On Game Interpretations for the Curvature Flow Equation and Its Boundary Problems

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Recently, there comes an interesting idea of interpreting the solution of a parabolic PDE as the value function of a deterministic two-person game. More precisely, a family of discrete-time, two-person games are constructed so that their value functions converge to the solution. This result is unusual since the value function of a deterministic control problem is supposed to be the solution of a first-order Hamilton-Jacobi equation.

We review this game-theoretic approach and adapt it to boundary problems. Our attention is mainly focused on the Neumann boundary problem of the two-dimensional curvature flow equation:

\begin{align}
\partial_t u - |\nabla u| \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) &= 0 \quad \text{in } \Omega \times (0, T), \\
\langle \nabla u(x, t), \nu(x) \rangle &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(x, T) &= u_0(x) \quad \text{in } \overline{\Omega},
\end{align}

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^2 \) and \( \nu(x) \) is the unit exterior normal to \( \partial \Omega \) at \( x \). The equation (1a) asserts that level sets of \( u \) are evolving by their curvature in \( \Omega \) while the boundary condition (1b) says each of them intersects perpendicularly with the boundary \( \partial \Omega \). To realize the boundary condition in the game setting, we utilize the planar billiard reflection on the boundary with proper modification. A comparison is given between our billiard semiflow and the solution of Skorokhod problem. The latter is known to connect with continuous time games for the Neumann boundary problems of first-order Hamilton-Jacobi equations.

We also consider more general oblique boundary conditions by devising an oblique billiard law. An example is given for the half plane case.