We consider $\alpha$-Gauss curvature flow with flat sides, which is given by the flow

$$\frac{\partial X}{\partial t}(x, t) = -K^\alpha(x, t)\nu(x, t)$$

where $\nu$ denotes the unit outward normal vector and $1/2 < \alpha \leq 1$. This flow is related to the deformation of 2-dimensional compact convex surfaces in $\mathbb{R}^3$ moving with collision from any random angle.

Let $\Sigma_0$ be a compact convex initial surface and $\alpha > 0$. Then there exists viscosity solution $\Sigma_t$ which has uniform Lipschitz bound for $0 < t < T_0$, [2]. For $\frac{1}{2} < \alpha \leq 1$, $\Sigma_t$ has a uniform $C^{1,1}$-estimate for $0 < t < T_0$, [2, 12]. The $C^\infty_0$-regularity of the strictly convex part of the surface and the smoothness of the interface between the strictly convex part and flat side have been studied for $\alpha = 1$ in [10]. For $n$-dimensional compact convex hypersurfaces and $\alpha \leq \frac{1}{n}$, it becomes more singular and the solution gets regular instantaneously. On the other hand, if $\alpha > \frac{1}{n}$, it becomes degenerate and the flat side of the hypersurface persists for a moment, [2, 4].

We assume that the initial surface $\Sigma_0$ has only one flat side, namely that at $t$ we have $\Sigma_t = \Sigma^1_t \cup \Sigma^2_t$ where $\Sigma^1_t$ is the flat side and $\Sigma^2_t$ is strictly convex part of $\Sigma_t$. The intersection between two regions is the free boundary $\Gamma_t = \Sigma^1_t \cap \Sigma^2_t$. Then the lower part of the surface $\Sigma_0$ can be written as a graph $z = f(x)$ and we can also write the lower part of $\Sigma_t$ as $z = f(x, t)$ for $x \in \Omega \subset \mathbb{R}^2$ where $\Omega$ is an open subset of $\mathbb{R}^2$.

In this talk, we prove that there exists smooth solution if $\Sigma_0$ is smooth and strictly convex and that there is $C^{1,1}$-viscosity solution before the collapsing time if $\Sigma_0$ is only convex. Furthermore, we show that $\Sigma^1_t$ will stay for a while. We also discuss $\Gamma_t$ remains smooth on $0 < t < T_0$ under the following conditions for the function $f$, where $T_0$ is the vanishing time of $\Sigma^1_t$.

**Condition 1.** Set $\Lambda(f) = \{f = 0\}$ and $\Gamma(f) = \partial \Lambda(f)$. 

\[\text{Department of Mathematics, Hokkaido University.}\]
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(I) (Nondegeneracy Condition)  
The function $f$ vanishes of the order $d_{st}(X, \Lambda(f))^\frac{\alpha-1}{2}$ and the interface $\Gamma(f)$ is strictly convex so that $\Gamma(f)$ moves with finite nondegenerate speed. Setting $g = (\beta f)^{\frac{1}{\beta}}$, we assume that at time $t = 0$ the function $g$ satisfies the following nondegeneracy condition: at $t = 0$,

$$0 < \lambda < |Dg(X)| < \frac{1}{\lambda} \quad \text{and} \quad 0 < \lambda^2 < D^2_{st}g(X) < \frac{1}{\lambda^2}$$

(2)

for all $X \in \Gamma_0$ and some positive number $\lambda > 0$, where $D^2_{st}$ denotes the second order tangential derivative at $\Gamma$. Then the initial speed of free boundary has the speed, at $t = 0$,

$$0 < \lambda^{\alpha-1} < |\gamma_t| < \frac{1}{\lambda^{4\alpha-1}}.$$  

(3)

(II) (Before Focusing of Flat Side)

Let $T$ be any number on $0 < T < T_0$, so that $\Sigma^1_t$ is non-zero. Since the area is non-zero, $\Sigma^1_t$ contains a disc $D_{\rho_0}$ for some $\rho_0 > 0$. We assume that

$$D_{\rho_0} = \{X \in \mathbb{R}^2 : |X| \leq \rho_0\} \subset \Sigma^1_t \quad \text{for} \ 0 \leq t \leq T_0.$$  

(4)

(III) (Graph on a Neighborhood of the Flat Side)

Without loss of generality, we assume that

$$\max_{x \in \Omega(t)} f(x, t) \geq 2, \quad 0 \leq t \leq T_0.$$  

(5)

where $\Omega(t) = \{X = (x, y) \in \mathbb{R}^2 : |Df(X, t)| < \infty\}$. Set

$$\Omega^p(t) = \{x \in \mathbb{R}^2 : f(x, y, t) \leq f(P)\}.$$  

(6)

The following is the first our main result. Let us assume $\frac{1}{2} < \alpha \leq 1$.

**Theorem 2.** For a compact convex initial surface $\Sigma_0$, any viscosity solution $\Sigma_t$ of (1) is $C^{1,1}$ for $0 < t < T_0$. Furthermore, $\Sigma^2_t$ is smooth for $0 < t < T_0$.

In [8], authors proved the following short time existence of $C^{\infty}_s$-solution with a flat side. From the conditions (2), our linearized equation is in the same class of operators considered in [8]. Hence their Schauder theory can apply to our linearized equation and then we get the short time existence by the application of implicit function theorem as in [8].
Theorem 3. [Short Time Regularity] [8] Let us assume that $g = (\beta f)^{\frac{1}{2}}$ is of class $C^{2+\gamma}$ up to the interface $z = 0$ at time $t = 0$, for some $0 < \gamma < 1$, and satisfies Conditions 1 for $f$. Then there exists a time $T > 0$ such that the equation (1) admits a solution $\Sigma(t)$ on $0 \leq t \leq T$. Moreover, the function $g = (\beta f)^{\frac{1}{2}}$ is smooth up to the interface $z = 0$ on $0 < t \leq T$. In particular, the interface $\Gamma(t)$ will be a smooth curve for all $t$ in $0 < t \leq T$.

Then we have the long time regularity of the solution.

Theorem 4. [Long Time Regularity] Under the assumptions of Theorem 3, $g = (\beta f)^{\frac{1}{2}}$ remains smooth up to the interface $z = 0$ on $0 < t < T$ for all $T < T_0$. In addition, the interface $\Gamma_t$ will be smooth curve for all $t$ in $0 < t < T_0$.

References


