THE MOTION OF WEAKLY INTERACTING PULSES IN REACTION-DIFFUSION SYSTEMS
DEDICATED TO PROFESSORS MASAYASU MIMURA AND TAKAAKI NISHIDA
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Abstract.
The interaction of stable pulse solutions on \(R^1\) is considered when distances between pulses are sufficiently large. We construct an attractive local invariant manifold giving the dynamics of interacting pulses in a mathematically rigorous way. The equations describing the flow on the manifold is also given in an explicit form. By it, we can easily analyze the movement of pulses such as repulsiveness, attractivity and/or the existence of bound states of pulses. Interaction of front solutions are also treated in a similar way.

Key words. interaction of pulses, interaction of fronts, reaction-diffusion systems

AMS subject classifications. 35B25, 35K57

1. Introduction.
Reaction diffusion systems have been widely treated in order to study temporal and/or spatial pattern formation problems for various phenomena. Among them, the research for the systems on one dimensional space has extensively progressed. Through the research, many important and interesting solutions describing patterns have been shown and analyzed. Traveling wave solutions are one of the typical examples among them, which stand for propagating spatially localized patterns. The existence and stability of traveling wave solutions have been shown for many models such as the Allen-Cahn equation [18], the FitzHugh-Nagumo system ([30], [17] and their references), the Gray-Scott model ([3]) and so on.

In this paper, we suppose the existence of a stable traveling wave or pulse solution and consider the interaction between them. This problem is crucially related to the pattern formation problem, specially to the time evolutional process of localized patterns. The typical example for this problem is the interaction of fronts in the Allen-Cahn equation

\[
\begin{align*}
  u_t &= \epsilon^2 u_{xx} + \frac{1}{2} u(1 - u^2), \quad t > 0, \quad -\infty < x < \infty.
\end{align*}
\]

(1.1)

(1.1) has a stable stationary front solution \(U(x) = \tanh \frac{x}{2\epsilon}\) satisfying \(U(\pm \infty) = \pm 1\) and \(U(0) = 0\), which represents a localized pattern. Since (1.1) has a translation invariance, functions \(U(x - l)\) and \(U(-x + l)\) are also stable stationary solutions of (1.1) for an arbitrarily fixed constant \(l \in R^1\). Solutions close to \(U(x - l)\) and \(U(-x + l)\) are called kink and antikink solutions respectively. Then, the problem which we will concern here is the interaction of these kink and antikink solutions. That is, we consider how the dynamics of solution is if the initial data \(u(0, x)\) is sufficiently close to \(U(x - x_1(0)) + U(-x + x_2(0)) - 1\) with \(x_1(0) < x_2(0)\). The dynamics is well known ([21], [2] and [15], [13]) that if \(\epsilon > 0\) is sufficiently small, the solution \(u(t, x)\) remains close to \(U(x - x_1(t)) + U(-x + x_2(t)) - 1\) and \(x_1(t), x_2(t)\) are approximately governed by

\[
\begin{align*}
  \dot{x}_1 &= 12\epsilon e^{-\frac{1}{4}h}, \\
  \dot{x}_2 &= -12\epsilon e^{-\frac{1}{4}h},
\end{align*}
\]

(1.2)

where \(h = x_2(t) - x_1(t)\). (1.2) describes the coarsening process of the localized patterns because (1.2) means the attractivity of kink-antikink front solutions.

Similar properties hold for the interaction of front solutions in competition-diffusion systems, which is stated in detail in Section 3.

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Another important example may be the dynamics of fronts in the Cahn-Hilliard equation. The equation describing the interaction of fronts shows the similar coarsening process to (1.2), which has been extensively studied ([1] and its references). But, we will not touch on it in this paper because it has a conservation law and is considered in a bounded interval. In fact, results in this paper are not directly applicable to the dynamics of fronts for the Cahn-Hilliard equation.

On the other hand, localized pulse like patterns in the Gray-Scott model exhibit self-replicating dynamics under some conditions and spatial patterns become complicated ([25], [24]). Also in this process, the interaction of localized patterns gives great influences to the splitting behaviors, which will be reported in the forthcoming paper [11], [12].

Another important and interesting example is the interaction of nerve impulses. Plural nerve impulses frequently run along a nerve axon at the same time depending on initial impulses. Then, the repulsiveness of nerve impulses plays an important role to transmit informations ([29]). On the other hand, there is a simplified model equation called the FitzHugh-Nagumo system describing the dynamics of nerve impulses. A traveling pulse solution of the model equation corresponds to the repulsive behaviors. In fact, the interaction of traveling pulse solutions in the FitzHugh-Nagumo system was shown to be repulsive under some assumptions by using a formal analysis and the specialty of the system ([4]). Other dynamics of interacting traveling pulses in the FitzHugh-Nagumo systems have also been reported in various settings (e.g. [31], [16]) while they are all formally derived results.

Thus, to consider the interaction of localized solutions gives the important informations on the evolutional process of patterns, but almost all of the results so far have been due to formal discussions except the interaction of kink-antikink front solutions for the Allen-Cahn equation.

The purpose of this paper is to give a general criterion on the dynamics of weakly interacting traveling wave or pulse solutions in mathematically rigorous way.

Let us consider reaction-diffusion systems of the form

\begin{equation}
\begin{aligned}
\dot{u}_t = D u_{xx} + F(u), & \quad t > 0, \quad -\infty < x < \infty, \\
\end{aligned}
\end{equation}

where $u \in \mathbb{R}^n$, $D = \text{diag}(d_1, d_2, \cdots, d_n)$ ($d_j \geq 0$) and $F$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$. We suppose for (1.3) that:

**H1)** There exist linearly stable equilibria $P^-$ and $P^+$ in the ODE

\begin{equation}
\begin{aligned}
\dot{u}_t = F(u), & \quad t > 0.
\end{aligned}
\end{equation}

**H2)** (1.3) has a traveling wave solution with velocity $\theta$ connecting from $P^-$ to $P^+$. That is, there exist a constant $\theta$, positive constants $\alpha$, $\beta$ and a function $P(z)$ satisfying the equation

\begin{equation}
\begin{aligned}
0 &= DP_{zz} - \theta P_z + F(P), \quad -\infty < z < \infty, \\
|P(z) - P^+| &\leq O(e^{-\alpha z}) \quad (z \to +\infty), \\
|P(z) - P^-| &\leq O(e^{\beta z}) \quad (z \to -\infty).
\end{aligned}
\end{equation}

We note that $u(t, x) = P(x + \theta t)$ is a solution of (1.3).

**H3)** Let a differential operator $L$ be

\begin{equation}
\begin{aligned}
Lv = Dv_{zz} - \theta v_z + F'(P(z))v, & \quad -\infty < z < \infty
\end{aligned}
\end{equation}

for $v(z) \in H^2(\mathbb{R})$. Then, the spectrum $\Sigma(L)$ of $L$ is given by $\Sigma(L) = \Sigma_0 \cup \{0\}$, where 0 is a simple eigenvalue and there exists a positive constant $\rho_0 > 0$ such that $\Sigma_0 \subset \{z \in C; \ \Re z < -\rho_0\}$.

H3) means the traveling wave solution $P(z)$ is stable in a linearized sense. We call $P(z)$ satisfying the assumptions H1) $\sim$ H3) for a constant $\theta$ stable traveling wave solution with velocity $\theta$. Many models of reaction-diffusion systems have stable traveling wave solutions in this sense.

Transforming (1.3) by $z = x + \theta t$, we have

\begin{equation}
\begin{aligned}
\dot{u}_t = \mathcal{L}(u),
\end{aligned}
\end{equation}
where $\mathcal{L}(u) = Du_{xx} - \theta u_x + F(u)$. We note that the stable traveling wave solution $P(z)$ is a stable stationary solution of (1.6).

Let $P(z)$ be a stable traveling wave solution with velocity $\theta$. Throughout this paper, we call $P(z)$ (stable) 1-pulse solution when $P^- = P^+$ and (stable) 1-front solution when $P^- \neq P^+$, respectively.

Let us consider the interaction of 1-pulse solutions, for example. In this case, we may assume $P^- = P^+ = 0$ without loss of generality, where $0 = (0, 0, \ldots, 0) \in \mathbb{R}^n$. If the initial data $u(0, x)$ of (1.3) is close enough to $P(x - y_1^+) + P(x - y_2^+)$ for sufficiently large $y_2^+ - y_1^+$, then it is shown by the main results in Section 2 that the solution $u(t, x)$ remains sufficiently close to the function

$$P(x + \theta t - y_1(t)) + P(x + \theta t - y_2(t))$$

as long as $y_2(t) - y_1(t)$ is large enough. $y_j(t)$ $(j = 1, 2)$ are essentially governed by the ordinary differential equation with initial data $y_j(0) = y_j^+$,

$$\begin{align*}
\dot{y}_1 &= - \langle \mathcal{L}(P(z) + P(z - h)), \phi^* \rangle_{L^2}, \\
\dot{y}_2 &= - \langle \mathcal{L}(P(z) + P(z + h)), \phi^* \rangle_{L^2},
\end{align*}$$

(1.7)

where $h = h(t) = y_2(t) - y_1(t)$ and $\phi^*$ is an eigenfunction corresponding to 0 eigenvalue of the adjoint operator $L^*$ of $L$ normalized by $\langle P_z, \phi^* \rangle_{L^2} = 1$. $(\cdot, \cdot)_{L^2}$ denotes the inner product in $L^2(\mathbb{R})$.

Remark 1.1. When $\theta = 0$ and the linearized operator $L$ is self-adjoint, corresponding results to (1.7) were obtained by Schatzman [28]. Sandstede [27] has got almost same results as this paper at the same time independently of the author.

When $P(z)$ converges $0$ such that

$$\begin{align*}
P(z) &\to e^{\alpha z} a^+ \ (z \to +\infty), \\
P(z) &\to e^{\beta z} a^- \ (z \to -\infty)
\end{align*}$$

for positive constants $\alpha, \beta$ and non-zero constant vectors $a^\pm$ in $\mathbb{R}^n$, we call exponentially monotone convergent. If $P(z)$ converges $0$ in an exponentially monotone way, then it is shown in Section 2 that the right hand side of (1.7) are written

$$\begin{align*}
\langle \mathcal{L}(P(z) + P(z - h)), \phi^* \rangle_{L^2} &= M_1 e^{-\beta h} (1 + O(e^{-\gamma h})), \\
\langle \mathcal{L}(P(z) + P(z + h)), \phi^* \rangle_{L^2} &= M_2 e^{-\alpha h} (1 + O(e^{-\gamma h})),
\end{align*}$$

(1.8)

as $h \to +\infty$ for some constants $M_j$ and $\gamma > 0$. Combining (1.7) and (1.8), we have

$$\dot{h} \sim M_1 e^{-\beta h} - M_2 e^{-\alpha h}. $$

(1.9)

Thus, the values of constants $M_j$ are important to determine whether 1-pulses are repulsive or attractive while it has been a difficult thing so far. In Section 2, main results will be stated and the formulas for the explicit form of $M_j$ are given. Applications to several examples including the FitzHugh-Nagumo system, the Gray-Scott model equations will be in Section 3. We shall show in the section that the constants $M_j$ can be calculated in explicit forms and we can know easily the interaction of 1-pulses.

The case of the interaction of stable 1-front solutions is treated in a similar manner to the case of stable 1-pulse solutions, but in a slightly different setting from pulses. It will be mentioned parallel to 1-pulses.

Proofs will be in Sections 4 and 5. The basic tools for proofs are integral manifold theory. To construct the unstable manifold, we use the analogous manner in [1] and [2], but the manner of the construction developed in this paper is fairly generalized to be applicable to quite a general reaction-diffusion systems as (1.3).

2. Main results.

Suppose the assumptions H1) $\sim$ H3).
2.1. Interaction of 1-pulses. In this subsection, we assume equilibria $P^\pm = 0$.

Let $P(z)$ be a stable 1-pulse solution of (1.3) with vorticity $\theta$ and fix it. Let $\phi^*(z)$ be the eigenfunction of $L^*$ normalized by $\langle P_z, \phi^* \rangle_{L^2} = 1$ as stated in Section 1. We consider (1.6) instead of (1.3). Arbitrarily fixing the number of considered 1-pulses, say $N+1$, we let

$$P(z; h) = P(z) + P(z-z_1) + \cdots + P(z-z_N),$$

where $h = (h_1, h_2, \ldots, h_N)$ for $h_j > 0$ and

$$z_j = z_j(h) = \sum_{k=1}^{j} h_k \quad (j = 1, \ldots, N).$$

Here, we set $z_0 = 0$ for convenience.

Let $\mathbb{E}(l)$ be the translation operator given by $(\mathbb{E}(l)v)(z) = v(z-l)$ for $v \in L^2(\mathbb{R}^l)$. Moreover, define the quantity

$$\delta(h) = \sup_{z \in \mathbb{R}^l} |\mathcal{L}(P(z; h))|,$$

the set

$$\mathcal{M}(h^*) = \left\{ \mathbb{E}(l)P(z; h) ; l \in \mathbb{R}^l, \min h > h^* \right\},$$

and functions

$$H_j(h) = \langle \mathcal{L}(P(z + z_j; h)), \phi^* \rangle_{L^2}$$

for $j = 0, 1, \ldots, N$. Here, $\min h$ means $\min \{h_1, h_2, \ldots, h_N\}$ for $h = (h_1, h_2, \ldots, h_N)$. Then, we have

**Theorem 2.1.** There exist positive constants $h^*$, $C_0$ and a neighborhood $U = U(h^*)$ of $\mathcal{M}(h^*)$ in $H^2(\mathbb{R}^l)^m$ such that if $u(0, \cdot) \in U$, then there exist functions $l(t) \in \mathbb{R}^l$ and $h(t) \in \mathbb{R}^N$ such that

$$\|u(t, \cdot) - \mathbb{E}(l(t))P(z; h(t))\|_\infty \leq C_0 \delta(h(t))$$

holds as long as $\min h(t) > h^*$, where $u(t, z)$ is a solution of (1.6). Functions $l(t) \in \mathbb{R}^l$ and $h(t) \in \mathbb{R}^N$ satisfy

$$\dot{h} = H(h) + O(\delta^2),$$

(2.2)

$$\dot{l} = -H_0(h) + O(\delta^2),$$

(2.3)

where $\delta = \delta(h(t))$ and $H = (H_0 - H_1, H_1 - H_2, \ldots, H_{N-1} - H_N)$.

**Remark 2.1.** Since $\mathcal{L}(P(z-z_j)) = 0$ and $\mathcal{L}(0) = 0$ hold for any $z_j$, we have $\delta(h) \to 0$ as $\min h \to \infty$. On the other hand, the magnitude of $\max H_j(h)$ is $O(\delta(h))$, which means $H(h)$ in (2.2) or (2.3) are necessarily dominant as long as $\min h$ is large enough.

**Remark 2.2.** Theorem 2.1 gives the motion of positions $y_j = l + z_j \; (j = 0, 1, \ldots, N)$ of the $j$-th pulse by

$$y_j = l + \dot{h}_1 + \cdots + \dot{h}_j$$

$$= -H_j(h) + O(\delta^2).$$

Consider the ordinary differential system consisting of the principal parts of (2.2)

$$\dot{h} = H(h).$$

**Theorem 2.2.** Suppose all of the elements $d_j$ of $D$ are positive. Then, there exist positive constants $C_0, C_1$ and $h^*$ such that if (2.4) has an equilibrium $\overline{h}$ satisfying $\min \overline{h} > h^*$ and the set of
eigenvalues $\Sigma(\mathbf{H}'(\mathbf{h})) \subset \{ z \in \mathcal{C} \mid \Re z < -C_0 \delta(\mathbf{h}) \}$, there exists a stable traveling wave pulse solution $\mathcal{P}(z + \delta t)$ of (1.6) such that

$$
\| \mathcal{P}(z) - \mathcal{P}(z; \mathbf{h}) \|_\infty \leq C_1 \delta(\mathbf{h})
$$

and $\delta = H_0(\mathbf{h}) + O(\delta^2(\mathbf{h}))$. Here, $\mathbf{H}'(\mathbf{h})$ denotes the linearized matrix of $\mathbf{H}$ with respect to $\mathbf{h}$.

If (2.4) has an equilibrium such that $h > h^*$ and the set of eigenvalues $\Sigma(\mathbf{H}'(\mathbf{h})) \subset \{ z \in \mathcal{C} \mid \Re z < -C_0 \delta(\mathbf{h}) \}$, then at least one eigenvalue of $\mathbf{H}'(\mathbf{h})$ is in $\{ z \in \mathcal{C} \mid \Re z > C_0 \delta(\mathbf{h}) \}$, there exists an unstable traveling wave pulse solution $\mathcal{P}(z + \delta t)$ of (1.6) such that

$$
\| \mathcal{P}(z) - \mathcal{P}(z; \mathbf{h}) \|_\infty \leq C_1 \delta(\mathbf{h})
$$

and $\delta = H_0(\mathbf{h}) + O(\delta^2(\mathbf{h}))$.

**Remark 2.3.** The results stated in Theorem 2.2 have been already obtained by Sandstede [26]. Theorem 2.2 gives another proof of the results by using the invariant manifold theory as mentioned in the proof.

**Remark 2.4.** The situation of Theorem 2.2 occurs e.g. in the case when the tail of 1-pulse $P(z)$ is oscillatory. This will be stated in Section 3.

Now, we can know the explicit forms of functions $H_j(\mathbf{h})$ and $\phi^*$ when $P(z)$ converges $0$ in an exponentially monotone way. We note here that we may generically assume the corresponding adjoint eigenfunction $\phi^*$ also converges $0$ in similar decaying rates. That is,

**Theorem 2.3.** Suppose $P(z)$ converges $0$ satisfying

(2.5)

$$
P(z) = e^{-\alpha z}(a^+ + O(e^{-\gamma z})) \quad (z \to +\infty),
$$

(2.6)

$$
P(z) = e^{\beta z}(a^- + O(e^{\gamma z})) \quad (z \to -\infty)
$$

for positive constants $\alpha$, $\beta$ and $\gamma$ and non-zero constant vectors $a^\pm \in \mathbb{R}^n$, and suppose $\phi^*$ also converges $0$ in an exponentially monotone way such that

(2.7)

$$
\phi^*(z) = e^{-\beta z}(b^+ + O(e^{\gamma z})) \quad (z \to +\infty),
$$

(2.8)

$$
\phi^*(z) = e^{\alpha z}(b^- + O(e^{\gamma z})) \quad (z \to -\infty)
$$

for non-zero constant vectors $b^\pm \in \mathbb{R}^n$. Then, functions $H_j(\mathbf{h})$ are represented by

(2.9)

$$
H_j(\mathbf{h}) = (M_\beta e^{-\beta h_j} + M_\alpha e^{-\alpha h_j}) \left( 1 + O(e^{-\gamma \min h_j}) \right)
$$

$$
(j = 1, 2, \cdots, N - 1),
$$

(10)

$$
H_0(\mathbf{h}) = M_\alpha e^{-\alpha h_1} \left( 1 + O(e^{-\gamma \min h_1}) \right),
$$

(11)

$$
H_N(\mathbf{h}) = M_\alpha e^{-\alpha h_N} \left( 1 + O(e^{-\gamma \min h_N}) \right),
$$

for a constant $\gamma > 0$ and the constants $M_\alpha$, $M_\beta$ are given by

(12)

$$
M_\beta = 2\beta \langle a^-, Db^- \rangle - \theta \langle a^-, b^+ \rangle,
$$

(13)

$$
M_\alpha = 2\alpha \langle a^+, Db^- \rangle + \theta \langle a^+, b^- \rangle,
$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$.

**Corollary 2.1.** Suppose $P(z)$ is a symmetric standing pulse, that is, $P(z)$ satisfies $P(z) = P(-z)$ with $\theta = 0$. If $P(z)$ converges $0$ in an exponentially monotone way, then $\alpha = \beta$, $a^+ = a^-$, say $a$, and $b^+ = -b^-$, say $b$, hold in (2.5) $\sim$ (2.8) of Theorem 2.3. Hence, it follows that

(14)

$$
H_j(\mathbf{h}) = M_\alpha \left( e^{-\alpha h_j} + e^{-\alpha h_j} \right) \left( 1 + O(e^{-\gamma \min h_j}) \right)
$$

$$
(j = 1, 2, \cdots, N - 1),
$$

(15)

$$
H_0(\mathbf{h}) = M_\alpha e^{-\alpha h_1} \left( 1 + O(e^{-\gamma \min h_1}) \right),
$$

(16)

$$
H_N(\mathbf{h}) = -M_\alpha e^{-\alpha h_N} \left( 1 + O(e^{-\gamma \min h_N}) \right),
$$

where $\gamma > 0$ and the constants $M_\alpha$, $M_\beta$ are given by

(17)

$$
M_\beta = 2\beta \langle a^-, Db^- \rangle - \theta \langle a^-, b^+ \rangle,
$$

(18)

$$
M_\alpha = 2\alpha \langle a^+, Db^- \rangle + \theta \langle a^+, b^- \rangle,
$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$. 


and the constant $M_0$ is given by

$$M_0 = 2\alpha \langle \mathbf{a}, D\mathbf{b} \rangle.$$  

**Remark 2.5.** Theorem 2.1 and Corollary 2.1 show that the movement of interacting symmetric pulses is just determined by the sign of the constant $M_0$ because the dynamics is given by

$$\dot{h}_j = M_0 \left(2e^{-\alpha h_j} - e^{-\alpha h_{j+1}} - e^{-\alpha h_{j-1}}\right) + o(\delta).$$  

(2.18) implies $M_0 > 0$ reads the repulsiveness and $M_0 < 0$ does attractivity of pulses.

**Remark 2.6.** In the situation of Corollary 2.1, $\delta = \delta(h) = O(e^{-\alpha \min h})$ holds. Hence, the term $M_0 \left(2e^{-\alpha h_j} - e^{-\alpha h_{j+1}} - e^{-\alpha h_{j-1}}\right)$ in (2.18) may not be necessarily dominant when there are more than three pulses and distances between pulses are extremely different. But, we can also prove on the movement of each $h_j$

$$\dot{h}_j = M_0 \left(2e^{-\alpha h_j} - e^{-\alpha h_{j+1}} - e^{-\alpha h_{j-1}}\right) + o \left(e^{-\alpha} + e^{-\alpha h_{j+1}} + e^{-\alpha h_{j-1}}\right)$$

(2.19)

for any distribution of pulses, which means the term $M_0 \left(2e^{-\alpha h_j} - e^{-\alpha h_{j+1}} - e^{-\alpha h_{j-1}}\right)$ is necessarily dominant for the movement of the $j$th pulse. This is proved by slightly modified manner of the proof of Theorem 2.1, but the detail will be mentioned in the forthcoming papers ([12], [14]) because the some parts of the proof are jointly worked in those papers.

### 2.2. Interaction of standing 1-fronts.

For the interaction of 1-fronts, we can consider only the case of the velocity $\theta = 0$. We use the same notations as in the previous subsection with $\theta = 0$, but use $x$ as the space variable instead of $z$ because of $x = z$ in this case.

Let $P(x)$ be a stable 1-front solution of (1.3) with $\theta = 0$ and fix it. Note that $P(-x)$ is also a stable 1-front solution connecting from $P^+$ to $P^-$. Assume the number of 1-fronts $N + 1 = N^+ + N^-$, where $N^+$ and $N^-$ are the numbers of 1-fronts of the shapes $P(x)$ and $P(-x)$, respectively. We note that either $N^+ = N^-$ or $N^+ - 1 = N^-$ holds. For $N + 1$ 1-fronts, we define

$$P(x; h) = P(x) + P(-(x - x_1)) + P(x - x_2) + \cdots + P \left((-1)^N (x - x_N)\right) - \{N^+ P^+ + (N^- - 1) P^-\}$$

if $N^+ = N^-$ and define

$$P(x; h) = P(x) + P(-(x - x_1)) + P(x - x_2) + \cdots + P \left((-1)^N (x - x_N)\right) - \{(N^+ - 1) P^+ + N^- P^-\}$$

if $N^+ - 1 = N^-$, where $h = (h_1, h_2, \cdots, h_N) \in \mathbb{R}^N$, $x_j = \sum_{k=1}^{j-1} h_k$ for $j = 1, \cdots, N$ and $x_0 = 0$. Define functions $H_j(h)$ ($j = 0, 1, \cdots, N$) by

$$H_j(h) = \langle \mathcal{L}(P(x + x_j; h), \phi^*((-1)^j x)) \rangle_{L^2}.$$

Then, we have

**Theorem 2.4.**

Theorems 2.1 and 2.2 hold in the same statements but

$$\dot{h}_j = (-1)^{j+1}(H_{j-1}(h) + H_j(h)) + O(\delta^2) \ (j = 1, 2, \cdots, N),$$

(2.20)

$$\dot{\nu} = -H_0(h) + O(\delta^2),$$

(2.21)

and

$$H = (H_0 + H_1, -(H_1 + H_2), \cdots, (-1)^{N+1}(H_{N-1} + H_N)).$$

**Theorem 2.5.** Suppose $P(x)$ converges $P^\pm$ as

$$P(x) - P^+ = e^{-\alpha_1^x} \left(\mathbf{a}^+ + O(e^{-\gamma_1^x})\right) \ (x \to +\infty),$$

(2.22)

$$P(x) - P^- = e^{\beta_1^x} \left(\mathbf{a}^- + O(e^{\gamma_1^x})\right) \ (x \to -\infty)$$

(2.23)
for positive constants $\alpha, \beta$ and $\gamma$ and non-zero constant vectors $a^\pm \in \mathbb{R}^n$, and suppose $\phi^*$ converges to 0 in an exponentially monotone way such that
\begin{align}
(2.24) & \quad \phi^*(x) = e^{-\alpha x}(b^+ + O(e^{-\gamma x}))(x \to +\infty), \\
(2.25) & \quad \phi^*(x) = e^{\beta x}(b^- + O(e^{\gamma x}))(x \to -\infty)
\end{align}
for non-zero constant vectors $b^\pm \in \mathbb{R}^n$. Then, functions $H_j(h)$ are represented by
\begin{align}
(2.26) & \quad H_{2j-1}(h) = (M^+e^{-\alpha h_{2j-1}} - M^-e^{-\beta h_{2j-1}}) \left(1 + O(e^{-\gamma \min h})\right) \\
(2.27) & \quad H_{2j}(h) = (M^+e^{-\alpha h_{2j+1}} - M^-e^{-\beta h_{2j+1}}) \left(1 + O(e^{-\gamma \min h})\right) \\
(2.28) & \quad H_0(h) = M^+e^{-\alpha h_1} \left(1 + O(e^{-\gamma \min h})\right), \\
(2.29) & \quad H_N(h) = \begin{cases} 
M^+e^{-\alpha h_N} \left(1 + O(e^{-\gamma \min h})\right) & (\text{if } N^+ = N^-), \\
M^-e^{-\beta h_N} \left(1 + O(e^{-\gamma \min h})\right) & (\text{if } N^+ - 1 = N^-)
\end{cases}
\end{align}
for a constant $\gamma' > 0$ and the constants $M^\pm$ are given by
\begin{align}
(2.30) & \quad M^+ = 2\alpha \langle a^+, Db^+ \rangle, \\
(2.31) & \quad M^- = 2\beta \langle a^-, Db^- \rangle.
\end{align}

3. Applications.
In this section, the notation $f(h) \sim g(h)$ stands for $f(h) = g(h)(1 + o(1))$ as $h \to \infty$.

3.1. Interaction of 1-fronts in the Allen-Cahn equation. In this subsection, we consider the interaction of 1-fronts of
\begin{align}
(3.1) & \quad u_t = \varepsilon^2 u_{xx} + f(u), \quad t > 0, \quad -\infty < x < \infty,
\end{align}
where $u \in \mathbb{R}^1$, $f(u) = \frac{1}{2}u(1 - u^2)$ and $\varepsilon$ is a sufficiently small positive parameter. Then, (3.1) has a stable standing 1-front $P(x)$ given by
\begin{align}
P(x) = \tanh \frac{x}{2\varepsilon}
\end{align}
as in Section 1. This 1-front is linearly stable and connecting $P^\pm = \pm 1$ (\cite{[18]}).

Let us consider the interaction of two 1-fronts $P(x - l)$ and $P(-(x - l - h))$ for $l, h > 0$ by using Theorems 2.4 and 2.5. Let $P(x; h) = P(x) + P(-(x - h)) - 1$. Then, $\delta(h) = \sup_{x \in \mathbb{R}^1} |\mathcal{L}(P(x; h))|$ is estimated as $O(e^{-\frac{x}{\varepsilon}})$ because the asymptotic form of $P(x)$ as $x \to \infty$ is
\begin{align}
(3.2) & \quad P(x) \to -2e^{-\frac{x}{\varepsilon}} + 1.
\end{align}
On the other hand, the eigenfunction $\phi^*(x)$ is easily obtained as
\begin{align}
\phi^*(x) = \frac{3\varepsilon}{2} P_2(x)
\end{align}
together with its asymptotic form
\begin{align}
(3.3) & \quad \phi^*(x) \to 3e^{-\frac{x}{\varepsilon}} (x \to \infty)
\end{align}
because the linearized operator with respect to $P(x)$ is self-adjoint. Thus, all of necessary quantities in Theorem 2.5 are given by

$$D = \varepsilon^2, \; \alpha = \frac{1}{\varepsilon}, \; a^+ = -2, \; b^+ = 3$$

from (3.2) and (3.3). Hence, $H_j(h)$ are calculated as

$$H_0(h), \; H_1(h) \sim M_e^{-\alpha h} = 2\alpha (a^+, D b^+) e^{-\alpha h}$$

$$= 2 \cdot \frac{1}{\varepsilon} \cdot (\varepsilon^2 \cdot 3) \cdot e^{-\varepsilon h}$$

$$= -12\varepsilon e^{-\frac{1}{\varepsilon} h}$$

and we get the equation of $l$ and $h$ as

$$ \begin{cases} i & = -H_0(h) + O(\delta^2(h)) \sim 12\varepsilon e^{-\frac{1}{\varepsilon} h}, \\ h & = (H_0(h) + H_1(h)) + O(\delta^2(h)) \sim -24\varepsilon e^{-\frac{1}{\varepsilon} h}. \end{cases} $$

Especially, the equation of the distance $h$ of two 1-fronts is that obtained by [13], [2] and [15].

3.2. Interaction of 1-pulses in the FitzHugh-Nagumo system. Let us consider the equation

$$ \begin{cases} u_t & = \varepsilon^2 u_{xx} + f(u) - v, \\ v_t & = \varepsilon(u - bv), \end{cases} $$

where $f(u) = u(1-u)(u-a)$ ($0 < a < \frac{1}{2}$), $b$ is small enough that two graphs $v = f(u)$ and $u = bv$ have no intersection except $0 = (0, 0)$. When $\varepsilon > 0$ is sufficiently small, it is known (e.g. [19, 30]) that (3.5) has a stable traveling 1-pulse solution, say $P(z) = (\Phi(z), \Psi(z))$, with velocity $\theta = \varepsilon c$ for a positive constant $c$ and converging to 0 in an exponentially monotone way. Transforming (3.5) by $z = x + \varepsilon t$, we have

$$ u_t = Du_{zz} - \varepsilon cu_z + F(u), $$

where $u = (u, v), \; D = \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & 0 \end{pmatrix}$ and $F(u) = \begin{pmatrix} f(u) - v \\ \varepsilon(u - bv) \end{pmatrix}$.

Let $c = c_0 + \varepsilon c_1 + O(\varepsilon^2)$. By the construction of $P(z)$ (e.g. [30]), the asymptotic form of $P(z)$ is easily known as

$$ P(z) \to \begin{cases} e^{K_1 x} a^- (z \to -\infty), \\ e^{-\alpha z} a^+ (z \to +\infty) \end{cases} $$

for a positive constant $K_1$, $a^- \in R^2$ and

$$ \alpha = \frac{1}{c_0} \left( b - \frac{1}{f'(0)} \right) + O(\varepsilon), $$

$$ a^+ = -K_2 \left( \frac{1}{f'(0)} \right) + O(\varepsilon) \in R^2 $$

for $K_2 > 0$. Hence, $\delta(h) = \sup_{z \in R^1} |L(P(z) + P(z - h))|$ is $O(\varepsilon^{-\alpha h})$.

On the other hand, the eigenfunction $\phi^*(z)$ of the adjoint operator of the linearized equation of (3.6) extensively studied in [4], [9]. By using the result, we can know the asymptotic form of $\phi^*$ as

$$ \phi^*(z) \to \begin{cases} e^{K_2 z} b^- (z \to -\infty), \\ e^{-\frac{1}{K_2} z} b^+ (z \to +\infty) \end{cases} $$
for $K_3 > 0$, $b^+ \in \mathbb{R}^2$ and

$$b^- = K_4 \left( \frac{2}{f'(0)} + O(\varepsilon^2) \right) \in \mathbb{R}^2$$

for $K_4 > 0$. Hence, Theorems 2.1 and 2.3 yield the equation of pulse distance $h$ as follows:

$$\dot{h} = -M_\alpha e^{-\alpha h} + M_\beta e^{-K_3 h} + O(\varepsilon^2(h)) \sim -M_\alpha e^{-\alpha h}.$$ 

Here, we ignored the second term in the above equation due to the smallness of $\varepsilon$, We can also obtain the explicit value of $M_\alpha$ by substituting (3.8), (3.9) and (3.10) into (2.12) as

$$M_\alpha = 2\alpha \langle a^+, Db^- \rangle + \varepsilon c \langle a^+, b^- \rangle$$

$$= \varepsilon c_0 K_2 K_4 f(0) + O(\varepsilon^2)$$

$$< 0.$$ 

This shows two 1-pulses in (3.5) interact repulsively.

### 3.3. Interaction of 1-pulses with oscillatory tails.

In this subsection, we will consider the interaction of two 1-pulses with oscillatory tails such that

$$P(z) \rightarrow \Re \left( e^{\lambda \pm z} a^\pm \right)$$

as $z \rightarrow \pm \infty$, where $a^\pm \in \mathbb{C}^n$ and $\lambda^\pm = -\alpha + i\nu^\pm$, $\lambda^- = \beta + i\nu^-$ for positive constants $\alpha$, $\beta$, and we assume either $\nu^\pm$ is not zero. We let $\nu^+ \neq 0$ for simplicity and suppose $\alpha < \beta$. This means the right tail of $P(z)$ converges $0$ slowly and oscillatorily. This setting has been extensively studied for the pulse solution of the FitzHugh-Nagumo system related to multi-pulse solutions (see e.g. [26] and the references).

Let us consider the interaction of two 1-pulses. Then, the equation describing the distance $h$ between 1-pulses is

$$\dot{h} = H_0 - H_1$$

as in Theorem 2.1. By quite a similar way to the proof of Theorem 2.3, we can show

$$H_0(h) = \Re \left( e^{-\lambda^- h} M^- \left( 1 + O(e^{-\gamma' h}) \right) \right),$$

$$H_1(h) = \Re \left( e^{\lambda^+ h} M^+ \left( 1 + O(e^{-\gamma' h}) \right) \right)$$

for a constant $\gamma' > 0$, where

$$M^+ = \int_{-\infty}^{\infty} e^{\lambda^+ z} \left( F'(P(z)) - F'(0) a^+, \phi^*(z) \right) dz,$$

$$M^- = \int_{-\infty}^{\infty} e^{\lambda^- z} \left( F'(P(z)) - F'(0) a^-, \phi^*(z) \right) dz.$$ 

Note that the constants $M^+$ and $M^-$ are well defined because the integrals giving their constants are given as the Fourier transformation because of the form of $\lambda^\pm$. Let $M^\pm = A^\pm + iB^\pm$. Then, we have

$$H_0(h) \sim e^{-\beta h} (A^- \cos \nu^- h - B^- \sin \nu^- h),$$

$$H_1(h) \sim e^{-\alpha h} (A^+ \cos \nu^+ h + B^+ \sin \nu^+ h).$$

Since $\alpha < \beta$ and $\delta(h) = O(e^{-\alpha h})$ in this case, the equation on $h$ is

$$\dot{h} = H_0 - H_1 + O(\delta^2(h)) \sim -H_1 \sim -e^{-\alpha h} (A^+ \cos \nu^+ h + B^+ \sin \nu^+ h)$$

for sufficiently large $h$. From (3.11), we easily find that stable and unstable equilibria appear alternatively in (3.11) satisfying the assumptions of Theorem 2.2 because $\delta(h) = e^{-\alpha h}$. This gives the existence and its stability of double pulse solutions by Theorem 2.2.

We have considered the interaction only of two 1-pulses in this subsection, but the method developed here is easily extended to arbitrarily many numbers of 1-pulses. Thus, we can easily give another proof on the existence and stability analysis of multiple-pulse solutions.
3.4. Interaction of 1-pulses in the Gray-Scott model on 1-D. We will show the repulsiveness of pulses in the Gray-Scott model equations on one dimension. The model equation here is:

\begin{align}
  u_t &= u_{xx} - uv^2 + A(1 - u), \\
  v_t &= Dv_{xx} - Bv + uv^2,
\end{align}

where \( A, B \) and \( Dv \) are positive constants. Doelman et. al. [3] showed the existence of a stable standing 1-pulse solution, say \( P(x) = (\Phi(x), \Psi(x)) \) under the assumptions:

\[ A = \epsilon^2 a, \quad B = \epsilon^4 b, \quad Dv = \epsilon^2 \]

for a small parameter \( \epsilon > 0 \) and positive constants \( a, b \) and \( 0 < \nu < 1 \). We consider in this subsection only the case \( \nu = \frac{1}{2} \). The profile \( P(x) \) is then given by ([3])

\begin{align}
  \Phi(x) &= \epsilon^\frac{3}{2} (p_0 + o(1)), \quad (x \sim 0), \\
  \Phi(x) &= \Phi_0(x) + o(1), \quad x > 0, \\
  \Psi(x) &= \epsilon^\frac{3}{2} (q_0(\eta) + O(\epsilon^{\frac{1}{2}}))
\end{align}

as \( \epsilon \to 0 \), where \( p_0 = 3b\sqrt{\frac{\epsilon}{a}}, q_0(\eta) = \frac{1}{2} \sqrt{\frac{a}{\epsilon}} \text{sech} \left( \frac{\sqrt{a}}{2} \eta \right) \). \( \eta = \epsilon^{\frac{3}{2}} x \) and \( \Phi_0 \) is the function satisfying:

\[ 0 = \frac{\partial^2 \Phi_0}{\partial x^2} + \epsilon^2 a (1 - \Phi_0), \quad \Phi_0(0) = \epsilon^\frac{3}{2} p_0, \quad \Phi_0(\infty) = 1. \]

Let us obtain the adjoint eigenfunction \( \phi^*(x) = (\phi^*(x), \psi^*(x)) \) corresponding to the zero eigenvalue of the adjoint operator. Here, we note that the following calculations are all easily justified by singular perturbation techniques though we treat it by the formal asymptotic expansion for simplicity.

The equation which should be satisfied by \( \phi^* \) is:

\begin{align}
  0 &= u_{xx} - (\Psi^2 + \epsilon^2 a)u + \Psi^2 v, \\
  0 &= \epsilon^2 v_{xx} - 2\Phi \Psi u + (2\Phi \Psi - \epsilon^\frac{3}{2} b) v
\end{align}

with \( u(0) = v(0) = 0 \) and \( u(\infty) = v(\infty) = 0 \) because \( \phi^* \) is a bounded odd function. It suffices to consider (3.14) and (3.15) only for \( x > 0 \). Transforming (3.14) and (3.15) by \( u = \epsilon^{\frac{3}{2}} \phi, \quad \Phi = \epsilon^{\frac{3}{2}} p, \quad v = \epsilon^{\frac{3}{2}} \Psi, \quad \Psi = \epsilon^{\frac{3}{2}} q \) and \( x = \epsilon^{\frac{3}{2}} \eta \), we have:

\begin{align}
  0 &= \phi_{\eta\eta} - (\epsilon q^2 + \epsilon^\frac{3}{2} a) \phi + q^2 \Psi, \\
  0 &= \psi_{\eta\eta} - 2\epsilon pq \phi + (2pq - b) \psi.
\end{align}

Suppose \( \phi = \phi_0 + o(1) \) and \( \psi = \psi_0 + o(1) \) as \( \epsilon \to 0 \). Then, \( \phi_0 \) and \( \psi_0 \) satisfy:

\begin{align}
  0 &= \frac{\partial^2 \phi_0}{\partial \eta^2} + q_0^2 \psi_0, \\
  0 &= \frac{\partial^2 \psi_0}{\partial \eta^2} + (2pq_0 - b) \psi_0
\end{align}

with \( \phi_0(0) = \psi_0(0) = 0 \). Substituting \( p_0 \) and \( q_0(\eta) \) into (3.19), we have:

\[ 0 = \frac{\partial^2 \psi_0}{\partial \eta^2} + b \left\{ 3 \text{sech}^2 \left( \frac{\sqrt{b}}{2} \eta \right) - 1 \right\} \psi_0 \]

which is easily solved (e.g. [22]) as \( \psi_0(\eta) = r M_1(\eta) \) for \( r \in \mathbb{R}^1 \), where

\[ M_1(\eta) = \text{sech}^2 \left( \frac{\sqrt{b}}{2} \eta \right) \tanh \left( \frac{\sqrt{b}}{2} \eta \right). \]
Hence, $\phi_0$ is obtained as

$$\phi_0(\eta) = rM_2(\eta),$$

where

$$M_2(\eta) = \int_0^{\eta} \left\{ - \int_0^{\eta'} q_0^2(\eta')M_1(\eta')d\eta' + K_5 \right\} d\eta'$$

and $K_5 = \int_0^{\infty} q_0^2(\eta)M_1(\eta)d\eta$. Let $\zeta^* = M_2(\infty)$, which is a positive constant.

As a consequence, we have the adjoint eigenfunction $\phi^*$ connecting $P^+$ and $v$, and there exists $c$ such that $P^+$ is repulsive.

Due to the positivity of $\phi^*(x)$, we consider here is the following system

$$\frac{\partial}{\partial x} \left( \phi^* r \right) = r \phi^* r \phi^*$$

which holds. This means the normalized condition $\langle P_x, \phi^* \rangle = 1$ yields $r = -r_0 \sqrt{\epsilon} < 0$ for $r_0 > 0$. Since the asymptotic forms $P(x)$ and $\phi^*(x)$ are respectively given by

$$P(x) \to e^{-\epsilon \sqrt{\pi} x} a, \quad \phi^*(x) \to e^{\epsilon \sqrt{\pi} x} b,$$

where $a = (-a^+,0)$ and $b = (\epsilon \sqrt{\pi} r \zeta^*,0)$ for a positive constant $a^+$, the constant $M_0$ in Corollary 2.1 is

$$M_0 = 2a \cdot (Da, b) = 2\sqrt{\alpha}(-a^+)\epsilon \sqrt{\pi} \zeta^* = 2\epsilon \sqrt{\pi} a^+ r_0 \zeta^* > 0$$

due to the positivity of $\zeta^*$ and $r_0$, where $\alpha = \epsilon \sqrt{\pi}$ and $D = \text{diag}(1, D_{11})$. Thus, we find the interaction is repulsive.

### 3.5. Interaction of 1-fronts in competition-diffusion systems

The model equation which we consider here is the following system

$$\begin{cases}
    u_t &= u_{xx} + u(1 - u - av), \\
    v_t &= dv_{xx} + v(c - bu - v),
\end{cases}$$

where $a$, $b$, $c$ and $d$ are positive constants. For (3.21), it is known by [20] that if $\frac{1}{a} < b$ is satisfied, then the kinetics of (3.21) is bistable, that is, (3.21) has two stable equilibria $P^+ = (0,c)$, $P^- = (1,0)$ and there exists $c = c_0 \in \left( \frac{1}{a}, b \right)$ such that (3.21) has a unique stable front $P(x)$ with velocity $\theta = 0$ connecting $P^\pm$. Fix $c = c_0$ in (3.21) and let $P(x) = (\Phi(x), \Psi(x))$ be the stable standing 1-front with $P(x) \to P^\pm$ as $x \to \pm \infty$. 

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Let us consider the interaction of two 1-fronts $P(x - l)$ and $P(-(x - l - h))$ for $l, h > 0$. First, we note that

\begin{equation}
\Phi_x(x) < 0, \Psi_x(x) > 0
\end{equation}

hold for $-\infty < x < \infty$ ([20]). This means the asymptotic form of $P(x)$ is

\begin{equation}
P(x) \to e^{-\alpha x} a^+ + P^+ (x \to \infty)
\end{equation}

for $\alpha > 0$ and a vector $a^+ \in \mathbb{R}^2$ written by $a^+ = (p, -q)$ for positive constants $p, q$.

On the other hand, it is also known in [20] that the eigenfunction $\phi^*(x) = (\phi^*(x), \psi^*(x))$ of the adjoint operator of the linearized equation of (3.21) satisfies

\begin{equation}
\phi^*(x) \cdot \psi^*(x) < 0.
\end{equation}

Since $\phi^*(x)$ is normalized by the condition $\langle P_x, \phi^* \rangle _{L^2} = 1$, (3.22) and (3.24) imply $\phi^*(x) < 0$ and $\psi^*(x) > 0$ for $-\infty < x < \infty$. Hence, we can find that the asymptotic form of $\phi^*(x)$ is

\begin{equation}
\phi^*(x) \to e^{-\alpha x} b^+ (x \to \infty),
\end{equation}

where $b^+ = (-r, s) \in \mathbb{R}^2$ for positive constants $r, s$. Thus, Theorems 2.4 and 2.5 yield

$$H_0(h), H_1(h) \sim M^+ e^{-\alpha h}$$

\begin{align*}
&= 2\alpha \langle a^+, Db^+ \rangle e^{-\alpha h} \\
&= 2 \cdot \alpha \cdot \left( \begin{pmatrix} p \\ -q \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} -r \\ s \end{pmatrix} \right) e^{-\alpha h} \\
&= -2\alpha(pr + dqs)e^{-\alpha h} < 0
\end{align*}

and the equation of the distance $h$

$$h \sim H_0(h) + H_1(h) \sim 2M^+ e^{-\alpha h} = -4\alpha(pr + dqs)e^{-\alpha h} < 0.$$  

This shows the attractivity of 1-fronts.

### 4. Proofs of Theorems.

In this section, we will devote ourselves to the proofs of theorems in Section 2.1. Theorems in Section 2.2 are all proved in quite a similar manner to this section and we omit the proofs.

In this and the following sections, $C$ denotes a generic positive constant independent of $h$ with sufficiently large $\min h$ and we take $h^*$ sufficiently large as required in following lemmas.

First, we consider the case that all of the elements of $D = \text{diag}(d_1, d_2, \ldots, d_n)$ are positive.

#### 4.1. Preliminaries for the proofs of Theorems 2.1, 2.2.

Let $X = \{L^2(R^1)\}^n$ with the norm $\| \cdot \|$ and

$$L(h)v = Dv_{zz} - \theta v_z + F'(P(;h))v,$$

$$L^*(h)v = Dv_{zz} + \theta v_z + ^tF'(P(;h))v$$

for $v \in \{H^2(R^1)\}^n$, where $^tF'$ means the transposed matrix of $F'$. These operators are sectorial in $X$.

**Proposition 4.1.** There exist positive constants $C$ and $h^*$ such that for $h$ with $\min h > h^*$, the operator $L(h)$ has $N + 1$ semi-simple eigenvalues $\{\lambda_j(h)\}_{j=0,\ldots,N}$ with $|\lambda_j(h)| \leq C\delta(h)$. Multiple eigenvalues are repeated as many times as their multiplicity indicates. Other spectra of $L(h)$ are in the left hand side of $z = -p_0$ for a positive constant $p_0$.

**Proof.** See e.g. [23] and [26].

Let $E(h)$ be the eigenspace corresponding to eigenvalues $\{\lambda_j(h)\}_{j=0,\ldots,N}$. The adjoint operator $L^*(h)$ has also similar $N + 1$ semi-simple eigenvalues $\{\lambda_j^*(h)\}_{j=0,\ldots,N}$ with $|\lambda_j^*(h)| \leq C\delta(h)$. Let $E^*(h)$ be the eigenspace corresponding to eigenvalues $\{\lambda_j^*(h)\}_{j=0,\ldots,N}$.
Proposition 4.2. \( E(h) \) and \( E^*(h) \) are spanned by \( N+1 \) functions \( \{ \phi_j(h)(\cdot) \} \) and \( \{ \phi_j^*(h)(\cdot) \} \) \((j = 0, \cdots, N)\) respectively such that for \( j = 0, \cdots, N \),

\[
\begin{align*}
\phi_j(h)(z) &= P_j(z - z_j) + O(\delta), \\
\phi_j^*(h)(z) &= \phi^*(z - z_j) + O(\delta),
\end{align*}
\]

hold, where \( \delta = \delta(h) \), \( z_j = z_j(h) \), and \( O(\delta) \) mean in this lemma \( \|O(\delta)\|_{H^2} \leq C\delta \).

Proofs of Propositions 4.1 and 4.2 will be given together in Section 5 for convenience while the proof of Proposition 4.1 has been already shown in some papers ([23] and [26]).

Now, we fix \( h^* > 0 \) large enough such that Propositions 4.1 and 4.2 hold.

Let operators \( Q(h) \) and \( R(h) \) be the projection from \( X \) to \( E(h) \) and \( R(h) = Id - Q(h) \) respectively, where \( Id \) is the identity on \( X \). Let \( E^\perp(h) = R(h)X \). Note that \( E^\perp(h) \) is characterized by

\[
E^\perp(h) = \{ v \in X ; \langle v, \phi_j^*(h) \rangle_{L^2} = 0 \ (j = 0, \cdots, N) \}.
\]

Let \( h^* = (h^*, h^*, \cdots, h^*) \), \( \hat{h} = (\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_N) \in \mathbb{R}^N \) with \( \min \hat{h} > h^* \) and \( \delta^* = \delta(h^*) \), \( \hat{\delta} = \delta(\hat{h}) \).

We define a map \( \Pi(h) \) from \( E^\perp(h) \) to \( E^\perp(\hat{h}) \) for \( h = (h_1, h_2, \cdots, h_N) \in \mathbb{R}^N \) with \( h_j > \hat{h}_j \) as follows:

Let \( \Theta(r;h) = rh + (1-r)\hat{h} \) \((0 \leq r \leq 1)\) and let

\[
S(r;h)v = - \sum_{j,k=0}^N (h_j - \hat{h}_j) \langle v, \frac{\partial}{\partial h_j} \phi_k^*(\Theta(r)) \rangle_{L^2} \phi_k(\Theta(r))
\]

for \( v \in X \), where \( \Theta(r) = \Theta(r;h) \). Define the map \( \Pi(h) \) by \( \Pi(h)v_0 = v(1) \), where \( v(r) \) is a solution of

\[
\begin{align*}
\begin{cases}
\frac{dv}{dr} &= S(r;h)v, \\
v(0) &= v_0 \in X.
\end{cases}
\end{align*}
\]

Lemma 4.1.

The map \( \Pi(h) \) is a homeomorphism from \( E^\perp(\hat{h}) \) to \( E^\perp(h) \).

Proof. Let \( v(r) \) be a solution of (4.5) with \( v_0 = v(1) \). We simply write \( Q(r) = Q(\Theta(r;h)) \), \( R(r) = R(\Theta(r;h)) \) and so on. Let \( v_1(r) = Q(r)v(r) \) and \( v_2(r) = R(r)v(r) \). Then, \( v_1(r) \) satisfies

\[
\frac{dv_1}{dr} = \frac{dQ}{dr}(r)v(r) + Q(r)\frac{dv}{dr}(r)
\]

\[
= \frac{dQ}{dr}(r)v(r) + Q(r)S(r)v(r)
\]

\[
= \frac{dQ}{dr}(r)(v_1(r) + v_2(r)) + Q(r)S(r)(v_1(r) + v_2(r))
\]

\[
= \frac{dQ}{dr}(r)(v_1(r) + v_2(r)) + S(r)v_1(r) - \frac{dQ}{dr}(r)v_2(r)
\]

\[
= \frac{dQ}{dr}(r)v_1(r) + S(r)v_1(r),
\]

because \( Q(r)S(r) = S(r) \) and

\[
S(r)|_{E^\perp(r)} = - \frac{dQ}{dr}(r)|_{E^\perp(r)}
\]

hold.

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Similarly, \( v_2(r) \) satisfies
\[
\frac{dv_2}{dr} = \frac{d}{dr} \{ R(r)v(r) \} \\
= \frac{d}{dr}(r)v(r) + R(r)\frac{dv}{dr}(r) \\
= -\frac{dQ}{dr}(r)v(r) + R(r)S(r)v(r) \\
= -\frac{dQ}{dr}(v_1(r) + v_2(r)),
\]
(4.7)
because \( \frac{dQ}{dr}(r) + \frac{dR}{dr}(r) = 0 \) and \( R(r)S(r) = 0 \) hold.

(4.6) and (4.7) mean \( v_1(r) \equiv 0 \) when \( v_1(0) = 0 \). This shows that \( v(r) = v_2(r) \in E^+(r) \), specially \( \Pi(h)v_0 = v(1) \in E^{+}(h) \).

The continuity of \( \Pi(h) \) and \( \Pi^{-1}(h) \) is obvious.

**Proof.** Fix \( l \in \mathbb{R} \), the continuity of \( \Pi(h) \) and \( \Pi^{-1}(h) \) is obvious.

Fix \( \rho_1 > 0 \) and define \( H(\hat{h}, \rho_1) = \{ h = (h_1, \cdots, h_N) \in \mathbb{R}^N \mid \hat{h}_j < h_j < \hat{h}_j + \rho_1 \}, \mathcal{M}(\hat{h}, \rho_1) = \{ P(l)P(z; \h) \mid l \in \mathbb{R}, h \in H(\hat{h}, \rho_1) \} \). Then, from the construction of \( \Pi(h) \), there exist a positive constant \( C_1 \) depending only on \( \rho_1 \) and independent of \( \hat{h} \) with \( \min \hat{h} > h^* \) for sufficiently large \( h^* \) such that for any \( h \in H(\hat{h}, \rho_1) \)
\[
\| \Pi(h) \|, \| \Pi^{-1}(h) \|, \| \frac{\partial}{\partial h_j} \Pi(h) \| \leq C_1,
\]
\[
\| \Pi(h) \|_{\infty}, \| \Pi^{-1}(h) \|_{\infty}, \| \frac{\partial}{\partial h_j} \Pi(h) \|_{\infty} \leq C_1,
\]
hold, where \( || \cdot ||_{\infty} \) is an operator norm with respect to the sup-norm \( || \cdot ||_{\infty} \) on \( \mathbb{R} \).

Let \( A = L(\hat{h}) \) and \( X^\omega \) be the space with the norm \( || \cdot ||_{\omega} \) defined by the fractional power \( A^\omega \) of \( A \) for \( \omega \in [0, 1] \). Hereafter, we fix \( \omega \) in \( \frac{1}{2} < \omega < 1 \) such that \( X^\omega \) is imbedded into \( BU^1(\mathbb{R}) \)(ref. [19]), where \( BU^k(\mathbb{R}) \) is the space consisting of uniformly continuous and bounded functions on \( \mathbb{R}^1 \) up to the \( k \) th order derivatives.

**Lemma 4.2.** There exists a neighborhood \( U = U(\hat{h}, \rho_1) \) of \( \mathcal{M}(\hat{h}, \rho_1) \) in \( X^\omega \) such that any \( v \in U \) is represented by
\[
v = \Xi(l)\{ P(z; \h) + \Pi(h)w \}
\]
for \( l \in \mathbb{R}^1, h \in H(\hat{h}, \rho_1) \) and \( w \in E^+\hat{h} \).

**Proof.** Fix \( l_0 \in \mathbb{R}^1, h_0 \in H(\hat{h}, \rho_1) \) arbitrarily and put \( v_0 = \Xi(l_0)P(z; h_0) \). We will show the existence of \( l, h \) and \( w \in E^\pm(\hat{h}) \) near \( l_0, h_0 \) and 0 for sufficiently small \( v \in X \) such that
\[
v + v_0 = \Xi(l)\{ P(\cdot, h) + \Pi(h)w \}.
\]
Since \( \Pi(h) \) is homeomorphic from \( E^\pm(\hat{h}) \) to \( E^\pm(h) \), it suffices to show
\[
\Xi(-l)(v + v_0) - P(\cdot; h) \in E^\pm(h).
\]
This is equivalent to
\[
0 = \langle \Xi(-l)(v + v_0) - P(\cdot; h), \phi_j^*(h) \rangle_{L^2} \\
= \langle v + v_0 - \Xi(l)P(\cdot; h), \Xi(l)\phi_j^*(h) \rangle_{L^2} \ (j = 0, 1, \cdots, N)
\]
Hence, defining
\[
V(l, h, v) = \langle v + v_0 - \Xi(l)P(\cdot; h), \Xi(l)\phi_j^*(h) \rangle_{L^2} \rangle_{j=0,1,\cdots,N} \in \mathbb{R}^{N+1},
\]
we can apply the implicit function theorem to the map $V$. First, we note $V(l_0, h_0, 0) = 0$ holds. On the other hand, Proposition 4.2 shows

\begin{align}
\langle P_z(z; h), \phi^*_j(h) \rangle_{L^2} &= 1 + O(\delta(h)), \\
\langle \frac{\partial}{\partial h_k} P(z; h), \phi^*_j(h) \rangle_{L^2} &= \begin{cases} 
-1 + O(\delta(h)) & (j \geq k), \\
O(\delta(h)) & (j < k).
\end{cases}
\end{align}

By this and the fact $\Xi(l) = -\Xi(l) \frac{\partial}{\partial z}$, we have

\begin{align*}
\frac{\partial V}{\partial l}(l_0, h_0, 0) &= \left( \langle \Xi(l_0) P_z(z; h_0), \Xi(l_0) \phi^*_j(h_0) \rangle_{L^2} \right)_{j=0,1,\ldots,N} \\
&= \left( \langle P_z(z; h_0), \phi^*_j(h_0) \rangle_{L^2} \right)_{j=0,1,\ldots,N} \\
&= (1 + O(\delta(h)))_{j=0,1,\ldots,N}.
\end{align*}

Similarly, we have for $k = (k_1, k_2, \ldots, k_N) \in \mathbb{R}^N$

\begin{align*}
\frac{\partial V}{\partial h}(l_0, h_0, 0) &= \left( \sum_{i=1}^N \left\langle -\Xi(l_0) \frac{\partial P}{\partial h_i} (\cdot, h_0), \Xi(l_0) \phi^*_j(h_0) \right\rangle_{L^2} k_i \right)_{j=0,1,\ldots,N} \\
&= \left( -\sum_{i=1}^N \left\langle \frac{\partial P}{\partial h_i} (\cdot, h_0), \phi^*_j(h_0) \right\rangle_{L^2} k_i \right)_{j=0,1,\ldots,N} \\
&= \left( \sum_{i=1}^N \frac{O(\delta(h))}{j} \right)_{j=0,1,\ldots,N}.
\end{align*}

Thus, we find $\frac{\partial V}{\partial (l, h)}(l_0, h_0, 0) = V_0 + O(\delta(h))$, a square matrix of $N + 1$ degree, where

\[
V_0 = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
1 & 1 & \cdots & 1
\end{pmatrix}.
\]

Since $V_0$ is a lower triangular matrix and invertible, $\frac{\partial V}{\partial (l, h)}(l_0, h_0, 0)$ is invertible for sufficiently small $\delta(h)$ and the implicit function theorem shows there exist $l = l(v)$ and $h = h(v)$ for small $v$ such that

\[
V(l(v), h(v), v) = 0.
\]

We transform the equation (1.6) of $u$ to that of $(w, l, h)$ by

\[
u(t, z) = \Xi(l) \{ P(z; h) + \Pi(h) w \}
\]

for $l \in \mathbb{R}^3$, $h \in H(\hat{h}, \rho_1)$ and $w \in E^+(\hat{h})$. Since $\Xi'(l) = -\Xi(l) \frac{\partial}{\partial z}$ holds, we have

\[
u_t = l\Xi'(l) \{ P(z; h) + \Pi(h) w \} + \Xi(l) \left( \frac{\partial}{\partial h} \{ P(z; h) + \Pi(h) w \} \hat{h} + \Pi(h) w_t \right)
\]

\[
= \Xi(l) \left( -\frac{l}{\partial z} \{ P(z; h) + \Pi(h) w \} + \frac{\partial}{\partial h} \{ P(z; h) + \Pi(h) w \} \hat{h} + \Pi(h) w_t \right)
\]

\[15\]
and
\[ L(u) = L(\Xi(l)\{P(z;h) + \Pi(h)w\}) = \Xi(l)L(P(z;h) + \Pi(h)w). \]

Hence, it follows that
\[ -i \frac{\partial}{\partial z} \{P(z;h) + \Pi(h)w\} + \frac{\partial}{\partial h} \{P(z;h) + \Pi(h)w\}h + \Pi(h)w = L(P(z;h) + \Pi(h)w) \]
and we have
\[
\begin{align*}
Q(h) & \left[ -i \frac{\partial}{\partial z} \{P(z;h) + \Pi(h)w\} + \frac{\partial}{\partial h} \{P(z;h) + \Pi(h)w\}h \right] \\
\Pi^{-1}(h)R(h) & \left[ -i \frac{\partial}{\partial z} \{P(z;h) + \Pi(h)w\} + \frac{\partial}{\partial h} \{P(z;h) + \Pi(h)w\}h \right] + w_t \\
& = Q(h)L(P(z;h) + \Pi(h)w),
\end{align*}
\]

\[ \Pi^{-1}(h)R(h) \left[ -i \frac{\partial}{\partial z} \{P(z;h) + \Pi(h)w\} + \frac{\partial}{\partial h} \{P(z;h) + \Pi(h)w\}h \right] + w_t = \Pi^{-1}(h)R(h)L(P(z;h) + \Pi(h)w). \]

Let \( \rho_2 > 0 \) and \( C_2 > 0 \) be constants such that if \( \|w\|_\omega < \rho_2 \) and \( h \in H(\hat{h}, \rho_1) \), then
\[ \|L(P(z;h) + \Pi(h)w) - L(P(z;h)) - L(h)\Pi(h)w\| \leq C_2|w|^2 \]
holds. We note here that \( \rho_2 \) is taken to be independent of \( \hat{h} \) and depending only on \( \rho_1 \).

Put
\[ W(\hat{h}, \rho_1, D_1) = \{w(\cdot) \in C \left( H(\hat{h}, \rho_1); E^{2,\omega}(\hat{h}) \cap X^\omega \right); \|w(h)\|_\omega < D_1 \delta(h) \}. \]

We determine \( D_1 \) later but suppose \( h^* \) is large enough so as to \( D_1 \delta(h) < \rho_2 \) for \( h \in H(\hat{h}, \rho_1) \) with \( \min \hat{h} > h^* \).

First, we consider (4.10). It follows that if \( \|w\|_\omega < D_1 \delta(h) \), then we have from (4.12)
\[
\begin{align*}
\langle L(P(z;h) + \Pi(h)w), \phi_j^*(h) \rangle_{L^2} & = \langle L(P(z;h)) + L(h)\Pi(h)w + O(\delta^2), \phi_j^*(h) \rangle_{L^2} \\
& = \langle L(P(z;h)), \phi_j^*(h) \rangle_{L^2} + \langle \Pi(h)w, L^*(h)\phi_j^*(h) \rangle_{L^2} + O(\delta^2) \\
& = \langle L(P(z;h)), \phi_j^*(h) \rangle_{L^2} + O(\delta^2) \\
& = H_j(h) + O(\delta^2),
\end{align*}
\]
where \( \delta = \delta(h) \). Here, we used the fact \( L^*(h)\phi_j^*(h) = O(\delta(h)) \).

**Lemma 4.3.**

Let \( \Pi(h)w = \frac{\partial}{\partial z} \{\Pi(h)w\} \) for \( w \in X^\omega \). Then,
\[ \|\Pi(h)w\| \leq C\|w\|_\omega \]
holds.

**Proof.** Let \( v(r) \) be the solution of (4.5) with \( v_0 = w \). We shall show \( \frac{\partial}{\partial z}v(r) \) is estimated at
\[ \|\frac{\partial}{\partial z}v(r)\| \leq C\|w\|_\omega. \]

Let \( \tilde{v}(r) = \frac{\partial}{\partial z}v(r) \). Differentiating the both sides of (4.5) by \( z \), we have
\[ \frac{d\tilde{v}}{dr} = \tilde{S}(r;h)v \]
with \( \tilde{v}(0) = \mathbf{w}_z \). Here,
\[
\tilde{S}(r; h)v = \frac{\partial}{\partial z} S(r; h)v \\
= - \sum_{j,k=0}^{N} (h_j - \tilde{h}_j) \langle \mathbf{v}, \frac{\partial}{\partial h_j} \phi_k^*(\Theta(r)) \rangle_{L^2} \frac{\partial}{\partial z} \phi_k(\Theta(r)),
\]
which is a bounded operator for \( v \in X \). Thus, we have from (4.14)
\[
\tilde{v}(r) = \mathbf{w}_z + \int_0^r \tilde{S}(r; h)v(r)dr,
\]
especially,
\[
\tilde{\Pi}(h)w = \tilde{v}(1) = \mathbf{w}_z + \int_0^1 \tilde{S}(r; h)v(r)dr.
\]
This gives the proof.

Since
\[
Q(h)v = \sum_{j=0}^{N} \langle \mathbf{v}, \phi_j^*(h) \rangle_{L^2} \phi_j(h)
\]
for \( v \in X \), (4.10) yields from (4.8), (4.9), (4.13) and Lemma 4.3,
\[
\begin{align}
-\hat{\mathbf{L}}(1 + O(\delta)) + \sum_{k=1}^{j} (-1 + O(\delta))\hat{h}_k &= H_j(h) + O(\delta^2) \ (j = 1, 2, \cdots, N), \\
-\hat{\mathbf{L}}(1 + O(\delta)) &= H_0(h) + O(\delta^2)
\end{align}
\]
if \( \mathbf{w} = \mathbf{w}(h) \in W(\hat{h}, \rho_1, D_1) \), where \( \delta = \delta(h) \). These and (4.10) imply that there exist functions
\( \tilde{H}_j = \tilde{H}(h, w) \ (j = 0, 1, \cdots, N) \) such that
\[
\begin{align}
\hat{\mathbf{L}} &= \tilde{H}_0(h, w) = -H_0(h) + O(\delta^2(h)), \\
\hat{h}_j &= \tilde{H}_j(h, w) \\
&= H_{j-1}(h) - H_j(h) + O(\delta^2(h)) \ (j = 1, 2, \cdots, N).
\end{align}
\]
Especially,
\[
\hat{\mathbf{L}}, \hat{h}_j = O(\delta(h))
\]
hold for \( \mathbf{w} \in W(\hat{h}, \rho_1, D_1) \). Similarly, it follows from (4.11) and (4.12) that
\[
w_z = A(h)\mathbf{w} + \tilde{G}(h, w)
\]
with \( \| \tilde{G} \| = O(\delta(h)) \) for \( h \in H(\hat{h}, \rho_1) \) and \( \mathbf{w} \in W(\hat{h}, \rho_1, D_1) \), where
\[
A(h) = \Pi^{-1}(h)L(h)\Pi(h), \\
\tilde{G}(h, w) = \Pi^{-1}(h)R(h)\left[ \mathbf{L}(P(z; h)) + L_2(w, w) \right] \\
+ \Pi_0 \frac{\partial}{\partial z} \{ P(z; h) + \Pi(h)w \} - \frac{\partial}{\partial h} \{ P(z; h) + \Pi(h)w \} \tilde{H},
\]
\[
L_2(w, w) = L_2(h, w)(w, w) \\
= \mathbf{L}(P(z; h) + \Pi(h)w) - \mathbf{L}(P(z; h)) - L(h)\Pi(h)w,
\]
\[
\tilde{H} = \tilde{H}(h, w) \\
= (\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_N).
\]
Lemma 4.4.

\[ \|A(h_1)w - A(h_2)w\| \leq C|h_1 - h_2| \cdot \|w\| \]

holds for \( h_j \in H(\hat{h}, \rho_1) \) \( (j = 1, 2) \) and \( w \in \{H^2(R^1)\}^n \).

Proof. From the definition of \( A(h_j) \), we have

\[
A(h_1)w - A(h_2)w = \Pi^{-1}(h_1)L(h_1)\Pi(h_1)w - \Pi^{-1}(h_2)L(h_2)\Pi(h_2)w
\]

(4.21)

\[
+ \Pi^{-1}(h_1)L(h_1)\{\Pi(h_1)w - \Pi(h_2)w\}
\]

(4.22)

\[
+ \Pi^{-1}(h_2)L(h_2)\Pi(h_2)w - \Pi^{-1}(h_2)L(h_2)\Pi(h_2)w
\]

(4.23)

for \( w \in \{H^2(R^1)\}^n \). We shall estimate (4.21), (4.22) and (4.23).

(4.23) is easily estimated by

\[
\|L(h_1)v - L(h_2)v\| = \|F'(P(z; h_1))v - F'(P(z; h_2))v\|
\leq C \sup_z \|F'(P(z; h_1)) - F'(P(z; h_2))\| \cdot \|v\|
\leq C|h_1 - h_2| \cdot \|v\|
\]

becuase \( \sup_z |P(z; h)|, \sup |P_z(z; h)| \leq C \) holds uniformly for any \( h \in R^N \), where \( v = \Pi(h_2)w \).

Next, we consider (4.22). Let \( v(r; h_j) \) be the solution of (4.5) with \( v(0; h_j) = w \). By the definition of \( \Pi(h_j) \) and \( v(0; h_j) = \Pi(h_1)w \), we have

\[
\Pi(h_1)w - \Pi(h_2)w = \{\Pi(h_1)w - \Pi(h_1)w\} - \{\Pi(h_2)w - \Pi(h_2)w\}
= \{v(1; h_1) - v(0; h_1)\} - \{v(1; h_2) - v(0; h_2)\}
= \int_0^1 \frac{dv}{dr}(r; h_1)dr - \int_0^1 \frac{dv}{dr}(r; h_2)dr
= \int_0^1 (S(r; h_1)v(r; h_1) - S(r; h_2)v(r; h_2))dr.
\]

Hence, it follows that

\[
\|\Pi^{-1}(h_1)L(h_1)\{\Pi(h_1)w - \Pi(h_2)w\}\|
\leq C \int_0^1 \|L(h_1)\{S(r; h_1)v(r; h_1) - S(r; h_2)v(r; h_2)\}\|dr
\leq C \int_0^1 \|L(h_1)S(r; h_1)v(r; h_1) - L(h_1)S(r; h_2)v(r; h_1)\|dr
+ C \int_0^1 \|L(h_1)S(r; h_2)v(r; h_1) - v(r; h_2)\|dr
\leq C \int_0^1 \|L(h_1)S(r; h_1) - L(h_1)S(r; h_2)\| \cdot \|v(r; h_1)\|dr
+ C \int_0^1 \|L(h_1)S(r; h_2)\| \cdot \|v(r; h_1) - v(r; h_2)\|dr
\leq C|h_1 - h_2| \cdot \|w\|
\]

for \( h_j \in H(\hat{h}, \rho_1) \) because \( L(h_1)S(r; h) \) is a bounded operator on \( X \) depending smoothly on \( h \in H(\hat{h}, \rho_1) \), \( r \in [0, 1] \) and \( v(r; h) \) is also a function depending smoothly on \( h \in H(\hat{h}, \rho_1) \), \( r \in [0, 1] \).

Finally, we estimate (4.21). Define the operator \( \Pi(\hat{r}; h) \) by \( \Pi(\hat{r}; h)v_0 = v(r; h) \), where \( v(r; h) \) is a solution of (4.5).
\textbf{Proposition 4.3.}

\[
\frac{d}{dr} \left\{ \hat{\Pi}^{-1}(r; h)v_0 \right\} = -\hat{\Pi}^{-1}(r; h)S(r; h)v_0
\]

holds for \( h \in H(\hat{h}, \rho_1) \), \( v_0 \in X \) and \( r \in [0, 1] \).

\textbf{Proof.} Let \( u(r) = \hat{\Pi}^{-1}(r; h)v_0 \). Then, we have

\[
v_0 = \hat{\Pi}(r; h)u(r)
\]

and therefore

\[
0 = \frac{d\hat{\Pi}}{dr}(r; h)u(r) + \hat{\Pi}(r; h)\frac{du}{dr}(r)
= S(r; h)\hat{\Pi}(r; h)u(r) + \hat{\Pi}(r; h)\frac{du}{dr}(r)
= S(r; h)v_0 + \hat{\Pi}(r; h)\frac{du}{dr}(r)
\]

holds. This implies the proof.

Similarly to the case of (4.22), we have from Proposition 4.3

\[
\Pi^{-1}(h_1)L(h_1)v - \Pi^{-1}(h_2)L(h_1)v
= \{ \Pi^{-1}(h_1)L(h_1)v - \Pi^{-1}(\hat{h})L(h_1)v \} + \{ \Pi^{-1}(\hat{h})L(h_1)v - \Pi^{-1}(h_2)L(h_1)v \}
= \int_0^1\frac{d}{dr}\left\{ \hat{\Pi}^{-1}(r; h_1)L(h_1)v \right\}dr - \int_0^1\frac{d}{dr}\left\{ \hat{\Pi}^{-1}(r; h_2)L(h_1)v \right\}dr
= -\int_0^1\hat{\Pi}^{-1}(r; h_1)S(r; h_1)L(h_1)vdr + \int_0^1\hat{\Pi}^{-1}(r; h_2)S(r; h_2)L(h_1)vdr,
\]

where \( v = \Pi(h_2)w \). Hence, it follows that

\begin{align*}
\| \Pi^{-1}(h_1)L(h_1)v - \Pi^{-1}(h_2)L(h_1)v \| \\
\leq \int_0^1 \| \hat{\Pi}^{-1}(r; h_1)S(r; h_1)L(h_1) - \hat{\Pi}^{-1}(r; h_2)S(r; h_2)L(h_1) \|dr\|v\| \\
\leq C\| h_1 - h_2 \| \cdot \| v \| \\
\leq C\| h_1 - h_2 \| \cdot \| w \|
\end{align*}

because the smooth dependence of \( \hat{\Pi}^{-1}(r; h)S(r; h)L(h_1) \) on the argument \( h \).

(4.24), (4.25) and (4.26) show the proof.

\textbf{Remark 4.1.} It is shown that

\begin{equation}
\| A(h)w - Aw \| \leq C\| h - \hat{h} \| \cdot \| w \|
\end{equation}

for \( h \in H(\hat{h}, \rho_1) \) and \( w \in X^w \) in quite a similar manner to this lemma but we use

\[
A(h)w - Aw = \Pi^{-1}(h)L(h)\Pi(h)w - L(\hat{h})w
= \Pi^{-1}(h)L(h)w - L(h)w
+ \Pi^{-1}(h)L(h)\Pi(h)w - w
+ L(h)w - L(\hat{h})w
\]

for \( h \in H(\hat{h}, \rho_1) \) and \( w \in X^w \).
Here, we note there exist $C_3 > 0$ (independent of $D_1$) and $C_4 = C_4(D_1) > 0$ (dependent on $D_1$) in (4.18) and (4.20) such that

\begin{align}
(4.28) & \quad \| \hat{H}(h, w) \| \leq C_3 \delta(h), \\
(4.29) & \quad \| \hat{H}(h, w) - \hat{H}(k, v) \| \leq C_4 \delta(h) \{ |h - k| + \| w - v \| \}, \\
(4.30) & \quad \| \tilde{G}(h, w) \| \leq C_3 \delta(h) (1 + (D_1 + D_2^2) \delta(h)), \\
(4.31) & \quad \| \tilde{G}(h, w) - \tilde{G}(k, v) \| \leq C_4 \{ \delta(h) + \delta(k) \} \{ |h - k| + \| w - v \| \},
\end{align}

hold for $h, k \in H(\hat{h}, \rho_1)$ and $w, v \in X^\omega$ with $\| w \|, \| v \| \leq D_1 \delta(h)$. We extend $\delta(h)$ for $h \not\in H(\hat{h}, \rho_1)$ such that $\delta(h) \leq \delta^* = \delta(h^*)$ holds for any $h \in \mathbb{R}^N$ and also extend $\tilde{H}$ and $\tilde{G}$ appropriately to the outside of $H(\hat{h}, \rho_1)$ such that (4.28) $\sim$ (4.31) hold for any $h, k \in \mathbb{R}^N$.

We shall construct an attractive invariant manifold of

\begin{align}
\begin{cases}
    h_t = \tilde{H}(h, w), \\
    w_t = A(h)w + \tilde{G}(h, w)
\end{cases}
\end{align}

for $h \in \mathbb{R}^N$ and $w \in W(\hat{h}, \rho_1, D_1)$. Since the resolvent of $A(h)$ satisfies

\begin{align}
(4.32) \quad \| (\lambda - A(h))^{-1} \| \leq \frac{C}{|\lambda + \rho_0|}
\end{align}

uniformly for $h \in H(\hat{h}, \rho_1)$ on $E^1(\hat{h})$, we may assume $A(h)$ is extended for $h \not\in H(\hat{h}, \rho_1)$ such that (4.32) holds for $h \in \mathbb{R}^N$. We also assume $A(h)$ is extended for $h \not\in H(\hat{h}, \rho_1)$ such that Lemma 4.4 and (4.27) hold for $h \in \mathbb{R}^N$.

Define $\delta(h, k) = \delta(h) + \delta(k)$ for $h, k \in \mathbb{R}^N$. Note that $\delta(h, k) \leq 2\delta^*$ holds. Let

$$\begin{align}
\tilde{W}(D_1, D_2) &= \{ w \in C \left( \mathbb{R}^N ; E^1(\hat{h}) \cap X^\omega \right) : \| w(h) \| \leq D_1 \delta(h), \\
& \quad \| w(h) - w(k) \| \leq D_2 \delta(h, k) |h - k| \text{ for } h, k \in \mathbb{R}^N \}
\end{align}$$

for a positive constant $D_2$. For $\sigma \in \tilde{W}(D_1, D_2)$, define $h(t) = h(t; \xi, \sigma)$ by the solution of

\begin{align}
\begin{cases}
    h_t = \tilde{H}(h, \sigma(h)), \\
    h(0) = \xi \in \mathbb{R}^N
\end{cases}
\end{align}

and define $T(t, s) = T(t, s; h(\cdot))$ by the evolution operator of

\begin{align}
(4.35) \quad w_t = A(h(t))w.
\end{align}

**Lemma 4.5.** There exist positive constants $\delta_0, C_5$ and $\gamma_1$ independent of $D_1$ such that if $h(t) \in C^1(\mathbb{R}; \mathbb{R}^N)$ satisfies $|h_t| \leq \delta_0$, then $\| T(t, s; h(\cdot))w \| \leq C_5 \max\{ (t - s)^{-\gamma_1}, 1 \} e^{-\gamma_1(t-s)} \| w \|$ holds for $w \in E^1(\hat{h})$.

**Proof.** (4.33), (4.27) and e.g. Theorem 7.4.2 of [19] directly give the proof.

---

Hereafter, we take $h^*$ large enough that $C_3 \delta^* \leq \delta_0$.

For $h(\cdot) \in C^1(\mathbb{R}; \mathbb{R}^N)$ with $|h_t| \leq C_3 \delta(h(t)) \leq \delta_0$, consider a bounded solution of

\begin{align}
(4.36) \quad w_t = A(h(t))w + \tilde{G}(h(t), w), \quad -\infty < t < +\infty.
\end{align}

**Lemma 4.6.** There exists a constant $D_1$ such that a bounded solution of (4.36), say $w(t; h(\cdot))$, uniquely exists satisfying

$$\| w(t; h(\cdot)) \| \leq D_1 \delta(h(t)).$$
Proof. Let $v(t)$ be a function satisfying
\[
\|v(t)\|_{\omega} \leq D_1 \delta(h(t)), \quad -\infty < t < +\infty
\]
and consider a bounded solution of
\[
\text{(4.37)} \quad \mathbf{w}_t = A(h(t))\mathbf{w} + \tilde{G}(h(t), v(t)).
\]
Solutions of (4.37) is represented as
\[
\mathbf{w}(t) = T(t, s)\mathbf{w}(s) + \int_{s}^{t} T(t, \eta)\tilde{G}(h(\eta), v(\eta))d\eta,
\]
where $T(t, s) = T(t, s; h(\cdot))$. Since $\|\mathbf{w}(t)\|_{\omega}$ is bounded as $t \to -\infty$, $\mathbf{w}(t)$ satisfies
\[
\text{(4.38)} \quad \mathbf{w}(t) = \int_{-\infty}^{t} T(t, s)\tilde{G}(h(s), v(s))ds.
\]
Let $W(t; v(\cdot))$ be the right hand side of (4.38). Then we have from Lemma 4.5
\[
\text{(4.39)} \quad \|W(t; v(\cdot))\|_{\omega}
\]
\[
\leq \int_{-\infty}^{t} C_5 \max\{\|(t-s)^{-\omega}, 1\}e^{-\gamma_1(t-s)}\|\tilde{G}(h(s), v(s))\|ds
\]
\[
\leq \int_{-\infty}^{t} C_5 \max\{\|(t-s)^{-\omega}, 1\}e^{-\gamma_1(t-s)}C_2\delta(h(s))\{1 + (D_1 + D_1^2)\delta(h(s))\}ds
\]
\[
\leq \int_{-\infty}^{t} C \max\{\|(t-s)^{-\omega}, 1\}e^{-\gamma_1(t-s)}\delta(h(s))ds \cdot \{1 + (D_1 + D_1^2)\delta^*\}
\]
\[
= C \int_{0}^{t} \max\{s^{-\omega}, 1\}e^{-\gamma_1^*s}\delta(h(t-s))ds \cdot \{1 + (D_1 + D_1^2)\delta^*\}
\]
\[
\leq C \int_{0}^{t} \max\{s^{-\omega}, 1\}e^{-\gamma_1^*s}ds \cdot \delta(h(t))\{1 + (D_1 + D_1^2)\delta^*\}
\]
\[
\leq C\delta(h(t))\{1 + (D_1 + D_1^2)\delta^*\}
\]
for a positive constant $\gamma_1^*$ with $0 < \gamma_1^* < \gamma_1$. Here, we used the monotone decrement of $\frac{d}{ds}\delta(h(t-s)) = O(\delta^2(h(t-s))) \leq \delta(h(t-s))O(\delta^*).$ Hence, we take $D_1$ and $h^*$ so large that
\[
C\{1 + (D_1 + D_1^2)\delta^*\} < D_1
\]
and we have
\[
\text{(4.40)} \quad \|W(t; v(\cdot))\|_{\omega} \leq D_1 \delta(h(t)).
\]
Let $\tilde{W}(D_1) = \{\mathbf{w} \in C\left(\mathbf{R}, E^{U(\tilde{h})} \cap X^{\omega}\right) : \|\mathbf{w}(t)\|_{\omega} \leq D_1 \delta(h(t))\}$. Then, (4.40) shows $W$ is a map from $\tilde{W}(D_1)$ into $\tilde{W}(D_1)$.

Now, we shall show $W$ is a contraction on $\tilde{W}(D_1)$. We have from (4.38)
\[
\|W(t; v(\cdot)) - W(t; v(\cdot))\|_{\omega}
\]
\[
\leq C \int_{-\infty}^{t} \|T(t, s)(\tilde{G}(h(s), v(s)) - \tilde{G}(h(s), v(s)))\|_{\omega}ds
\]
\[
\leq C \int_{-\infty}^{t} \|T(t, s)(\tilde{G}(h(s), v(s)) - \tilde{G}(h(s), v(s)))\|_{\omega}ds
\]
\[
\begin{align*}
&\leq C \int_{-\infty}^{t} \max\{(t-s)^{-\omega}, 1\} e^{-\gamma(t-s)}\|G(h(s), w(s)) - \tilde{G}(h(s), v(s))\|ds \\
&\leq C \int_{-\infty}^{t} \max\{(t-s)^{-\omega}, 1\} e^{-\gamma(t-s)}\delta(h(s))\|w(s) - v(s)\|_{\omega}ds \\
&\leq C \int_{0}^{t} \max\{s^{-\omega}, 1\} e^{-\gamma(s)} \{ e^{-\gamma(t-s)}\delta(h(t-s)) ds \cdot \sup_{t} \|w(t) - v(t)\|_{\omega} \\
&\leq C\delta(h(t)) \sup_{t} \|w(t) - v(t)\|_{\omega} \\
&\leq C\delta^{*} \sup_{t} \|w(t) - v(t)\|_{\omega}.
\end{align*}
\]

This shows \( W \) is a contraction on \( \tilde{W}(D_{1}) \) if \( h^{*} \) is large enough, which completes the proof.

Fix \( D_{1} \) such that Lemma 4.6 holds.

Define

\[
J(\sigma)(\xi) = w(0; h(\cdot; \xi, \sigma))
\]

for \( \xi \in \mathbb{R}^{N} \) and \( \sigma \in \tilde{W}(D_{1}, D_{2}) \). Then,

\[
(4.41) \quad \|J(\sigma)(\xi)\|_{\omega} \leq D_{1}\delta(h(0; \xi, \sigma)) = D_{1}\delta(\xi)
\]

holds by the definition and Lemma 4.6.

We shall estimate \( \|J(\sigma)(\xi_{2}) - J(\sigma)(\xi_{1})\|_{\omega} \) for \( \xi_{2}, \xi_{1} \in \mathbb{R}^{N} \) and \( \sigma \in \tilde{W}(D_{1}, D_{2}) \).

**Lemma 4.7.** If \( h_{2}, h_{1} \in C^{1}(\mathbb{R}; \mathbb{R}^{N}) \) with \( \left| \frac{d}{dt} h_{j} \right| \leq C_{3}\delta(h_{j}(t)) \) satisfy

\[
|h_{2}(t) - h_{1}(t)| \leq \zeta e^{\gamma_{2}\rho^{2}|t|}
\]

for positive constants \( \zeta \) and \( \gamma_{2} \), then

\[
\|w(t; h_{2}(\cdot)) - w(t; h_{1}(\cdot))\|_{\omega} \leq \delta(h_{1}(t), h_{2}(t))C\zeta e^{\gamma_{2}\rho^{2}|t|}
\]

holds.

**Proof.** Let \( w_{j}(t) = w(t; h_{j}(\cdot)) \) \((j = 1, 2)\). Since \( w_{j} \) satisfy

\[
w_{j}(t) = \int_{-\infty}^{t} T(t, s)\tilde{G}(h_{j}(s), w_{j}(s))ds
\]

from (4.39), we have by using similar arguments to (4.39)

\[
\begin{align*}
&\|w_{2}(t) - w_{1}(t)\|_{\omega} \\
&\leq C \int_{-\infty}^{t} \|T(t, s)\{\tilde{G}(h_{2}(s), w_{2}(s)) - \tilde{G}(h_{1}(s), w_{1}(s))\}\|_{\omega}ds \\
&\leq C \int_{-\infty}^{t} \max\{(t-s)^{-\omega}, 1\} e^{-\gamma(t-s)}\|\tilde{G}(h_{2}(s), w_{2}(s)) - \tilde{G}(h_{1}(s), w_{1}(s))\|ds \\
&\leq C \int_{-\infty}^{t} \max\{(t-s)^{-\omega}, 1\} e^{-\gamma(t-s)}\delta(h_{2}(s), h_{1}(s))\{\|h_{2}(s) - h_{1}(s)\| + \|w_{2}(s) - w_{1}(s)\|_{\omega}\}ds \\
&\leq C\delta(h_{2}(t), h_{1}(t)) \int_{-\infty}^{t} \max\{(t-s)^{-\omega}, 1\} e^{-\gamma(t-s)}\zeta e^{\gamma_{2}\rho^{2}|t|}ds \\
&\quad + C\delta(h_{2}(t), h_{1}(t)) \int_{-\infty}^{t} \max\{(t-s)^{-\omega}, 1\} e^{-\gamma(t-s)}\|w_{2}(s) - w_{1}(s)\|_{\omega}ds \\
&\leq C\delta(h_{2}(t), h_{1}(t)) \int_{0}^{t} \max\{s^{-\omega}, 1\} e^{-\gamma(t-s)}\zeta e^{\gamma_{2}\rho^{2}|t-s|}ds
\end{align*}
\]
\[ +C\delta(h_2(t), h_1(t)) \int_{-\infty}^t \max\{t-s, 1\} e^{-\gamma_1(t-s)} e^{2\delta^*\|s\|} ds \cdot \sup_t e^{-\gamma_2\delta^*\|t\|} \|w_2(t) - w_1(t)\|_\omega \]

\[ \leq C\delta(h_2(t), h_1(t)) \int_0^\infty \max\{s^{-\omega}, 1\} e^{-(\gamma'_2 - \gamma_2\delta^*)s} ds \]

\[ +C\delta(h_2(t), h_1(t)) \int_0^\infty \max\{s^{-\omega}, 1\} e^{-\gamma'_1 s} e^{2\delta^*\|t-s\|} ds \cdot \sup_t e^{-\gamma_2\delta^*\|t\|} \|w_2(t) - w_1(t)\|_\omega \]

\[ \leq C\delta(h_2(t), h_1(t)) e^{2\delta^*\|t\|} \{\xi + \sup_t e^{-\gamma_2\delta^*\|t\|} \|w_2(t) - w_1(t)\|_\omega\} \]

for a positive constant \(0 < \gamma'_1 < \gamma_1\). This yields

\[ \sup_t e^{-\gamma_2\delta^*\|t\|} \|w_2(t) - w_1(t)\|_\omega \leq C\delta(h_2(t), h_1(t)) \{\xi + \sup_t e^{-\gamma_2\delta^*\|t\|} \|w_2(t) - w_1(t)\|_\omega\}, \]

which completes the proof.

Let \(h_j(t) = h(t; \xi_j, \sigma)\) for \(j = 1, 2\) and \(\sigma \in \overline{W}(D_1, D_2)\). Let \(w_j(t) = \sigma(h_j(t))\).

**Lemma 4.8.** \(h_j(t)\) for \(j = 1, 2\) defined above satisfy

\[ |h_2(t) - h_1(t)| \leq e^{\delta^* C_4(1 + 2\delta^* D_2)|t|} |\xi_2 - \xi_1|. \]

**Proof.** Let \(h_3(t) = h_2(t) - h_1(t)\). From (4.29), we have

\[ \left| \frac{d}{dt} h_3 \right| \leq \delta^* C_4 \{|h_3(t)| + \|w_2(t) - w_1(t)\|_\omega\}. \]

Since

\[ \|w_2(t) - w_1(t)\|_\omega = \|\sigma(h_2(t)) - \sigma(h_1(t))\|_\omega \leq 2\delta^* D_2 |h_3(t)| \]

holds, it follows that

\[ \left| \frac{d}{dt} h_3 \right| \leq \delta^* C_4(1 + 2\delta^* D_2) |h_3|. \]

This gives the proof.

From Lemmas 4.7 and 4.8,

\[ \|w(t; h_2(\cdot)) - w(t; h_1(\cdot))\|_\omega \leq \delta(h_2(t), h_1(t)) C|\xi_2 - \xi_1| e^{\delta^* C_4(1 + 2\delta^* D_2)|t|} \]

holds. Therefore, we have

\[ \|J(\sigma)(\xi_2) - J(\sigma)(\xi_1)\|_\omega = \|w(0; h_2(\cdot)) - w(0; h_1(\cdot))\|_\omega \]

\[ \leq \delta(h_2(t), h_1(t)) C|\xi_2 - \xi_1| \]

\[ \leq D_2 \delta(h_2(t), h_1(t)) |\xi_2 - \xi_1| \]

by taking \(D_2\) appropriately large. (4.41) and (4.42) imply that

\[ J(\sigma) : \overline{W}(D_1, D_2) \to \overline{W}(D_1, D_2). \]

We shall show \(J\) is a contraction map on \(\overline{W}(D_1, D_2)\). For given \(\sigma_1, \sigma_2 \in \overline{W}(D_1, D_2)\) and \(\xi \in \mathbb{R}^N\), we let \(h_j(t) = h(t; \xi, \sigma_j)\) and \(w_j(t) = \sigma_j(h_j(t))\) for \(j = 1, 2\).
Lemma 4.9.

\[ |h_2(t) - h_1(t)| \leq \frac{1}{1 + 2\delta^*D_2} ||\sigma_2 - \sigma_1||e^{\delta^*C_4(1 + 2\delta^*D_2)|t|}, \]

holds, where \( ||\cdot|| \) denotes \( ||\sigma|| = \sup_{\xi \in \mathbb{R}^n} ||\sigma(\xi)||_\omega \) for \( \sigma \in \overline{W}(D_1, D_2) \).

Proof. Let \( h_3(t) = h_2(t) - h_1(t) \). Then, from (4.29), we have

\begin{align*}
|\frac{d}{dt} h_3| & \leq C_4\delta^*|h_3| + \|w_2(t) - w_1(t)||\omega \\
& = C_4\delta^* \{ |h_3| + ||\sigma_2(h_2(t)) - \sigma_1(h_1(t))|| \}. 
\end{align*}

Here,

\begin{align*}
||\sigma_2(h_2(t)) - \sigma_1(h_1(t))|| & \leq ||\sigma_1(h_2(t)) - \sigma_1(h_1(t))|| + ||\sigma_2(h_2(t)) - \sigma_1(h_2(t))|| \\
& \leq D_2\delta(h_2(t), h_1(t))|h_2(t) - h_1(t)| + ||\sigma_2 - \sigma_1|| \\
& \leq 2D_2\delta^*|h_2(t) - h_1(t)| + ||\sigma_2 - \sigma_1||
\end{align*}

holds. Substituting this to (4.44), we have

\[ |\frac{d}{dt} h_3| \leq C_4\delta^*(1 + 2\delta^*D_2)|h_3| + C_4\delta^*||\sigma_2 - \sigma_1||. \]

This yields

\[ |h_3(t)| \leq \int_0^{|t|} e^{C_4\delta^*(1 + 2\delta^*D_2)(|t| - s)}ds \cdot C_4\delta^*||\sigma_2 - \sigma_1|| \\
= \frac{1}{1 + 2\delta^*D_2} \left( e^{C_4\delta^*(1 + 2\delta^*D_2)|t|} - 1 \right) ||\sigma_2 - \sigma_1|| \\
\leq \frac{1}{1 + 2\delta^*D_2} ||\sigma_2 - \sigma_1|| e^{C_4\delta^*(1 + 2\delta^*D_2)|t|}. \]

\[ \Box \]

Lemmas 4.7 and 4.9 imply

\[ ||w(t; h_2(\cdot)) - w(t; h_1(\cdot))||_\omega \leq \frac{2\delta^*C}{1 + 2\delta^*D_2} ||\sigma_2 - \sigma_1|| e^{C_4\delta^*(1 + 2\delta^*D_2)|t|}. \]

This directly shows

\begin{align*}
||J(\sigma_2)(\xi) - J(\sigma_1)(\xi)||_\omega & = ||w(0; h_2(\cdot)) - w(0; h_1(\cdot))||_\omega \\
& \leq \frac{2\delta^*C}{1 + 2\delta^*D_2} ||\sigma_2 - \sigma_1|| \\
& \leq C\delta^*||\sigma_2 - \sigma_1||.
\end{align*}

Hence, \( J \) is a contraction and there uniquely exists \( \tilde{\sigma} \in \overline{W}(D_1, D_2) \) satisfying \( J(\tilde{\sigma}) = \tilde{\sigma} \).

Let \( \overline{M}(\hat{h}, \rho_1) = \{ \Xi(l)[P(z; h) + \Pi(h)\tilde{\sigma}(h)]; l \in \mathbb{R}^l, h \in H(\hat{h}, \rho_1) \} \). Then, from the construction of \( \tilde{\sigma} \), we can show easily that \( \overline{M}(\hat{h}, \rho_1) \) is positively invariant with respect to the flow of (1.6) as long as \( h(t) \in H(\hat{h}, \rho_1) \), where \( h(t) \) is a solution of (4.34) with \( \sigma = \tilde{\sigma} \). Smoothness and an exponential attractivity of \( \overline{M}(\hat{h}, \rho_1) \) together with the existence of asymptotic phase are also shown in quite a similar manner to Section 9 of [19]. Here, we specially note that the attractivity is determined only by the estimate of semigroup \( e^{-\lambda h(t)} \). This implies that the set \( \overline{M}(\hat{h}, \rho_1) \) has an attractivity uniformly for \( \hat{h} \) with sufficiently large min \( \hat{h} > h^* \). That is, we have the following result now.
THEOREM 4.1. There exist positive constants $h^*, \rho_1, \gamma_3, \eta_1$ and $M_3$ such that for any $\hat{h}$ with $\min \hat{h} \geq h^*$ and any $u(0, \cdot)$ with $\text{dist}\{u(0, \cdot), \hat{M}(\hat{h}, \rho_1)\} < \eta_1$, there exist functions $l(t)$ and $h(t)$ which are the solutions of (4.17), (4.18) with $w = \hat{\sigma}(h)$ such that

$$\|u(t, \cdot) - \Xi(l(t))\{P(\cdot, h(t)) + \Pi(h(t))\hat{\sigma}(h(t))\}\|_\omega \leq M_3 e^{-\gamma_3 t}\text{dist}\{u(0, \cdot), \hat{M}(\hat{h}, \rho_1)\}$$

holds as long as $h(t) \in H(\hat{h}, \rho_1)$, where $u(t, z)$ is a solution of (1.6).

By using this theorem, we shall show Theorems 2.1 and 2.2.

4.2. Proof of Theorem 2.1. Let $\rho_3 = \frac{1}{2} \rho_1$. For any $k = (k_1, k_2, \ldots, k_N) \in Z_+^N$, define $\tilde{h}_j(k) = h^* + k_j \rho_3$ and $\tilde{h}(k) = (\tilde{h}_1(k), \tilde{h}_2(k), \ldots, \tilde{h}_N(k))$, where $Z_+ = \{0, 1, \ldots\}$. Then, $(h^*, \infty)^N = \bigcup_{k \in Z_+^N} H(\tilde{h}(k), \rho_1)$ holds. Theorem 4.1 shows that there exist, say $\hat{\sigma} = \hat{\sigma}(\cdot; k)$, and $\hat{M}(\hat{h}(k), \rho_1)$ for any $k \in Z_+^N$ such that the solution $u(t, \cdot)$ of (1.6) satisfies

$$\|u(t, \cdot) - \Xi(l(t))\{P(\cdot, h(t)) + \Pi(h(t))\hat{\sigma}(h(t); k)\}\|_\omega \leq M_3 e^{-\gamma_3 t}\text{dist}\{u(0, \cdot), \hat{M}(\hat{h}, \rho_1)\},$$

and therefore,

$$\|u(t, \cdot) - \Xi(l(t))P(\cdot, h(t))\|_\omega \leq M_3 e^{-\gamma_3 t}\text{dist}\{u(0, \cdot), \hat{M}(\hat{h}, \rho_1)\} + \|\Xi(l(t))\Pi(h(t))\hat{\sigma}(h(t); k)\|_\omega$$

$$\leq M_3 e^{-\gamma_3 t}\text{dist}\{u(0, \cdot), \hat{M}(\hat{h}, \rho_1)\} + C\delta(h(t))$$

$$\leq C\delta(h(t))$$

holds for sufficiently large $t > 0$ because $\hat{\sigma}(\cdot; k) \in \hat{W}(D_1, D_2)$. Since we may assume $C\delta(h(t)) \leq C\delta^* \leq \eta_1$, the solution $u(t, \cdot)$ is in an attractive region of $\hat{M}(\tilde{h}(k), \rho_1)$ for a certain $k$. That is, there exists $k \in H(\tilde{h}(k), \rho_1)$ for any $u(t, \cdot)$ with $h(t) \in (h^*, \infty)^N$ such that $u(t, \cdot)$ stays close to $\hat{M}(\tilde{h}(k), \rho_1)$. This completes the proof.

4.3. Proof of Theorem 2.2. Let $\tilde{h}$ be the equilibrium stated in the theorem. Since $\min \tilde{h} > h^*$ is satisfied, there exist a $\tilde{v} \in (h^*, \infty)^N$ such that $\tilde{h} \in H(\tilde{h}(k), \rho_1)$ and there exist a function $\tilde{\sigma}(h) = \hat{\sigma}(h, \tilde{h})$ which gives an attractive invariant set of (4.32) with $w = \tilde{\sigma}(h)$ by Theorem 4.1. As

$$\tilde{H}(h, \tilde{\sigma}(h)) = H(h) + O(\delta^2(h))$$

holds from (4.18), it is easily shown by the implicit function theorem that $\tilde{H}(h, \tilde{\sigma}(h))$ has a stable equilibrium $h^\dagger$ satisfying

$$h^\dagger = \tilde{h} + O(\delta(\tilde{h})).$$

Defining $l^\dagger = \tilde{h}_0(h^\dagger, \tilde{\sigma}(h^\dagger))$ and $\tilde{P}(\cdot) = P(\cdot, h^\dagger) + \Pi(h^\dagger)\tilde{\sigma}(h^\dagger)$, we see

$$\Xi(l^\dagger, t)\tilde{P}(\cdot) = \tilde{P}(z - l^\dagger t)$$

gives the stable traveling wave pulse solution stated in the statement.

Unstable traveling wave pulse solution is constructed in quite a similar manner.

4.4. Proofs of theorems when $D$ has zero elements. In this subsection, we use same notations and symbols as those in previous subsections as long as they are not specially noticed.

When $D$ includes zero in the elements, the operator $L(h)$ is not sectorial and the fractional powerspace $X^\omega$ imbedded into $BU^1(\tilde{R})$ is not defined, and hence, the transformation of (4.10) and (4.11) is not applicable. We reconstruct a new map instead of $\Pi(h)$.

Let the base space $X = BU^1(\tilde{R})$ with sup-norm and let an operator $K(h)$ be

$$K(h)v = \sum_{j=0}^N \langle v, \frac{\partial}{\partial z} \phi_j^*(h) \rangle_{L^2} \phi_j(h)$$
for $v \in X$. Fix $\hat{l} \in \mathbb{R}^1$ arbitrarily and define a map $\Lambda(l, h)$ by $\Lambda(l, h) v_0 = v(\hat{l})$, where $v(\hat{l})$ is a solution of

$$
\begin{align*}
\frac{dv}{dr} &= \Xi(r) K(h) \Xi(-r)v, \\
v(\hat{l}) &= v_0 \in X.
\end{align*}
$$

Let $\hat{\Pi}(l, h) = \Lambda(l, h) \Pi(h)$ and $E^{+}(l, h) = \Xi(l) E^{+}(h)$. Then, it is easily shown that the map $\hat{\Pi}(l, h)$ is homeomorphic from $E^{+}(\hat{l}, \hat{h})$ to $E^{+}(l, h)$ by (4.3) and (4.4). Moreover, it is also shown by the construction that $\hat{\Pi}(l, h)$ is a bounded operator on $X$ up to their first derivatives with respect to $l$ and $h$. Hence, the transformation

$$
v = \Xi(l) P(z; h) + \hat{\Pi}(l, h) w
$$

for $(l, h) \in H(\hat{l}, \hat{h}, \rho_1)$ and $w \in E^{+}(\hat{l}, \hat{h})$, where $H(\hat{l}, \hat{h}, \rho_1) = \{(l, h) \in \mathbb{R}^{N+1}; \; \hat{l} < l < \hat{l} + \rho_1, \; h \in H(\hat{h}, \rho_1)\}$ yields the equations of $(l, h, w)$ as (4.17), (4.18) and (4.20)

\begin{align*}
\dot{l} &= H_0(l, h, w) = -H_0(h) + O(\hat{\delta}^2), \\
\dot{h}_j &= H_j(l, h, w) = H_{j-1}(h) - H_j(h) + O(\hat{\delta}^2), \\
\dot{w}_l &= A(l, h) w + \tilde{G}(l, h, w)
\end{align*}

for $w \in E^{+}(\hat{l}, \hat{h})$ with $\|w\|_{\infty} = O(\hat{\delta})$, where $A(l, h) = \hat{\Pi}^{-1}(l, h) \Xi(l) L(h) \hat{\Pi}(l, h)$ and so on. Then, quite a similar way to the previous subsections, we can show the existence of $\tilde{\sigma}(l, h) \in E^{+}(\hat{l}, \hat{h})$ and a positively locally invariant attractive manifold given by

$$
\tilde{M}(\hat{l}, \hat{h}, \rho_1) = \{\Xi(l) P(z; h) + \hat{\Pi}(l, h) \tilde{\sigma}(l, h); \; (l, h) \in H(\hat{l}, \hat{h}, \rho_1)\}
$$

such that $\|\tilde{\sigma}(l, h)\|_{\infty} \leq C\delta(h)$ and

$$
\|u(t, \cdot) \{\Xi(l) P(z; h) + \hat{\Pi}(l, h) \tilde{\sigma}(l, h)\}\|_{\infty} \leq C \gamma \text{dist}\{u(0, \cdot), \tilde{M}(\hat{l}, \hat{h}, \rho_1)\}
$$

holds as long as $(l, h) \in H(\hat{l}, \hat{h}, \rho_1)$ if $u(0, \cdot)$ is sufficiently close to $\tilde{M}(\hat{l}, \hat{h}, \rho_1)$. In the proof of these results, we used the same estimate for the operator $A(l, h)$ as in Lemma 4.5 with $\omega = 0$ though $A$ generates only the $C^0$ semigroup. In fact, the estimate in Lemma 4.5 with $\omega = 0$ is obtained only from the result in Lemma 4.4 and the exponential decay of $e^{tA}$ with respect to $t > 0$, which has been extensively studied in related works on the stability of the 1-pulse solution for the FitzHugh-Nagumo systems (e.g. [5] ~ [8], [17],[30]).

The rest of the proof is the same as subsection 4.2.

4.5. Proof of Theorem 2.3. From the assumptions (2.5), (2.6) on $P(z)$, constants $\alpha$, $\beta$ and vectors $a^{\pm}$ satisfy

\begin{align*}
\alpha^2 Da^{\pm} + \theta a a^{\pm} + F'(0) a^{\pm} &= 0, \\
\beta^2 Da^{-} - \theta a^{-} + F'(0) a^{-} &= 0.
\end{align*}

Hence, it is easily seen that there exist non zero eigenvectors $b^{\pm}$ satisfying

\begin{align*}
\alpha^2 Db^{-} + \theta a b^{-} + F'(0) b^{-} &= 0, \\
\beta^2 Db^{+} - \theta a b^{+} + F'(0) b^{+} &= 0.
\end{align*}

(4.51) and (4.52) give the asymptotic form of $\phi^*$ as $x \to \mp \infty$, respectively, which shows (2.7) and (2.8) hold.

We fix $j^*$ ($1 \leq j^* \leq N - 1$) arbitrarily and show (2.9) for $j = j^*$. For $j = 0$ and $N$, we can prove (2.10), (2.11) in quite a similar way and omit the proof.
Let \( h_{\text{min}} = \min h \) and
\[
\mu = \frac{\beta}{2(\alpha + \beta)} h_{\text{min}}.
\]

Defining \( I_1 = (-\infty, -h_j^* + \mu] \cup [h_{j^*+1} - \mu, \infty) \) and \( I_2 = (-h_j^* + \mu, h_{j^*+1} - \mu) \), we consider
\[
H_j^*(h) = \int_{-\infty}^\infty \langle \mathcal{L}(P(z + j^*; h)), \phi^*(z) \rangle \, dz = \int_{I_1 \cup I_2} \langle \mathcal{L}(P(z + j^*; h)), \phi^*(z) \rangle \, dz.
\]

Lemma 4.10.
\[
\int_{I_1^*} \left| \langle \mathcal{L}(P(z + j^*; h)), \phi^*(z) \rangle \, dz \right| \leq C \left( e^{-\alpha h_j^*} + e^{-\beta h_{j^*+1}} \right) e^{-\gamma h_{\text{min}}}
\]
for a constant \( \gamma_4 > 0 \).

Proof.

Let \( I_1 = I_1^- \cup I_1^+ \), where \( I_1^- = (-\infty, -h_j^* + \mu] \) and \( I_1^+ = [h_{j^*+1} - \mu, \infty) \). We divide \( I_1^- \) moreover into \( I_1^- = I_1^{-L} \cup I_1^{-R} \), where \( I_1^{-L} = (-\infty, -h_j^*] \) and \( I_1^{-R} = (-h_j^*, -h_{j^*+1}] \). On \( I_1^{-L}, \phi^*(z) \) has the estimate (2.8) and we have
\[
\int_{I_1^{-L}} \langle \mathcal{L}(P(z + j^*; h)), \phi^*(z) \rangle \, dz \leq C \delta(h) \int_{I_1^{-L}} |\phi^*(z)| \, dz \leq C \delta(h)e^{-\alpha h_j^*} \left( 1 + O(e^{-\gamma h_j^*}) \right) \leq C \delta(h)e^{-\alpha h_j^*}.
\]

On the other hand, on \( I_1^{-R} \),
\[
|P(z + j^*; h) - P(z + h_j^*)| \leq C \left( e^{-\alpha(z+h_j^*+h_{j^*+1})} + e^{\beta z} \right)
\]
and for \( j \neq j^* - 1 \),
\[
|P(z + j^* - z_j)| \leq C \left( e^{-\alpha(z+h_j^*+h_{j^*+1})} + e^{\beta z} \right)
\]
hold. Hence, we have for \( z \in I_1^{-R} \),
\[
\mathcal{L}(P(z + j^*; h)) = \mathcal{L}(P(z + j^*; h)) - \sum_{j=0}^N \mathcal{L}(P(z + j^* - z_j))
\]
\[
= F(P(z + j^*; h)) - \sum_{j=0}^N F(P(z + j^* - z_j))
\]
\[
= F(P(z + h_j^*)) + F'(P(z + h_j^*)) \sum_{j \neq j^* - 1} P(z + j^* - z_j)
\]
\[
- F(P(z + h_j^*)) - \sum_{j \neq j^* - 1} F'(0)P(z + j^* - z_j)
\]
\[
+ O \left( e^{-2\alpha(z+h_j^*+h_{j^*+1})} + e^{2\beta z} \right)
\]
\[27\]
Thus, it follows that

\[ F'(P(z+h_j)) - F'(0) = \sum_{j \neq j^* - 1}^N P(z + z_j - z_j) + O\left(e^{-2\alpha(z+h_j, + h_{j^* - 1})} + e^{2\beta}\right) \]

\[ = O\left(e^{-\alpha(z+h_j, + h_{j^* - 1})} + e^{\beta}\right). \]

Thus, it follows that

\[ \int_{I_i} \langle \mathcal{L}(P(z + z_j; h)), \phi^*(z) \rangle \, dz \]

\[ \leq C \int_{I_i} \left( e^{-\alpha(z+h_j, + h_{j^* - 1})} + e^{\beta}\right) \cdot |\phi^*(z)| \, dz \]

\[ \leq C \int_{I_i} \left( e^{-\alpha(z+h_j, + h_{j^* - 1})} + e^{\beta}\right), e^{\alpha z} (1 + O(e^{\gamma z})) \, dz \]

\[ \leq C \left( e^{-\alpha h_j,} \left( e^{-\alpha h_{j^* - 1}} + e^{-\beta h_{j^*} + (\alpha + \beta)\mu} \right) \right) \]

\[ \leq C e^{-\alpha h_j,} \left\{ \frac{\beta}{\alpha + \beta} h_{\text{min}} e^{-\alpha h_{j^* - 1}} + e^{-\frac{1}{2}\beta h_{\text{min}}} \right\} \]

\[ \leq C e^{-\alpha h_j,} e^{-\gamma h_{\text{min}}} \]

for a constant \( \gamma_5 > 0 \) because \( h_{\text{min}} \leq h_{j^* - 1} \) and (4.53). Since \( \delta(h) \leq C e^{-\gamma h_{\text{min}}} \) holds for \( \gamma^* = \min\{\alpha, \beta\} \), (4.54) and (4.55) yield

\[ \int_{I_i} \langle \mathcal{L}(P(z + z_j; h)), \phi^*(z) \rangle \, dz \leq C e^{-\alpha h_j,} e^{-\gamma h_{\text{min}}} \]

for a constant \( \gamma_6 > 0 \).

Similarly, it is shown that

\[ \int_{I_i} \langle \mathcal{L}(P(z + z_j; h)), \phi^*(z) \rangle \, dz \leq C e^{-\beta h_j,} e^{-\gamma h_{\text{min}}} \]

for a constant \( \gamma_7 > 0 \).

(4.56) and (4.57) complete the proof. 

**Lemma 4.11.**

\[ \int_{I_i} \langle \mathcal{L}(P(z + z_j; h)), \phi^*(z) \rangle \, dz \]

\[ = \left( M_a e^{-\alpha h_j,} + M_b e^{-\beta h_j, + 1} \right) \left\{ 1 + O\left(e^{-\gamma h_{\text{min}}}\right) \right\} \]

for a constant \( \gamma_8 > 0 \).

**Proof.**

From assumptions (2.5) and (2.6),

\[ P(z + z_j; h) = P(z) + \sum_{j=0}^{j^* - 1} e^{-\alpha(z+z_j, - z_j)} a^+ + \sum_{j=j^* + 1}^N e^{\beta(z-z_j, + z_j)} a^+ \]

\[ + O(e^{-\alpha \gamma(z+h_j,)} + e^{(\beta + \gamma)(z-h_{j^*}, + 1)}) \]

\[ = P(z) + \sum_{j=0}^{j^* - 1} e^{-\alpha(z+z_j, - z_j)} a^+ + \sum_{j=j^* + 1}^N e^{\beta(z-z_j, + z_j)} a^+ \]

\[ + O(e^{-\alpha \gamma(z+h_j,)} + e^{(\beta + \gamma)(z-h_{j^*}, + 1)}) \]

\[ = P(z) + \sum_{j=0}^{j^* - 1} e^{-\alpha(z+z_j, - z_j)} a^+ + \sum_{j=j^* + 1}^N e^{\beta(z-z_j, + z_j)} a^+ \]

\[ + O(e^{-\alpha \gamma(z+h_j,)} + e^{(\beta + \gamma)(z-h_{j^*}, + 1)}) \]

\[ = P(z) + \sum_{j=0}^{j^* - 1} e^{-\alpha(z+z_j, - z_j)} a^+ + \sum_{j=j^* + 1}^N e^{\beta(z-z_j, + z_j)} a^+ \]

\[ + O(e^{-\alpha \gamma(z+h_j,)} + e^{(\beta + \gamma)(z-h_{j^*}, + 1)}) \]
holds on $I_2$. We let the right hand side of (4.58) be $P(z) + PL(z) + PR(z) + g(z)$. Then, we have

\[(4.59)\]

\[
\mathcal{L}(P(z + zm), h)) = \mathcal{L}(P(z + zm), h)) - \sum_{j=0}^{N} \mathcal{L}(P(z + zm, zj)) = F(P(z + zm, h)) - \sum_{j=0}^{N} F(P(z + zm, zj)) = F(P(z)) + F'(P(z))(PL(z) + PR(z) + g(z)) - \{F(P(z)) + F'(0)(PL(z) + PR(z) + g(z))\} + O(e^{-2\alpha z + h} + e^{2\beta(z - h)})
\]

\[
= \{F'(P(z)) - F'(0)\}(PL(z) + PR(z) + g(z)).
\]

Here, we assumed $\gamma < \gamma^*$ and wrote $O(g(z))$ by $g(z)$ again.

It is easily seen that

\[(4.60)\]

\[
\left| \int_{I_2} \langle g(z), \phi^*(z) \rangle \, dz \right| \leq C(e^{-\alpha h} + e^{-\beta h}) e^{-\gamma h_{\min}}
\]

for a constant $\gamma_0 > 0$.

Now, we shall calculate

\[(4.61)\]

\[
\int_{I_2} \langle \{F'(P(z)) - F'(0)\}(PL(z) + PR(z)), \phi^*(z) \rangle \, dz.
\]

Substituting the definitions of $PL(z)$ and $PR(z)$ into (4.61), we have

\[(4.62)\]

\[
\int_{I_2} \langle \{F'(P(z)) - F'(0)\}(PL(z) + PR(z)), \phi^*(z) \rangle \, dz = M_a' \int \int_{I_2} e^{-\alpha z} \langle \{F'(P(z)) - F'(0)\}a^+, \phi^*(z) \rangle \, dz,
\]

\[= M_a' \int \int_{I_2} e^{-\alpha z} \langle \{F'(P(z)) - F'(0)\}a^+, \phi^*(z) \rangle \, dz,
\]

\[M_b' = \int \int_{I_2} e^{\beta z} \langle \{F'(P(z)) - F'(0)\}a^-, \phi^*(z) \rangle \, dz.
\]

**Proposition 4.4.**

\[(4.63)\]

\[
\left| \int_{I_1} e^{-\alpha z} \langle \{F'(P(z)) - F'(0)\}a^+, \phi^*(z) \rangle \, dz \right| \leq Ce^{-\gamma_0 h_{\min}},
\]

\[(4.64)\]

\[
\left| \int_{I_1} e^{\beta z} \langle \{F'(P(z)) - F'(0)\}a^-, \phi^*(z) \rangle \, dz \right| \leq Ce^{-\gamma_0 h_{\min}}
\]

hold for a constant $\gamma_0 > 0$.

**Proof.**

We shall prove (4.63). On $I_1$, $P(z)$ and $\phi^*(z)$ have estimates (2.6) and (2.8), respectively. Hence,

\[
\left| \langle \{F'(P(z)) - F'(0)\}a^+, \phi^*(z) \rangle \right| \leq Ce^{\beta z} \cdot e^{\alpha z}
\]

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holds and therefore, we have
\[
\left| \int_{t_1} e^{\beta z} \langle \{F'(P(z)) - F'(0)\}a^-, \phi^*(z) \rangle dz \right|
\leq C \left| \int_{-\infty}^{e^{-h_j + \mu}} e^{-\alpha z} e^{\beta z} e^{\alpha z} dz \right|
\leq Ce^{h_j + \mu}
\]
(4.64) is shown in quite a same manner.

Proposition 4.4 implies that

(4.65) \[ M'_a = \int_{-\infty}^{\infty} e^{-\alpha z} \langle \{F'(P(z)) - F'(0)\}a^+, \phi^*(z) \rangle dz + O(e^{-\gamma_0 h_{\min}}), \]
(4.66) \[ M'_b = \int_{-\infty}^{\infty} e^{\beta z} \langle \{F'(P(z)) - F'(0)\}a^-, \phi^*(z) \rangle dz + O(e^{-\gamma_0 h_{\min}}), \]

PROPOSITION 4.5.

(4.67) \[ M_a = \int_{-\infty}^{\infty} e^{-\alpha z} \langle \{F'(P(z)) - F'(0)\}a^+, \phi^*(z) \rangle dz, \]
(4.68) \[ M_b = \int_{-\infty}^{\infty} e^{\beta z} \langle \{F'(P(z)) - F'(0)\}a^-, \phi^*(z) \rangle dz \]
hold.

Proof.

First, we will consider (4.67). Since \( \phi^*(z) \) and \( a^+ \) satisfy \( L^*(h)\phi^* = 0 \) and (4.49), we have
\[
\langle F'(P(z))a^+, \phi^*(z) \rangle = \langle a^+, F'(P(z))\phi^*(z) \rangle
= \langle a^+, D\phi^* + \theta \phi^* \rangle,
\]
\[
\langle F'(0)a^+, \phi^*(z) \rangle = -\langle a^+ D a^+ + \theta a^+, \phi^*(z) \rangle
= -\langle a^+, a^2 D \phi^* + \theta a^* \rangle.
\]

Hence, \( M_a \) is

(4.69) \[ M_a = \int_{-\infty}^{\infty} e^{-\alpha z} \langle a^+, a^2 D \phi^* + \theta a^* - D\phi^* - \theta \phi^* \rangle dz. \]

We calculate each term of the right hand side of (4.69). Note that (2.8) and
\[
\phi^*_z(z) \rightarrow \alpha e^{\alpha z} (b^- + O(e^{\gamma z}))
\]
hold as \( z \rightarrow -\infty \). Then by integration by parts, we see

(4.70) \[ \int_{-\infty}^{\infty} e^{-\alpha z} \langle a^+, D\phi^* \rangle dz
= \left[ e^{-\alpha z} \langle a^+, D\phi^* \rangle \right]_{-\infty}^{\infty} + \alpha \int_{-\infty}^{\infty} e^{-\alpha z} \langle a^+, D\phi^* \rangle dz
= -\alpha \langle a^+, D b^- \rangle + \alpha \left[ e^{-\alpha z} \langle a^+, D\phi^* \rangle \right]_{-\infty}^{\infty} + \alpha^2 \int_{-\infty}^{\infty} e^{-\alpha z} \langle a^+, D\phi^* \rangle dz
= -2\alpha \langle a^+, D b^- \rangle + \alpha^2 \int_{-\infty}^{\infty} e^{-\alpha z} \langle a^+, D\phi^* \rangle dz. \]

(30)
where \( B \) is taken sufficiently large and hence \( \delta(h) \) is small enough.

Define the operator \( \mathcal{D} \).

Proof. Throughout this section, we assume \( \min h \) is taken sufficiently large and hence \( \delta(h) \) is small enough.

Let

\[
L_j v = Dv_{zz} - \theta v_z + F'(P(z - z_j))v,
\]

\[
\tilde{L}v = Dv_{zz} - \theta v_z + F'(0)v
\]

for \( v \in \{H^2(B^1)\}^n \), where \( z_j = \sum_{k=1}^j h_k \ (j = 1, 2, \cdots, N) \) and \( z_0 = 0 \) as introduced in the top part of Section 2. Note that each \( L_j \) has simple zero eigenvalue together with the eigenfunction \( P_j(z - z_j) \). Now, we define a set of cut-off functions \( \{\chi_j(z); \ j = 0, 1, \cdots, N\} \) such that \( 0 \leq \chi_j(z) \leq 1, \sum_{j=0}^N \chi_j(z) \equiv 1 \) and

\[
\chi_j(z) = \begin{cases} 
1, & z \in \Omega_j^1 = \left( z_j - \frac{1}{2}h_j, z_j + \frac{1}{2}h_j + 1 \right), \\
0, & z \in \Omega_j^2 = \left( -\infty, z_j - \frac{1}{2}h_j \right) \cup \left( z_j + \frac{1}{2}h_j + 1, \infty \right)
\end{cases}
\]

for \( j = 1, 2, \cdots, N \) and 1 and

\[
\chi_0(z) = \begin{cases} 
1, & z \in \Omega_0^1 = \left( -\infty, z_0 + \frac{1}{2}h_1 - 1 \right), \\
0, & z \in \Omega_0^2 = \left( z_0 + \frac{1}{2}h_1 + 1, \infty \right),
\end{cases}
\]

\[
\chi_N(z) = \begin{cases} 
1, & z \in \Omega_N^1 = \left( z_N - \frac{1}{2}h_N + 1, \infty \right), \\
0, & z \in \Omega_N^2 = \left( -\infty, z_N - \frac{1}{2}h_N - 1 \right).
\end{cases}
\]

Lemma 5.1. The spectrum \( \Sigma(L(h)) \) of \( L(h) \) consists of sets \( \Sigma_1 \cup \Sigma_2 \) such that \( \Sigma_1 \subset \{\lambda \in C; |\lambda| \leq C\sqrt{\delta(h)}\} \) and \( \Sigma_2 \subset \{\lambda \in C; \Re \lambda < -\rho_1\} \) for positive constants \( C \) and \( \rho_1 \).

Proof. Define the operator \( D(\lambda) = \sum_{j=0}^N \chi_j(\lambda - L_j)^{-1} \), and let \( \tilde{u} = D(\lambda)f \), \( u_j = (\lambda - L_j)^{-1}f \) for \( \lambda \in \rho(L) \) and \( f \in X \), where \( \rho(L) \) is the resolvent set of \( L \), i.e. \( \rho(L) = C \setminus \Sigma(L) \). Note that \( \Sigma(L_j) = \Sigma(L) \) for \( j = 0, 1, \cdots, N \).

First, we consider \( (\lambda - L(h))u \) in \( L^2(\Omega_j^1) \). For \( z \in \Omega_j^1 \), \( L(h) \) is given by

\[
L(h) = L_j + B_j,
\]

where \( B_j = B_j(z; h) = F'(P(z; h)) - F'(P(z - z_j)) \) is a matrix operator with \( |B_j| \leq O(e^{-\alpha(z - z_j - 1)} + e^{\beta(z - z_j + 1)}) \leq O(e^{-\frac{1}{2}\alpha h_j} + e^{\frac{1}{2}\beta h_j + 1}) \leq O(\sqrt{\delta}) \). Since \( \tilde{u} = u_j \) on \( \Omega_j^1 \), we have

\[
(5.1) \quad (\lambda - L(h))\tilde{u} = (\lambda - L(h))u_j = (\lambda - L_j)u_j + B_ju_j = f + B_ju_j = f + B_j\tilde{u}.
\]
Let $\Omega_j = (z_j - \frac{1}{2}h_j, z_j + \frac{1}{2}h_j)$ and let $\Omega_j^b = (z_j - \frac{1}{2}h_j - 2, z_j - \frac{1}{2}h_j + 2)$. We shall consider $(\lambda - L(h))$ in $L^2(\Omega_j^b)$. Since $P(z; h) = O(e^{-\alpha(z-z_j^1-1)} + e^{\beta(z-z_j^1)})$, $|P(z; h)| \leq O(e^{-\alpha h_j} + e^{-\beta h_j}) \leq O(\sqrt{\delta})$ holds for $z \in \Omega_j^b$ and $|P(z; h)| \leq O(\sqrt{\delta})$ holds for $z \in \Omega_j^b$. Hence, $L(h)$ is expressed by $L(h) = \tilde{L} + \tilde{B}$, where $\tilde{B} = \tilde{B}(x) = F'(P(z; h)) - F'(0)$ with the estimate $|\tilde{B}(x)| \leq O(\sqrt{\delta})$ on $\Omega_j^b$.

On the other hand, $L_k$ are also represented by

$$L_k = \tilde{L} + \tilde{B}_k$$

with $|\tilde{B}_k(x)| \leq O(\sqrt{\delta})$ on $\Omega_j^b$ and therefore

$$(\lambda - \tilde{L})u_k - \tilde{B}_k u_k = f$$

hold for $k = j - 1, j$, where $\tilde{B}_k = \tilde{B}_k(x) = F'(P(z - z_k))$. Since all the spectra of $\tilde{L}$ is in the left hand side of imaginary axis by the assumption H1, we may assume $(\lambda - \tilde{L})$ is invertible in $X$ for $\lambda \in C$ with $\Re \lambda > -\rho_0$. Let $\tilde{u} = (\lambda - \tilde{L})^{-1} f$. Since $u_k$ $(k = j - 1, j)$ satisfy (5.2) on $\Omega_j^b$ with small $O(\sqrt{\delta})$ perturbations $\tilde{B}_k$ and $u_k$ are bounded on $\Omega_j^b$, we have

$$(5.3) \quad \|u_k - \tilde{u}\|_{H^1(\Omega_j^b)} \leq O(\sqrt{\delta}) (\|\tilde{u}\| + \|u_k\| + \|\tilde{u}\|).$$

Let $g_k = u_k - \tilde{u}$. By (5.2), (5.3), it follows on $\Omega_j^b$ that

$$(5.4) \quad (\lambda - L(h)) \tilde{u} = (\lambda - \tilde{L} - \tilde{B}) \tilde{u} = f + R_{j-1} u_{j-1} + R_{j} u_{j} + \tilde{R} \tilde{u} + \tilde{R} \tilde{u}$$

for certain bounded operators $R_{j-1}$, $R_{j}$ and $\tilde{R}$ in $L^2(\Omega_j^b)$ with the norms estimated at $O(\sqrt{\delta})$.

(5.1) and (5.4) show that

$$(5.5) \quad (\lambda - L(h)) \tilde{u} = f + R_{j-1} u_{j-1} + R_{j} u_{j} + \tilde{R} \tilde{u} + \tilde{R} \tilde{u}$$

holds in $X$. Here, we indentify all operators such as $R_{j}$ defined on $L^2(\Omega_j^b)$ as bounded operators extended on $X$ with the norms $O(\sqrt{\delta})$. Since $||(\lambda - L_k)^{-1}|| \leq \frac{C}{|\lambda|}$ and $||(\lambda - \tilde{L})^{-1}|| \leq \frac{C}{|\lambda|}$ hold for $\lambda \in \rho(L)$ and a constant $C > 0$, it follows that

$$(5.6) \quad ||R_{j}(\lambda - L_k)^{-1}||, \ |R_{j}(\lambda - L_k)^{-1}||, \ |\tilde{R}D(\lambda)|| \leq \frac{C_5 \sqrt{\delta}}{|\lambda|}$$

for a constant $C_5 > 0$, which means $Id + R_{j-1}(\lambda - L_{j-1})^{-1} + R_{j}(\lambda - L_{j})^{-1} + \tilde{R}(\lambda - \tilde{L})^{-1} + \tilde{R}D(\lambda)$ is invertible and bounded if $|\lambda| > 8C_5 \sqrt{\delta}$. This yields that $(\lambda - L(h))$ is invertible and the inverse is given by

$$(5.7) \quad (\lambda - L(h))^{-1} = D(\lambda) \{Id + R_{j-1}(\lambda - L_{j-1})^{-1} + R_{j}(\lambda - L_{j})^{-1} + \tilde{R}(\lambda - \tilde{L})^{-1} + \tilde{R}D(\lambda)\}^{-1}$$

for $\lambda \in \rho(L)$ with $|\lambda| > 8C_5 \sqrt{\delta}$ and $\Re \lambda > -\rho_4$ for a constant $\rho_4 > 0$.

Let $\Gamma$ be a closed circle surrounding $\Sigma_1$ in the region $\{\lambda \in C; \ Re \lambda > -\rho_4\}$. Then, the projection $Q = Q(h)$ from $X$ to $E(h)$ is given by

$$Q(h) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - L(h))^{-1} d\lambda,$$
where \( i = \sqrt{-1} \) and \( E(h) \) is the eigenspace corresponding to the spectral set \( \Sigma_1 \). Now, we may take \( \Gamma = \{ \lambda \in C; |\lambda| = \rho_5 \} \) for a constant \( \rho_5 > 0 \).

Let \( Q_j \) be the projection from \( X \) to \( \text{Ker}L_j = \text{span}\{P_z(z - z_j)\} \), that is,

\[
Q_j u = \langle u, \phi^*(\cdot - z_j) \rangle_{L^2} P_z(\cdot - z_j),
\]

and define \( \tilde{Q} = \sum_{j=0}^{N} Q_j \). Let \( \tilde{Q} \) be the projection operator from \( X \) to \( \text{span}\{P_z(z - z_j); j = 0, 1, \cdots, N\} \) such that \( \tilde{Q} = \tilde{Q} + o(1) \) as \( m \rightarrow \infty \). In fact, we can easily construct such a projection \( \tilde{Q} \) satisfying \( \tilde{Q} = \tilde{Q} + O(\sqrt{\delta}) \).

**Lemma 5.2.**

\[
\|Q(h) - \tilde{Q}\| = O(\sqrt{\delta})
\]

holds.

**Proof.** On \( \Gamma \), \( \|R_{j-1}(\lambda - L_{j-1})^{-1} + R_j(\lambda - L_j)^{-1} + \tilde{R}(\lambda - \tilde{L})^{-1} + \tilde{R}D(\lambda)\| \leq \frac{4C_5 \sqrt{\delta}}{|\lambda|} \leq \frac{4C_5 \sqrt{\delta}}{\rho_5} \leq O(\sqrt{\delta}) \)

holds by (5.6). Hence, \( \{Id + R_{j-1}(\lambda - L_{j-1})^{-1} + R_j(\lambda - L_j)^{-1} + \tilde{R}(\lambda - \tilde{L})^{-1} + \tilde{R}D(\lambda)\}^{-1} \) is expanded as

\[
\{Id + R_{j-1}(\lambda - L_{j-1})^{-1} + R_j(\lambda - L_j)^{-1} + \tilde{R}(\lambda - \tilde{L})^{-1} + \tilde{R}D(\lambda)\}^{-1} = Id + G
\]

with \( \|G\| \leq O(\sqrt{\delta}) \), and from (5.7)

\[
(\lambda - L(h))^{-1} = D(\lambda)(Id + G)
\]

holds for \( \lambda \in \Gamma \).

Let \( f \in C^\infty_0(\Omega_j^\theta) \). Since \( D(\lambda) = (\lambda - L_j)^{-1} \) on \( \Omega_j^\theta \) and \( Q_j = \frac{1}{2\pi i} \int_\Gamma (\lambda - L_j)^{-1}d\lambda \), we have

\[
\langle Q(h)u, f \rangle_{L^2} = \frac{1}{2\pi i} \int_\Gamma \langle (\lambda - L_j)^{-1}(Id + G)u, f \rangle_{L^2}
\]

\[
= \langle Q_j u + G_j u, f \rangle_{L^2},
\]

where \( G_j = \frac{1}{2\pi i} \int_\Gamma (\lambda - L_j)^{-1}Gd\lambda = O(\sqrt{\delta}) \). Since \( f \) is arbitrary,

\[
Q(h) = Q_j + G_j
\]

holds in \( L^2(\Omega_j^\theta) \).

Next, consider \( Q(h) \) in \( L^2(\Omega_j^\theta) \). Let \( f \in C^\infty_0(\Omega_j^\theta) \). Then, it follows

\[
\langle Q(h)u, f \rangle_{L^2} = \frac{1}{2\pi i} \int_\Gamma \langle D(\lambda)(Id + G)u, f \rangle_{L^2}
\]

\[
= \frac{1}{2\pi i} \int_\Gamma \sum_{j=0}^{N} \langle \lambda_j(\lambda - L_j)^{-1}(Id + G)u, f \rangle_{L^2} d\lambda
\]

\[
= \frac{1}{2\pi i} \int_\Gamma \sum_{j=0}^{N} \langle \lambda_j(\lambda - L_j)^{-1}(Id + G)u, \chi_j f \rangle_{L^2} d\lambda
\]

\[
= \sum_{j=0}^{N} \langle (Q_j + G_j)u, \chi_j f \rangle_{L^2}
\]

\[
= \langle \sum_{j=0}^{N} \chi_j(Q_j + G_j)u, f \rangle_{L^2}.
\]

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This shows
\[ Q(h) = \sum_{j=0}^{N} \chi_j(Q_j + G_j) \text{ on } \Omega_j^i \]
and specially, shows
\[ Q(h) = O(\sqrt{\delta}) \]
in \( L^2(\Omega_j^i) \).
\( \tilde{Q} \) satisfies the same estimates as (5.9) and (5.10), so that the lemma is proved [1]

Lemma 5.2 implies that the spaces \( E(h) = Q(h)X \) and \( \tilde{Q}X = \text{span}\{P_z(z - z_j); \ j = 0, \cdots, N\} \) are homeomorphic each other and hence \( \text{dim} \ E(h) = N + 1 \) holds.
Let a projection \( R(h) \) be \( \text{Id} - Q(h) \) and \( E^+(h) = R(h)X \).

**LEMMA 5.3.**
\[ \| L(h)^{-1}|_{E^+(h)} \| \leq C \]
holds uniformly for \( h \) with sufficiently large \( \min h \).

**Proof.** Let \( f(\lambda) = \langle (\lambda - L(h))^{-1}u, v \rangle_{L^2} \) for \( u \in E^+(h) \) and \( v \in X \). Then, \( f(\lambda) \) is holomorphic function of \( \lambda \in D(\Gamma) = \{ \lambda \in C; \ |\lambda| < p_0 \} \) because \( L(h) \) is invertible on the space \( E^+(h) \). Since \( f(\lambda) \) has an estimate
\[ |f(\lambda)| \leq \frac{C_7}{|\lambda|} \| u \| \cdot \| v \| \]
by (5.7) for a constant \( C_7 > 0 \), \( |f(\lambda)| \leq \frac{C_7}{\rho_5} \| u \| \cdot \| v \| \) holds on \( \Gamma \). Therefore, the maximum principle for the holomorphic functions yields
\[ |f(\lambda)| \leq \frac{C_7}{\rho_5} \| u \| \cdot \| v \| \]
for any \( \lambda \in D(\Gamma) \) and specially for \( \lambda = 0 \), which give the proof. [1]

Since \( L(h)P_z(z - z_j) = O(\delta) \) hold, we have \( L(h)R(h)P_z(z - z_j) = O(\delta) \). Lemma 5.3 yields
\[ R(h)P_z(z - z_j) = O(\delta). \]
Let \( \psi_j = Q(h)P_z(z - z_j) \). Then, \( \psi_j \in E(h) \) and we see
\[ \psi_j = P_z(z - z_j) - R(h)P_z(z - z_j) = P_z(z - z_j) + O(\delta) \]
by (5.11). Since \( \psi_0, \cdots, \psi_N \) are obviously linearly independent and \( \text{dim} E(h) = N + 1 \), these give the basis of \( E(h) \), that is,
\[ E(h) = \text{span}\{\psi_0, \cdots, \psi_N\}. \]

Thus, we know \( \Sigma_1 \subset \{ \lambda \in C; |\lambda| \leq C\delta \} \) for a constant \( C > 0 \) because \( Q(h)\psi_j = O(\delta) \).

For the adjoint operator \( L^+(h) \), quite a similar properties to \( L(h) \) hold. That is, there exist \( \{\phi_0^*, \cdots, \phi_N^*\} \) such that \( \phi^* = \phi^*(z - z_j) + O(\delta) \) and \( E^+(h) = \text{span}\{\phi_0^*, \cdots, \phi_N^*\} \). Since
\[ \langle \psi_j, \phi_k^* \rangle_{L^2} = \begin{cases} 1 + O(\delta) & (j = k), \\ O(\delta) & (j \neq k), \end{cases} \]
we can easily construct \( \{\phi_0, \cdots, \phi_N\} \) in Propotitions 4.1 and 4.2 by slightly modifying \( \{\psi_0, \cdots, \psi_N\} \).
In this section, we will state several extensions of the results of this paper.

First, we would like to mention that the method of proofs developed in this paper can be easily extended to higher dimensional problems. In fact, we will show the repulsiveness of the spike solutions for the Gierer-Meinhardt models on two dimensional space in the forthcoming paper [14].

On the other hand, the idea based on pulse interactions contributes powerfully to analyze various transient behaviors. For example, we will see in the forthcoming papers [10], [11] that the approach of weakly interacting pulses stated in this paper is very useful to analyze the self-replicating behaviors appearing in the Gray-Scott model and/or the reflection of traveling pulses in some reaction-diffusion systems.

Finally, we emphasize that our problem (1.3) in this paper is easily extended to the problem with small perturbations like
\[
\epsilon u_t = Du_{xx} + F(u) + \epsilon g(u, u_x)
\]
for small \( \epsilon \). The problem of this type includes many important problems such as the bifurcation problems of homoclinic and/or heteroclinic orbits. Let us only consider the problems related to heteroclinic orbits here. Suppose the equation (6.1) with \( \epsilon = 0 \) has a stable 1-front solution, say \( P(x) \), which satisfies \( \theta = 0 \) and \( P(x) \to e^{-\alpha |x|}u^\pm + P^\pm \) as in Section 2.2. \( P(x) \) is a heteroclinic solution for the unperturbed equation. Then, the solution \( u(t, x) \) of (6.1) remains close to \( P(x - l) \) and
\[
\dot{P} = -\epsilon \left( g(P(x - l), P_x(x - l)), \phi^*(x - l) \right)_{L^2} + O(\epsilon^2)
\]
holds for the constant \( C = \left( g(P(x), P_x(x)), \phi^*(x) \right)_{L^2} \) since \( g \) includes no space variable \( x \). This is easily proved in quite a similar way to this paper. Thus, we can know what kind of traveling front bifurcates by the perturbation \( \epsilon g \).

The bifurcation of homoclinic solutions is also dealt with similarly. Let us consider the solution (6.1) with the initial function \( u_0(x) \) close to \( P(x - l_0) + P(-x + l_0 + h_0) - P^- \) for sufficiently large \( h_0 \). Then we can show that the solution \( u(t, x) \) remains close to \( P(x - l(t)) + P(-x + l(t) + h(t)) \) and
\[
\dot{l} = -M^+ e^{-\alpha h} - \epsilon C + O(\delta^2 + \epsilon^2),
\]
\[
\dot{h} = 2M^+ e^{-\alpha h} + 2\epsilon C + O(\delta^2 + \epsilon^2)
\]
hold, where \( M^+ \) is the constant stated in Theorem 2.5. Then, we can know directly from (6.3) that when \( M^+ > 0(\leq 0) \) and \( C < 0(\geq 0) \), (6.3) has one (un)stable equilibrium, which means the existence of (un)stable pulse solution corresponding to the homoclinic orbit with respect to \( P^- \). Thus, we find the stability of bifurcating traveling pulse solution crucially depends on the repulsiveness \( (M^+ > 0) \) or attractivity \( (M^+ < 0) \) of pulses.

We have found that the approach of weakly interacting pulses stated in this paper is very effective also to the study of bifurcation problems for homoclinic orbits in ODEs. In this field, there have been a lot of important works from the dynamical system point of view ([23] and see the references). For this research, the relation between the theories for dynamical systems in ODE and the method in this paper is left as an important problem to be clarified.

REFERENCES

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