Singular perturbation problems for nonlinear elliptic equations in degenerate settings

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0. Introduction

Singular perturbation problems for nonlinear elliptic equations has been studied by many mathematicians. Especially the following singularly perturbed nonlinear Schrödinger equations is well studied since the pioneering work of Floer-Weinstein [FW]:

\[
\begin{cases}
-\varepsilon^2 \Delta u + V(x)u = g(u) & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\] (0.1)_\varepsilon

Here \( N \geq 1 \), \( g(s) \in C(\mathbb{R}, \mathbb{R}) \) is a function with a subcritical growth, \( V(x) \in C(\mathbb{R}^N, \mathbb{R}) \) is a positive function and \( 0 < \varepsilon \ll 1 \). Among solutions of (0.1)_\varepsilon, we are interested in concentrating families \( (u_\varepsilon) \) of solutions, which have the following behavior:

(i) \( u_\varepsilon(x) \) has a local maximum at \( x_\varepsilon \in \mathbb{R}^N \) and \( x_\varepsilon \) converges to some \( x_0 \in \mathbb{R}^N \) as \( \varepsilon \to 0 \).

(ii) rescaled function \( v_\varepsilon(y) = u_\varepsilon(\varepsilon y + x_\varepsilon) \) converges as \( \varepsilon \to 0 \) to a solution \( \omega(y) \in H^1(\mathbb{R}^N) \) of the limit equation:

\[
-\Delta \omega + V(x_0)\omega = g(\omega), \ \omega > 0 \ \text{in } \mathbb{R}^N, \ \omega \in H^1(\mathbb{R}^N).
\] (0.2)

The limit equation (0.2) plays important roles in the study of (0.1)_\varepsilon. When solutions of (0.2) are unique up to translation and non-degenerate, we can use Lyapunov-Schmidt reduction method and we can reduce (0.1)_\varepsilon to a finite dimensional problem and interesting family of solutions with multiple concentrating points are constructed. See [ABC, DKW, KW, O1, O2, W] and references therein. However, uniqueness and non-degeneracy of solutions of (0.2) is verified only restricted classes of nonlinearities including \( g(u) = u^p \) \((1 < p < \frac{N+2}{N-2})\).
There are a lot of efforts to relax the non-degenerate condition using variational methods, especially for general nonlinearities. See [BJ, BT2, BT3, DPR, DF1, DF2, DF3, G, JT] and references therein.

In this talk, I would like to talk about another class of a singular perturbation problem, in which a domain depends on singular perturbation parameter $\varepsilon \in (0,1)$:

\[
\begin{aligned}
-\Delta u &= u^p, \quad u > 0 \quad \text{in } \Omega_\varepsilon, \\
      u &= 0 \quad \text{on } \partial \Omega_\varepsilon.
\end{aligned}
\]  

Here $1 < p < \frac{N+2}{N-2}$ ($N \geq 3$), $1 < p < \infty$ ($N = 2$) and $\Omega_\varepsilon \subset \mathbb{R}^k \times \mathbb{R}^\ell$ ($N = k + \ell$) is given by

\[\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^\ell; (\varepsilon x, y) \in \Omega_1\} = \bigcup_{x \in \mathbb{R}^k} \{\{x\} \times D_{\varepsilon x}\}.\]

Here

\[\Omega_1 = \bigcup_{z \in \mathbb{R}^k} \{\{z\} \times D_z\}\]

and $D_z \subset \mathbb{R}^\ell$ be a family of bounded smooth domains which depends on $z \in \mathbb{R}^k$ smoothly.

Such a problem naturally appeared when we studied a problem in an expanding tubular type domain in [BT4] (c.f. [DY, ACP]). We would like to give a partial answer to the following questions:

(i) Where the peaks appear?

(ii) What happens if the section depends on the location $z \in M$.

1. Setting of our problem

1.1. Domain $\Omega_\varepsilon$

First we give a precise definition of the domain $\Omega_\varepsilon$. We assume that $\Omega_1$ satisfies the following conditions.

(\Omega1) $D \subset \mathbb{R}^\ell$ is a bounded domain with a smooth boundary $\partial D$.

(\Omega2) $\varphi(z, y) : \mathbb{R}^k \times \overline{D} \rightarrow \mathbb{R}^\ell$ is a smooth map such that

(i) For $z \in \mathbb{R}^k$, set $D_z = \varphi(z, D)$. Then

\[\varphi(z, \cdot) : \overline{D} \rightarrow \overline{D}_z \quad \text{is a diffeomorphism for each } z \in \mathbb{R}^k.\]

(ii) All derivatives of $\varphi(z, y)$ is bounded in $\mathbb{R}^k \times \overline{D}$ and there exists $C_0 > 0$ such that

\[\det \left[ \frac{\partial \varphi}{\partial y} (z, y) \right] \geq C_0 \text{ on } \mathbb{R}^k \times \overline{D}.\]
We set
\[
\Omega_1 = \bigcup_{z \in \mathbb{R}^k} \{z\} \times D_z, \\
\Omega_\varepsilon = \bigcup_{x \in \mathbb{R}^k} \{x\} \times D_{\varepsilon x} = \{(x, \varphi(\varepsilon x, y)); \ x \in \mathbb{R}^k, \ y \in D\} \text{ for } \varepsilon \in (0, 1].
\]

1.2. Variational formulation

We consider
\[
\left\{
\begin{array}{l}
-\Delta u = u^p, \ u > 0 \text{ in } \Omega_\varepsilon, \\
u = 0 \text{ on } \partial \Omega_\varepsilon.
\end{array}
\right. \tag{\ast}_\varepsilon
\]

This problem is reduced to a problem finding a critical point of
\[
u \mapsto \int_{\Omega_\varepsilon} \frac{1}{2} |\nabla u|^2 - \frac{1}{p + 1} u^{p+1} \, dx \, dy; \ H^1_0(\Omega_\varepsilon) \to \mathbb{R}.
\]

Using a transformation appeared in (\Omega1)–(\Omega3), it can be written as a functional:
\[
I_\varepsilon(u) = \int_{\mathbb{R}^k \times D} F_\varepsilon(\varepsilon x, y, \nabla u, u) \, dx \, dy \in C^1(H^1_0(\mathbb{R}^k \times D), \mathbb{R}),
\]
\[
F_\varepsilon(z, y, \nabla u, u) = \left(\frac{1}{2} |\nabla_z u + \varepsilon B(z, y) \nabla_y u|^2 + \frac{1}{2} |A(z, y) \nabla_y u|^2 \right.
\]
\[
- \frac{1}{p + 1} u^{p+1} \right) \det \left[ \frac{\partial \varphi}{\partial y}(z, y) \right],
\]

where \(A(z, y), B(z, y)\) are matrices defined using \(\varphi(z, y)\). We also set the limit functional at \(z \in \mathbb{R}^k\) by
\[
L(z, u) = \int_{\mathbb{R}^k \times D} F_0(z, y, \nabla u, u) \, dx \, dy \in C^1(H^1_0(\mathbb{R}^k \times D), \mathbb{R}).
\]

Here \(F_0(z, y, \nabla, u)\) is defined by setting \(\varepsilon = 0\) in the definition of \(F_\varepsilon(z, y, \nabla, u)\).

We note that \(L(z, u)\) plays a role of the limit functional of \(I_\varepsilon(u)\). In fact, for \(u(x, y) \in C_0^\infty(\mathbb{R}^k \times D)\) and \(z \in \mathbb{R}^k\), we have
\[
I_\varepsilon(u(x - \frac{z}{\varepsilon}, y)) \to L(z, u) \quad \text{as } \varepsilon \to 0.
\]

We also note that \(L(z, u)\) is corresponding to the following limit problem:
\[
\left\{
\begin{array}{l}
-\Delta u = u^p, \ u > 0 \text{ in } \mathbb{R}^k \times D_z, \\
u = 0 \text{ on } \mathbb{R}^k \times \partial D_z.
\end{array}
\right. \tag{\ast\ast}_z
\]
1.3. Properties of the limit problem

It is known that the limit problem has the following properties:

1° Solutions of (**) has symmetry $u(x, y) = u(|x|, y)$ after a suitable shift in $x$ and set of solutions

$$S_z = \{\omega(|x|, y); \omega \neq 0, \ D_u L(z, \omega) = 0\}$$

is compact in $H^{1}_{0,s}(\mathbb{R}^k \times D)$ for all $z \in \mathbb{R}^k$.

2° For each $z$, (**)$_z$ has a least energy solution; we denote the least energy level by $m(z)$:

$$m(z) = \inf\{L(z, \omega); \omega \in S_z\}.$$

Moreover $m(z): \mathbb{R}^k \to \mathbb{R}$ is continuous.

In general, least energy solutions are not unique.

3° $m(z)$ has a property:

$$D_z \subset D_{z'} \implies m(z') \leq m(z).$$

We refer to Gidas-Ni-Nirenberg [GNN] and Byeon [B] for the symmetry of solutions. We also refer to Esteban [E] and Byeon-Tanaka [BT1] for the existence of least energy solutions. We also note that the natural space to deal with $L(z, u)$ is the following space

$$H^{1}_{0,s}(\mathbb{R}^k \times D) = \{u(x, y) \in H^{1}_{0}(\mathbb{R}^k \times D); u(x, y) = u(|x|, y)\}$$

by the property 1°. In what follows, we regard $L(z, u)$ is a functional defined on $\mathbb{R}^k \times H^{1}_{0,s}(\mathbb{R}^k \times D)$.

2. Our results

First we recall a non-existence result due to Esteban-Lions [EL]. In the following theorem, which is a special case of [EL], we denote the outward normal vector of $\Omega_\varepsilon$ at $(x, y) \in \partial \Omega_\varepsilon$ by $N(x, y) \in \mathbb{R}^N$.

**Theorem 1 (Esteban-Lions [EL]).** If $\Omega_\varepsilon$ is monotone in one direction, that is, there is a vector $T \in \mathbb{R}^N$ satisfying

$$(N(x, y), T) > 0 \quad \text{for all } (x, y) \in \partial \Omega_\varepsilon,$$

then $(*)_\varepsilon$ does not have non-trivial solutions.

From this result, we cannot expect the existence of concentrating solution in monotone parts of $\Omega_1$.

2.1. Concentration at a thick part

First we deal with thick parts of $\Omega_1$. By the property 3°, we have the following property (2.1) in the following theorem at a thick part $O$ of $\Omega_1$. 

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Theorem 2. Suppose that a bounded open set \( O \subset \mathbb{R}^k \) satisfies
\[
\inf_{z \in O} m(z) < \inf_{z \in \partial O} m(z). \tag{2.1}
\]
Then for \( \varepsilon > 0 \) small, \((*)_\varepsilon\) has a positive solution \( u_\varepsilon(x, y) \) concentrating in \( O \). More precisely, any sequence \((\varepsilon_n)\) with \( \varepsilon_n \to 0 \) has a subsequence \((\varepsilon_{n_j})\), \((x_{n_j}) \subset \mathbb{R}^k \) and \((z_0, \omega_0) \in O \times H^1_{0,s}(\mathbb{R}^k \times D)\) such that
\[
\begin{align*}
    u_{\varepsilon_{n_j}}(x - x_{n_j}, y) &\to \omega_0(x, y) \text{ in } H^1_0(\mathbb{R}^k \times D), \\
    \varepsilon_{n_j}x_{n_j} &\to z_0 \in O.
\end{align*}
\]
Here \((z_0, \omega_0)\) is a critical point of \( L(z, u) \), i.e., \( D_z L(z_0, \omega_0) = 0 \), satisfying
\[
L(z_0, \omega_0) = m(z_0) = \inf_{z \in O} m(z).
\]

2.2. Concentration at a thin part

Next we consider thin parts of \( \Omega_1 \). Thin parts correspond to high energy solutions and we need more assumptions.

Condition \((E)\). For a bounded open set \( O \subset \mathbb{R}^k \), we say that \( O \) satisfies \((E)\) if and only if
\[
\left( \frac{\partial \varphi}{\partial z}(z, y)n(z), \nu(z, y) \right) > 0 \text{ for all } z \in \partial O \text{ and } y \in \partial D.
\]
Here \( n(z) \in \mathbb{R}^k \) \((\nu(z, y) \in \mathbb{R}^k \text{ resp.})\) is a unit outward normal vector of \( O \) \((D_z \text{ resp.})\) at \( z \in \partial O \) \((\varphi(z, y) \in \partial D_z \text{ resp.})\).

Remark. If \( O \subset \mathbb{R}^k \) satisfies \((E)\), then we have for some \( \delta_0 > 0 \)
\[
D_z \subset D_{z+tn(z)} \text{ for } z \in \partial O \text{ and } t \in [0, \delta_0].
\]

Under \((E)\) we have the following existence result.

Theorem 3. Assume that \( O \subset \mathbb{R}^k \) satisfies \((E)\). Then for \( \varepsilon > 0 \) small, \((*)_\varepsilon\) has a positive solution \( u_\varepsilon(x, y) \) concentrating in \( O \). More precisely, any sequence \((\varepsilon_n)\) with \( \varepsilon_n \to 0 \) has a subsequence \((\varepsilon_{n_j})\), \((x_{n_j}) \subset \mathbb{R}^k \) and \((z_0, \omega_0) \in O \times H^1_{0,s}(\mathbb{R}^k \times D)\) such that
\[
\begin{align*}
    u_{\varepsilon_{n_j}}(x - x_{n_j}, y) &\to \omega_0(x, y) \text{ in } H^1_0(\mathbb{R}^k \times D), \\
    \varepsilon_{n_j}x_{n_j} &\to z_0 \in O.
\end{align*}
\]
Here \((z_0, \omega_0)\) is a critical point of \( L(z, u) \).
Remark. In Theorem 3, we only have

$$L(z_0, \omega_0) \geq \max_{z \in \mathcal{O}} m(z).$$

We conjecture that the equality does not hold in general and $\omega_0$ is not a least energy solution of the limit problem. In contrast, for a singular perturbation problem for NLS, we have

$$L(z_0, \omega_0) = \max_{z \in \mathcal{O}} m(z).$$

3. Our approach

To show our Theorems 2–3, we take the following approach:

Step 1: Analysis of the limit problem.

We introduce a minimax method to a suitably modified limit functional $\tilde{L}(z, u) \in C^1(O \times H_{0, s}^1(\mathbb{R}^k \times D), \mathbb{R})$:

$$b = \inf_{\gamma \in \Gamma} \max_{(s, z) \in [0, 1] \times \mathcal{O}} \tilde{L}(\gamma(s, z)).$$

We show

$$K_b = \{(z, \omega) \in O \times H_{0, s}^1(\mathbb{R}^k \times D); L(z, \omega) = b, DL(z, \omega) = 0\}$$

is non-empty and compact in $O \times H_{0, s}^1(\mathbb{R}^k \times D)$.

Here we use a deformation argument in a manifold with a boundary, which is due to Majer [M]. We note that the condition (E) and the compactness of the sets $S_z$ enable us to introduce a modification $\tilde{L}(z, u) \in C^1(O \times H_{0, s}^1(\mathbb{R}^k \times D), \mathbb{R})$ of $L(z, u)$ such that

(i) for $(z, u) \in O \times H_{0, s}^1(\mathbb{R}^k \times D)$, $D\tilde{L}(z, u) = 0$ if and only if $DL(z, u) = 0$;

(ii) for $(z, u) \in \partial O \times H_{0, s}^1(\mathbb{R}^k \times D)$ and $\lambda \geq 0$

$$D\tilde{L}(z, u) \neq -\lambda(n(z), 0).$$

Step 2: Construction of a critical point $u_\varepsilon \in H_{0, s}^1(\mathbb{R}^k \times D)$ of $I_\varepsilon(u)$ related to $K_b$.

We try to find a family $(u_\varepsilon)$ of critical points of $I_\varepsilon(u)$ such that for some $(z_0, \omega_0) \in K_b$

$$\varepsilon X(u_\varepsilon) \rightarrow z_0,$$

$$u_\varepsilon(x - X(u_\varepsilon), y) \rightarrow \omega_0.$$

Here $X(u) : H_{0, s}^1(\mathbb{R}^k \times D) \rightarrow \mathbb{R}^k$ is a center of mass of $u \in H_{0, s}^1(\mathbb{R}^k \times D)$. To find such a family, we develop a new local deformation argument in a neighborhood of

$$A_b^{(\varepsilon)} = \{\omega(x - \frac{z}{\varepsilon}, y); (z, \omega) \in K_b\},$$
which is an extension of the arguments in \([\text{BT2, BT3}]\).

References


