WEAKLY COUPLED SYSTEMS OF
THE INFINITY LAPLACE EQUATIONS

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1. Introduction

The talk is based on the results in [14] of the recent collaboration with Hung V. Tran. We consider the Dirichlet problem for the weakly coupled systems of the infinity Laplace equations:

\[
\begin{align*}
-\Delta_\infty u_i + \sum_{j=1}^{m} c_{ij}(u_i - u_j) &= 0 \quad \text{in } U \text{ for } i = 1, \ldots, m \\
u_i &= g_i \quad \text{on } \partial U \text{ for } i = 1, \ldots, m,
\end{align*}
\]

where \( U \) is a bounded domain with a smooth boundary in \( \mathbb{R}^n \), and \((c_{ij})_{i,j=1}^{m}\) is a given constant matrix which describes the generator of an irreducible continuous-time Markov chain with \( m \) states satisfying

\[c_{ij} > 0 \text{ for } i \neq j, \quad \text{and } \sum_{j=1}^{m} c_{ij} = 0,\]

and \( g_i \in C(\partial U) \) are given functions for \( i = 1, \ldots, m \). Here \( u_i \) are unknown functions and the operator \( \Delta_\infty \) is the so-called game infinity Laplacian, i.e., for a smooth function \( f \),

\[
\Delta_\infty f := \frac{\text{tr}(Df \otimes Df D^2f)}{|Df|^2} = \sum_{i,j=1}^{n} f_{x_i} f_{x_j} f_{x_i x_j} |Df|^2.
\]

The study of the infinity Laplacian began with pioneer works by Aronsson [2, 3] to understand a so-called absolutely minimizing Lipschitz function. More precisely, the equation arises in the \( L^\infty \) calculus of variations as the Euler–Lagrange equation for properly interpreted minimizers of all of energy functionals \( u \mapsto \|Du\|_{L^\infty(V)} \) for all open sets \( V \subseteq U \). Aronsson achieved existence results and pointed out that we cannot expect the classical solutions in general. However, he could not prove uniqueness and stability results. It turned out that the theory of viscosity solution is an appropriate instrument for the study of infinity Laplacian. Jensen [11] gave fundamental results on the comparison principle and hence uniqueness of the single infinity Laplace equation in the viscosity solution sense, and generated considerable interest in the theory. Nowadays, there are a great number of works related to the infinity Laplace equation.

In the talk, I present (i) Derivation, (ii) Characterization of solutions by comparison with “generalized cones” for systems. If time permitted, I want to discuss the application of comparison with “generalized cones”, which is a property of blow up limits.
2. Derivation

Peres, Schramm, Sheffield, and Wilson [15] showed that the infinity Laplace equation arises in the study of certain two-player, zero-sum stochastic games. They introduced a random-turn game called $\varepsilon$-tug-of-war, in which two players try to move a token in an open set $U$ toward a favorable spot on the boundary $\partial U$ corresponding to a given payoff function $g$ on $\partial U$. Inspired by this work, we derive the system of the infinity Laplace equation (1.1).

Let $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, which is the place where the game is played by two persons, player I and player II. Suppose that there are $m$ modes: mode 1, ..., mode $m$, and $m$ corresponding the number of given functions $g_i \in C(\partial U)$ for $i = 1, \ldots, m$. We call $g_i$ the payoff function on the boundary of $U$ corresponding to mode $i$ for $1 \leq i \leq m$. We consider the following two-player, zero-sum game.

Fix a number $\varepsilon > 0$, a token $x_0 := x \in U$, and a mode $m_0 := i \in \{1, \ldots, m\}$. Suppose that both players start the game at position $x_0 = x$ and mode $m_0 = i$, and have the same position and mode all the time. At each time step $t_k := \varepsilon^2 k$ for $k \in \mathbb{N}$, the players toss a fair coin and the winner of the toss is allowed to choose a next token $x_k \in \overline{B}(x_{k-1}, \varepsilon) \cap U$, and the mode is switched from $m_{k-1}$ to mode $m_k = j$ for any $j \in \{1, \ldots, m\}$ with the probability which is determined by a piecewise-deterministic Markov process introduced by Davis [8]. The change from modes to modes with the starting point $m_0 = i$ is determined by a continuous-time Markov chain on $[0, \infty)$: $\nu(0) = i$, and for $\Delta s > 0$,

$$\mathbb{P}(\nu(s + \Delta s) = j \mid \nu(s) = i) = \frac{c_{ij}}{2} \Delta s + o(\Delta s) \quad \text{as } \Delta s \to 0 \text{ for } i \neq j,$$

(2.1)

where $o : [0, \infty) \to [0, \infty)$ is a function satisfying $o(r)/r \to 0$ as $r \to 0$. After $k$ steps, if $x_k \in U$ then the game moves to step $k + 1$. Otherwise, if $x_k \in \partial U$ then the game ends and player II pays the payoff $g_{m_k}(x_k)$ to player I as they are at mode $m_k = \nu(t_k)$. Notice that the change of modes is determined solely by the Markovian chain (2.1), and is not determined by the two players. In particular, $\nu(t_k)$ can take any value in $\{1, \ldots, m\}$ with probability determined by (2.1). The expected payoff is

$$\mathbb{E}_i [g_{\nu(t_k)}(x_k)].$$

A strategy for a player is a way of choosing the players’ next move as a function of all previous information (played moves, all known coin tosses and known states.) It is a map from the set of partially played games to moves (or in the case of a random strategy, a probability distribution on moves.) Usually, one would think of a good strategy as being Markovian, i.e., as a map from the current state to the next move. However, in some settings, it is also useful to allow more general strategies that take into account the history.

We consider the value which the players get. Of course player I wants to maximize the expected payoff, while player II wants to minimize it in this tug-of-war game. Let $S_I$ and $S_{II}$ be the strategies of player I and player II, respectively, and then we define the cost functions by

$$J^\varepsilon_i (S_I, S_{II})(x) := \begin{cases} 
\mathbb{E}_{S_I, S_{II}} \mathbb{E}_i [g_{\nu(t_k)}(x_k)] & \text{if the game terminates with probability one}, \\
-\infty & \text{otherwise,}
\end{cases}$$

where $x$ and $i$ are the starting point and mode of the game. The value of the game for player I is then defined as

$$u^\varepsilon_i (x) := \sup_{S_I} \inf_{S_{II}} J^\varepsilon_i (S_I, S_{II})(x).$$
In the talk, I show that the limit of $u^\varepsilon_t$ as $\varepsilon \to 0$ satisfies the system (1.1).

### 3. Characterization of solutions

Henceforth we only consider the simple system with two equations and we assume $c_{12} = c_{12} = 1, c_{11} = c_{22} = -1$.

For the single infinite Laplace equation

$$ -\Delta u = 0 \quad \text{in } U, \tag{3.1} $$

Crandall, Evans and Gariepy [7] realized that comparison with cones characterizes subsolutions and supersolutions of (3.1), and nowadays it is well-known that this plays important roles in the establishment of regularity results of solutions of (3.1). See [16, 9, 10]. In this section, we derive “generalized cones” for systems and establish comparison with “generalized cones”.

We first present one way to find the class of particular solutions of (3.1), and that cones are solutions of (3.1) everywhere except the vertices. Let us find radially symmetric solution $u$ of (3.1), i.e.

$$ u(x) = \eta(|x|), $$

where $\eta : [0, \infty) \to \mathbb{R}$ is some smooth function. We calculate, for $x \neq 0$,

$$ Du(x) = \eta'(|x|) \cdot \frac{x}{|x|}, $$

$$ D^2 u(x) = \eta''(|x|) \cdot \frac{x \otimes x}{|x|^2} + \eta'(|x|) \cdot \left( I - \frac{x \otimes x}{|x|^2} \right) \frac{1}{|x|^2}. $$

Plug these into (3.1) to get that

$$ -\eta''(r) = 0, $$

which implies that $\eta(r) = ar + b$ for any $a, b \in \mathbb{R}$. From these calculations, we establish that the cones

$$ u(x) = a|x - x_0| + b \quad \text{for any } x_0 \in \mathbb{R}^n, \text{ and } a, b \in \mathbb{R} \tag{3.2} $$

are solutions of (3.1) in $U \setminus \{x_0\}$.

Following the idea above, we first find particular solutions of (1.1) in the form of cones’ like. We consider $u_i$ radially symmetric of the form

$$ u_i(x) = \eta_i(|x|), $$

where $\eta_i : [0, \infty) \to \mathbb{R}$ are smooth functions for $i = 1, 2$. Assume that $(u_1, u_2)$ is a solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$. Then $(\eta_1, \eta_2)$ satisfies

$$ \begin{cases} -\eta_1'' + \eta_1 - \eta_2 = 0 & \text{in } (0, \infty), \\ -\eta_2'' + \eta_2 - \eta_1 = 0 & \text{in } (0, \infty). \end{cases} \tag{3.3} $$

Solving this system of ordinary differential equations with arbitrary initial data at 0, we get that, for $s > 0$,

$$ \begin{cases} \eta_1(s) = C_1 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}} + as + b, \\ \eta_2(s) = -C_1 e^{\sqrt{s}} - C_2 e^{-\sqrt{s}} + as + b, \end{cases} $$

where $C_1, C_2, a, b$ are arbitrary constants.

We can then easily check that the pair $(\psi_1, \psi_2)$ defined by

$$ \psi_1(x) := C_1 e^{\sqrt{s} |x - x_0|} + C_2 e^{-\sqrt{s} |x - x_0|} + a|x - x_0| + b, $$

$$ \psi_2(x) := -C_1 e^{\sqrt{s} |x - x_0|} - C_2 e^{-\sqrt{s} |x - x_0|} + a|x - x_0| + b, \tag{3.4} $$
is a solution of (1.1) in $\mathbb{R}^n \setminus \{x_0\}$ for any $x_0 \in \mathbb{R}^n, C_1, C_2, a, b \in \mathbb{R}$. We call $(\psi_1, \psi_2)$ a pair of “generalized cones”.

We introduce the notion of comparison with “generalized cones” following the single case.

**Definition 1 (Comparison with “Generalized Cones”)**. (i) A pair $(u_1, u_2) \in C(\overline{U})^2$ enjoys comparison with “generalized cones” from above in $U$ if $(u_1, u_2)$ satisfies that for any $x_0 \in U$ and $r > 0$ such that $\overline{B}(x_0, r) \subset U$,

$$\text{if } u_i \leq \psi_i \text{ on } \partial B(x_0, r) \cup \{x_0\} \text{ for } i = 1, 2, \text{ then } u_i \leq \psi_i \text{ on } \overline{B}(x_0, r) \text{ for } i = 1, 2,$$

for any choices of $C_1, C_2, a, b \in \mathbb{R}$.

(ii) A pair $(u_1, u_2) \in C(\overline{U})^2$ enjoys comparison with “generalized cones” from below in $U$ if $(u_1, u_2)$ satisfies that for any $x_0 \in U$ and $r > 0$ such that $\overline{B}(x_0, r) \subset U$,

$$\text{if } u_i \geq \psi_i \text{ on } \partial B(x_0, r) \cup \{x_0\} \text{ for } i = 1, 2, \text{ then } u_i \geq \psi_i \text{ on } \overline{B}(x_0, r) \text{ for } i = 1, 2,$$

for any choices of $C_1, C_2, a, b \in \mathbb{R}$.

In the case of the single equation (3.1), to characterize subsolutions by using comparison with cone, one could choose in (3.2)

$$a := \max_{|y-x_0|=r} \frac{u(y) - u(x)}{r}, \quad b := u(x_0).$$

For comparison with “generalized cones” for systems, we need to appropriately choose $C_1, C_2, a, b$ in (3.4). In order to do so, we introduce the following notations. For $x_0 \in U$, $r > 0$ such that $\overline{B}(x_0, r) \subset U$, we set

$$M_i(x_0, r) := \max_{|y-x_0|=r} u_i(y),$$

$$C_1(x_0, r) := \frac{-(u_1(x_0) - u_2(x_0))e^{-\sqrt{2}r}}{2(e^{\sqrt{2}r} - e^{-\sqrt{2}r})} + \frac{M_1(x_0, r) - M_2(x_0, r)}{2(e^{\sqrt{2}r} - e^{-\sqrt{2}r})},$$

$$C_2(x_0, r) := \frac{(u_1(x_0) - u_2(x_0))e^{\sqrt{2}r}}{2(e^{\sqrt{2}r} - e^{-\sqrt{2}r})} - \frac{M_1(x_0, r) - M_2(x_0, r)}{2(e^{\sqrt{2}r} - e^{-\sqrt{2}r})},$$

$$a(x_0) := \frac{M_1(x_0, r) + M_2(x_0, r) - (u_1(x_0) + u_2(x_0))}{2},$$

$$b(x_0) := \frac{u_1(x_0) + u_2(x_0)}{2}.$$

Here is one of the main theorems of [14]:

**Theorem 3.1 (Characterization of Subsolutions and Supersolution of (1.1)).**

Let $(u_1, u_2) \in C(\overline{U})^2$. The pair $(u_1, u_2)$ is a viscosity subsolution (resp., supersolution) of (1.1) if and only if $(u_1, u_2)$ satisfies comparison with “generalized cones” from above (resp., below).

4. **LINEARITY OF BLOW-UP LIMITS**

Take $x_0 \in U$ and $R > 0$ such that $\overline{B}(x_0, R) \subset U$. For each $r > 0$ sufficiently small, set

$$v_i^r(x) = \frac{u_i(x_0 + rx) - u_i(x_0)}{r}, \quad \text{for } |x| \leq \frac{R}{r}, \ i = 1, 2.$$

Clearly $\{v_i^r\}$ is precompact in $C(B(0, R))$. Thus for any sequence $\{r_j\}_{j \in \mathbb{N}}$ with $r_j \to 0$ as $j \to \infty$, we can pass to a subsequence if necessary and get $v_i^{r_j} \to v_i$ in $\text{Lip}(\mathbb{R}^n)$ locally uniformly in $\mathbb{R}^n$ as $j \to \infty$. We call $v_i$ a blow-up limit of $u_i$. We now prove that all of
blow-up limits $v_i$ are affine. Notice that $(v_1, v_2)$ here really depends on the subsequence we take. In general, a pair $(v_1, v_2)$ of blow-up limits depends on the choice of subsequences and it might not be unique.

Let us recall the literature on regularity results for the single infinity Laplace equation here. Note first that in all of these papers, the result on affine blow-up limits [7] plays an important role. Savin [16] showed that this blow-up limit is unique and achieved $C^1$ regularity for solutions in case $n = 2$. Evans and Savin [9] then established $C^{1, \alpha}$ regularity for solutions in this setting. The proofs in [16, 9] depend highly on the geometry of the 2-dimensional space and cannot be extended to the case with $n \geq 3$. Recently, Evans and Smart [10] used the nonlinear adjoint method to prove that this blow-up limit is unique, which yields the differentiability everywhere of solutions for all $n \geq 2$. The questions on $C^1$ and $C^{1, \alpha}$ regularity, however, are still open for $n \geq 3$.

Here is another main result of [14].

**Theorem 4.1.** All of blow up limits of $u_i$ are affine for $i = 1, 2$.

**References**


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