(x,\eta_p^*(\delta_E(x))), \quad \widetilde{E}_{K,e} = \bigcup_{x\in E} B(x,\eta_e^*(\delta_E(x))).$$

Theorem 4. *Let* $p \ge 1$*. There is a positive constant* A *depending only on* N*,* K *and* p *such that*

$$|\widetilde{E}_{K,p}| \le AC_{K,p}(E),$$
$$|\widetilde{E}_{K,e}| \le Ae_K(E).$$

Theorem 4 has an application to the tangential boundary behavior of harmonic functions. We shall later give Theorem 5, a generalization of Theorem 4, in connection with Nagel-Stein approach regions ([10]). We shall introduce a notion of "thin sets" to obtain precise description of the tangential boundary behavior of harmonic functions given as the Poisson integral of certain potentials. We shall combine it with Theorem 5 and observe that [9, Theorem 2.9] follows.

The plan of this article is as follows: We prove Theorems 1–3 in the next section. Since the proofs are similar, we shall give a complete proof only for Theorem 3. For Theorems 1 and 2 we refer the reader to [6]. In Section 3 we shall prove Theorem 4 by using the usual covering lemma. We shall also indicate that if we invoke the covering lemma due to Nagel-Stein [10], then we obtain Theorem 5, a generalization of Theorem 4. In Section 4 we shall give some applications of Theorem 5 to the boundary behavior of harmonic functions.

2. Proof of Theorems 1–3

We prepare an elementary lemma.

Lemma 1. *Let $0 < 2r \le R$. Suppose $x \notin B(x_0, R)$ and let $\rho = \operatorname{dist}(x, B(x_0, r))$. Then*

$$|B(x,\rho) \cap B(x_0,R)| \ge A|B(x_0,R)|,$$

where A depends only on the dimension.

Proof. Let x_1 be the point on the line segment connecting x_0 and x such that $|x_1 - x_0| = \frac{3}{4}R$. It is easy to see that

$$\rho = |x - x_0| - r = |x - x_1| + \frac{3}{4}R - r \ge |x - x_1| + \frac{1}{4}R.$$

We observe that

$$B(x_1, \frac{1}{4}R) \subset B(x,\rho) \cap B(x_0,R),$$

since if $y \in (x_1, \frac{1}{4}R)$, then

$$|x - y| < |x - x_1| + \frac{1}{4}R \le \rho \text{ and } |x_0 - y| < |x_0 - x_1| + \frac{1}{4}R = R.$$

Thus the required inequality follows. □

Proof of Theorem 3. For simplicity we write E_j for $E \cap B(x_j, r_j)$. It is sufficient to show that $\sum e_K(E_j) \le A e_K(E)$. Let μ_j be the e_K-equilibrium measure for E_j, i.e.

$$\mu_j \text{ is concentrated on } E_j,$$

$$\int K * \mu_j d\mu_j = 1,$$

$$\|\mu_j\|^2 = e_K(E_j).$$

Let $\mu_j^* = e_K(E_j)^{1/2}\mu_j$ and let $\mu^* = \sum \mu_j^*$. We observe that

$$\int K * \mu_j^* d\mu_j^* = \|\mu_j^*\| = e_K(E_j) \text{ and } \|\mu^*\| = \sum e_K(E_j). \tag{1}$$

We put

$$f_j = \frac{\|\mu_j^*\|}{|B(x_j, \eta_e^*(r_j))|}\chi_{B(x_j,\eta_e^*(r_j))},$$

$d\mu_j' = f_j dx$ and $\mu' = \sum \mu_j'$. Then $\|\mu_j'\| = \|\mu_j^*\| = e_K(E_j)$ and $f_j \le \chi_{B(x_j,\eta_e^*(r_j))}$, since $|B(x_j, \eta_e^*(r_j))| \ge |B(x_j, \eta_e(r_j))| = e_K(B(x_j, r_j)) \ge e_K(E_j) = \|\mu_j^*\|$. We observe from the disjointness of $B(x_j, \eta_e^*(r_j))$ that $\sum f_j \le \sum \chi_{B(x_j,\eta_e^*(r_j))} \le 1$. Hence

$$K * \mu' = K * \left(\sum f_j\right) \le K * 1 = \int K dx < \infty. \tag{2}$$

Let us compare $K * \mu^*$ and $K * \mu'$. Suppose first $x \notin B(x_j, \eta_e^*(r_j))$. We apply Lemma 1 with $x_0 = x_j$, $R = \eta_e^*(r_j)$ and $r = r_j$. Let $\rho_j = \operatorname{dist}(x, B(x_j, r_j))$. We have

$$\begin{aligned} K * \mu_j'(x) &\ge \int_{B(x,\rho_j)\cap B(x_j,\eta_e^*(r_j))} K(x-y)d\mu_j'(y) \\ &\ge K(\rho_j)\mu_j'(B(x,\rho_j) \cap B(x_j, \eta_e^*(r_j))) \\ &= K(\rho_j)\|\mu_j^*\|\frac{|B(x,\rho_j) \cap B(x_j, \eta_e^*(r_j))|}{|B(x_j, \eta_e^*(r_j))|} \\ &\ge AK(\rho_j)\|\mu_j^*\|. \end{aligned}$$

Obviously, $K * \mu_j^*(x) \le K(\rho_j)\|\mu_j^*\|$, whence

$$K * \mu_j^*(x) \le AK * \mu_j'(x). \tag{3}$$

Now suppose $x \notin \bigcup B(x_j, \eta_e^*(r_j))$. Then (3) holds for all j, so that by (2)

$$K * \mu^*(x) = \sum K * \mu_j^*(x) \le A \sum K * \mu_j'(x) = AK * \mu'(x) \le A.$$

Suppose $x \in B(x_j, \eta_e^*(r_j))$. Then, by the disjointness of $\{B(x_j, \eta_e^*(r_j))\}$, we have $x \notin \bigcup_{i\ne j} B(x_i, \eta_e^*(r_i))$. Hence (2) and (3) yield

$$K * \mu^*(x) = K * \mu_j^*(x) + \sum_{i\ne j} K * \mu_i^*(x) \le K * \mu_j^*(x) + A.$$

Therefore

$$\begin{aligned}\int K*\mu^* d\mu^* &= \int_{\mathbb{R}^N\setminus\cup B(x_j,\eta_e^*(r_j))} K*\mu^* d\mu^* + \sum\int_{B(x_j,\eta_e^*(r_j))} K*\mu^* d\mu^* \\ &\le A\|\mu^*\| + \sum\int_{B(x_j,\eta_e^*(r_j))} K*\mu_j^* d\mu^* \\ &= A\|\mu^*\| + \sum\int_{B(x_j,\eta_e^*(r_j))} K*\mu_j^* d\mu_j^* \\ &= (A+1)\sum e_K(E_j),\end{aligned}$$

where the last equality follows from (1).

Now the proof is easy. Let

$$\widetilde{\mu} = \Big(\sum e_K(E_j)\Big)^{-1/2}\mu^*.$$

We have

$$\int K*\widetilde{\mu}\, d\widetilde{\mu} \le A,$$

$$\|\widetilde{\mu}\| = \Big(\sum e_K(E_j)\Big)^{1/2}.$$

Obviously $\widetilde{\mu}$ is concentrated on E. Hence by definition

$$e_K(E) \ge A\|\widetilde{\mu}\|^2 = A\sum e_K(E_j).$$

Thus the required inequality follows. The theorem is proved. □

The proofs of Theorems 1 and 2 can be carried out in a similar way with the help of Lemma 1 and the following dual definition of C_K and $C_{K,p}$ (cf. [8, Theorem 14]). For details we refer to [6].

Theorem A. *Let E be an analytic set. Then*

$$C_K(E) = \sup\{\|\mu\| : \mu \text{ is concentrated on } E, K*\mu \le 1 \text{ on } \mathbb{R}^N\}.$$

Theorem B. *Let E be an analytic set. Then*

$$C_{K,p}(E) = \sup\{\|\mu\|^p : \mu \text{ is concentrated on } E, \|K*\mu\|_q \le 1\},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

3. Proof of Theorem 4

Proof of Theorem 4. Since the proofs are similar, we shall prove only the first inequality. Take an arbitrary compact subset F of $\widetilde{E}_{K,p}$. By the usual covering lemma we can find $x_j \in E$ such that

$$\begin{aligned}&F \subset \bigcup B(x_j, 5\eta_p^*(r_j)),\\ &\{B(x_j, \eta_p^*(r_j))\} \text{ is disjoint},\\ &r_j = \delta_E(x_j).\end{aligned}$$

Let $E' = \bigcup B(x_j, r_j)$. By definition this is a subset of E. We apply Theorems 1 and 2 for $B(x_j, r_j)$ and E'. We obtain

$$\sum C_{K,p}(B(x_j, r_j)) \le AC_{K,p}(E') \le AC_{K,p}(E).$$

On the other hand we have

$$|F| \le \sum |B(x_j, 5\eta_p^*(r_j))| = A \sum |B(x_j, \eta_p^*(r_j))|.$$

It is easy to see that

$$|B(0, \eta_p^*(r))| \approx |B(0, \eta_p(r))| = C_{K,p}(B(0, r)).$$

Hence

$$|F| \le AC_{K,p}(E).$$

Since F is an arbitrary compact subset of $\widetilde{E}_{K,p}$, the required inequality follows. The theorem is proved. □

In Theorem 4 we have considered the enlargement based on balls. We can replace balls by the so-called Nagel-Stein regions. Let Ω be a set in $\mathbb{R}_+^{N+1}$ with $\overline{\Omega} \cap \partial\mathbb{R}_+^{N+1} = \{0\}$. Put $\Omega(y) = \{x : (x, y) \in \Omega\}$. We say that Ω satisfies the Nagel-Stein condition (abbreviated to (NS)), if

(i) $|\Omega(y)| \le Ay^N$ with $A = A(\Omega)$;
(ii) there is $\alpha > 0$ such that

$$(x_1, y_1) \in \Omega \text{ and } |x - x_1| < \alpha(y - y_1) \implies (x, y) \in \Omega.$$

Obviously, the nontangential cone $\Gamma = \{(x, y) : |x| < y\}$ satisfies (NS). The section $\Gamma(y)$ is the open ball with center at 0 and radius y. So, $\Omega(y)$ may be regarded as an extension of a ball. For E we put

$$\widetilde{E}_{K,p;\Omega} = \bigcup_{x \in E} \left(x - \Omega(\eta_p^*(\delta_E(x)))\right).$$

This is a generalization of $\widetilde{E}_{K,p}$.

Theorem 5. *Let Ω satisfy (NS). Then*

$$|\widetilde{E}_{K,p;\Omega}| \le AC_{K,p}(E),$$

where $A > 0$ depends only on N, K, p and Ω.

Theorem 5 can be proved in a similar way with the help of the covering lemma due to Nagel-Stein [10, pp.90–92]. For details we refer to [6].

4. Boundary behavior of harmonic functions

In what follows we are interested in the boundary behavior of harmonic functions in $\mathbb{R}^{N+1}_+$. Hereafter we let $1 < p < \infty$. Following the idea in [2] and [7] we introduce the notion of thinness at the boundary. For a set $E \subset \mathbb{R}^{N+1}_+$ we put $E_t = \{(x,y) : 0 < y < t\}$ and $E^* = \bigcup_{(x,y)\in E} B(x,y)$. We recall that $B(x,y)$ is the N-dimensional ball with center at x and radius y, so that the set E^* is a set on the boundary $\mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$. We shall combine the above notation and write

$$E^*_t = \bigcup_{(x,y)\in E, 0<y<t} B(x,y).$$

Definition. Let $E \subset \mathbb{R}^{N+1}_+$. We say that E is $C_{K,p}$-thin at $\partial\mathbb{R}^{N+1}_+$ if

$$\lim_{t\to 0} C_{K,p}(E^*_t) = 0.$$

Remark. If E is $C_{K,p}$-thin at $\partial\mathbb{R}^{N+1}_+$, then the essential projection of E

$$\{x : \text{ for any } t > 0 \text{ there is a positive number } y < t \text{ such that } (x,y) \in E\ \}$$

is of $C_{K,p}$-capacity 0, and hence of measure 0.

From Theorem 5 we have

Theorem 6. *Suppose Ω satisfies (NS). Let $\Omega_{K,p} = \{(x,y) : x \in \Omega(\eta^*_p(y))\}$. If E is $C_{K,p}$-thin at $\partial\mathbb{R}^{N+1}_+$, then*

$$\left|\bigcap_{t>0}\{x : (x+\Omega_{K,p}) \cap E_t \neq \emptyset\}\right| = 0.$$

In other words, for almost all $x \in \partial\mathbb{R}^{N+1}_+$, $x + \Omega_{K,p}$ lies eventually outside E, i.e., there is $t = t_x > 0$ such that $E_t \cap (x + \Omega_{K,p}) = \emptyset$.

Proof. It is not so difficult to see that

$$\{x \in \mathbb{R}^N : (x+\Omega_{K,p})\cap E \neq \emptyset\} \subset \bigcup_{x\in E^*} (x - \Omega_{K,p}(\delta_{E^*}(x))) = \bigcup_{x\in E^*} (x - \Omega(\eta^*_p(\delta_{E^*}(x))))$$

(see [6, Lemma 2]). Hence Theorem 5 yields

$$|\{x \in \mathbb{R}^N : (x+\Omega_{K,p}) \cap E \neq \emptyset\}| \le AC_{K,p}(E^*).$$

Apply this inequality for E_t replacing E. Then the definition of thinness implies that

$$|\{x : (x+\Omega_{K,p}) \cap E_t \neq \emptyset\}| \le AC_{K,p}(E_t^*) \to 0 \text{ as } t \to 0.$$

Thus the theorem follows. □

Remark. It is not so difficult to see that $\eta_p^*(r)/r \to \infty$ as $r \to 0$ (cf. [1]). Hence $\Omega_{K,p}$ is a tangential region.

For a function f on $\partial\mathbb{R}_+^{N+1}$ we denote by $PI(f)$ its Poisson integral. In [6] we have proved

Theorem 7. *Let $\Omega \subset \mathbb{R}_+^{N+1}$ and suppose $\overline{\Omega} \cap \partial\mathbb{R}_+^{N+1} = \{0\}$. Suppose $f \in L^p(\mathbb{R}^N)$. Then there is a set $E \subset \mathbb{R}_+^{N+1}$ such that E is $C_{K,p}$-thin at $\partial\mathbb{R}_+^{N+1}$ and that*

$$\lim_{P \to x, P \in (x+\Omega)\setminus E} PI(K*f)(P) = K*f(x) \tag{4}$$

for $C_{K,p}$-a.e. $x \in \partial\mathbb{R}_+^{N+1}$, i.e. there is a set $F \subset \partial\mathbb{R}_+^{N+1}$ such that $C_{K,p}(F) = 0$ and (4) holds for $x \in \partial\mathbb{R}_+^{N+1} \setminus F$.

As a corollary to Theorems 6 and 7 we have the following theorem. This is a generalization of [9, Theorem 2.9].

Corollary. *Let $\Omega \subset \mathbb{R}_+^{N+1}$ and suppose Ω satisfies (NS). Suppose $f \in L^p(\mathbb{R}^N)$. Then*

$$\lim_{P \to x, P \in x+\Omega_{K,p}} PI(K*f)(P) = K*f(x)$$

for almost all $x \in \partial\mathbb{R}_+^{N+1}$.

Remark. In the proof of Theorem 7 we use the nontangential maximal function, which is of type (p,p) for $p > 1$ but not of type $(1,1)$. Hence the assumption $1 < p < \infty$ is necessary. However, we can show similar results even for $p = 1$ under an additional assumption on the kernel K. In fact, if

$$K(r) \approx r^{-N} \int_0^r K(t) t^{N-1} dt \text{ for small } r > 0,$$

then the same conclusions of Theorem 7 and Corollary hold. For details we refer to [6].

Remark. The approach region $\Omega_{K,p}$ in Theorem 7 is in some sense most tangential. If one consider less tangential approach regions, then one may obtain boundary limit theorems with smaller boundary exceptional sets (corresponding to F in Theorem 7) which can be measured by the Hausdorff measure. This problem was considered in [5].

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