Heat equation with a nonlinear boundary condition and uniformly local $L^r$ spaces

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This is joint work with my student Ryuichi Sato (Tohoku University) and it is concerned with the heat equation with a nonlinear boundary condition,

$$
\begin{aligned}
\partial_t u &= \Delta u, & x &\in \Omega, \ t > 0, \\
\nabla u \cdot \nu(x) &= |u|^{p-1}u, & x &\in \partial\Omega, \ t > 0, \\
u(x, 0) &= \varphi(x), & x &\in \Omega,
\end{aligned}
$$

(1)

where $N \geq 1$, $p > 1$, $\Omega$ is a smooth domain in $\mathbb{R}^N$, $\partial_t = \partial/\partial t$ and $\nu = \nu(x)$ is the outer unit normal vector to $\partial\Omega$. For any $\varphi \in BUC(\Omega)$, problem (1) has a unique solution $u \in C^2(\Omega \times (0, T]) \cap C^1(\overline{\Omega} \times (0, T]) \cap BUC(\overline{\Omega} \times [0, T])$ for some $T > 0$ and the maximal existence time $T(\varphi)$ of the solution can be defined. If $T(\varphi) < \infty$, then

$$
\limsup_{t \to T(\varphi)} \|u(t)\|_{L^\infty(\Omega)} = \infty
$$

and we call $T(\varphi)$ the blow-up time of the solution $u$.

Problem (1) has been studied in many papers from various points of view (see e.g. [1]–[5], [7]–[11], [12]–[17], [18], [19], [20] and references therein) while there are few results related to the dependence of the blow-up time on the initial function even in the case $\Omega = \mathbb{R}^N_+$. We remark that the blow-up time for problem (1) cannot be chosen uniform for all initial functions lying in a bounded set of $L^r(\mathbb{R}^N_+)$ with $1 \leq r \leq N(p-1)$.

For $1 \leq r < \infty$ and $\rho > 0$, let $L^r_{uloc, \rho}(\Omega)$ be the uniformly local $L^r$ space in $\Omega$ equipped with the norm

$$
\|f\|_{r, \rho} := \sup_{x \in \Omega} \left( \int_{\Omega \cap B(x, \rho)} |f(y)|^r \, dy \right)^{1/r}.
$$

We denote by $\mathcal{L}^r_{uloc, \rho}(\Omega)$ the completion of bounded uniformly continuous functions in $\Omega$ with respect to the norm $\|\cdot\|_{r, \rho}$, that is,

$$
\mathcal{L}^r_{uloc, \rho}(\Omega) := \overline{BUC(\Omega)}^{:\|\cdot\|_{r, \rho}}.
$$

We set $L^\infty_{uloc, \rho}(\Omega) = L^\infty(\Omega)$ and $\mathcal{L}^\infty_{uloc, \rho}(\Omega) = BUC(\Omega)$.

In this talk we prove the local existence and the uniqueness of the solutions of problem (1) with initial functions in $\mathcal{L}^r_{uloc, \rho}(\Omega)$, and study the dependence of the blow-up time on the initial functions. As an application of the main results of this paper, we study the asymptotic behavior of the blow-up time $T(\varphi)$ with $\varphi = \lambda \psi$ as $\lambda \to 0$ or $\lambda \to \infty$ and show the validity of our arguments. Furthermore, we obtain a lower estimate of the blow-up rate of the solutions.
Throughout this talk we assume that $\Omega \subset \mathbb{R}^N$ is a uniformly regular domain of class $C^1$. For any $x \in \mathbb{R}^N$ and $\rho > 0$, define

$$B(x, \rho) := \{ y \in \mathbb{R}^N : |x - y| < \rho \}, \quad \Omega(x, \rho) := \Omega \cap B(x, \rho), \quad \partial \Omega(x, \rho) := \partial \Omega \cap B(x, \rho).$$

By the trace inequality for $W^{1,1}(\Omega)$-functions and the Gagliardo-Nirenberg inequality we can find $\rho_* \in (0, \infty]$ with the following properties.

- There exists a positive constant $c_1$ such that

$$\int_{\partial \Omega(x, \rho)} |v| \, d\sigma \leq c_1 \int_{\Omega(x, \rho)} |\nabla v| \, dy$$

for all $v \in C^1_0(B(x, \rho))$, $x \in \Omega$ and $0 < \rho < \rho_*$.  

- Let $1 \leq \alpha, \beta < \infty$ and $\sigma \in [0, 1]$ be such that

$$\frac{1}{\alpha} = \sigma \left( \frac{1}{2} - \frac{1}{N} \right) + (1 - \sigma) \frac{1}{\beta}. \quad (3)$$

Assume, if $N \geq 2$, that $\alpha \neq \infty$ or $N \neq 2$. Then there exists a constant $c_2$ such that

$$\|v\|_{L^\alpha(\Omega(x, \rho))} \leq c_2 \|v\|_{L^\beta(\Omega(x, \rho))}^{1 - \sigma} \|\nabla v\|_{L^2(\Omega(x, \rho))}^\sigma$$

for all $v \in C^1_0(B(x, \rho))$, $x \in \Omega$ and $0 < \rho < \rho_*$.  

We remark that, in the case

$$\Omega = \{(x', x_N) \in \mathbb{R}^N : x_N > \Phi(x')\},$$

where $N \geq 2$ and $\Phi \in C^1(\mathbb{R}^{N-1})$ with $\|\nabla \Phi\|_{L^\infty(\mathbb{R}^{N-1})} < \infty$, (3) and (4) hold with $\rho_* = \infty$. Inequalities (2) and (4) are used to treat the nonlinear boundary condition.

Next we state the definition of the solution of (1).

**Definition 1** Let $0 < T \leq \infty$ and $1 \leq r < \infty$. Let $u$ be a continuous function in $\overline{\Omega} \times (0, T]$. We say that $u$ is a $L^r_{uloc}(\Omega)$-solution of (1) in $\Omega \times [0, T]$ if

- $u \in L^\infty(\tau, T : L^\infty(\Omega)) \cap L^2(\tau, T : W^{1,2}(\Omega \cap B(0, R)))$ for any $\tau \in (0, T)$ and $R > 0$,
- $u \in C([0, T) : L^r_{uloc, \rho}(\Omega))$ with $\lim_{\rho \downarrow 0} \|u(t) - \varphi\|_{r, \rho} = 0$ for some $\rho > 0$,
- $u$ satisfies

$$\int_0^T \int_{\Omega} \{-u\partial_t \phi + \nabla u \cdot \nabla \phi\} \, dyds = \int_0^T \int_{\partial \Omega} |u|^{r-1} u \phi \, d\sigma ds$$

for all $\phi \in C^\infty_0(\mathbb{R}^N \times (0, T))$.  

Here $d\sigma$ is the surface measure on $\partial \Omega$. Furthermore, for any continuous function $u$ in $\overline{\Omega} \times (0, T)$, we say that $u$ is a $L^r_{uloc}(\Omega)$-solution of (1) in $\Omega \times [0, T)$ if $u$ is a $L^r_{uloc}(\Omega)$-solution of (1) in $\Omega \times [0, \eta]$ for any $\eta \in (0, T)$.  

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Now we are ready to state the main results of this talk. Let \( p_* = 1 + 1/N \).

**Theorem 1** Let \( N \geq 1 \) and \( \Omega \subset \mathbb{R}^N \) be a uniformly regular domain of class \( C^1 \). Let \( p_* \) satisfy (2) and (4). Then, for any \( 1 \leq r < \infty \) with

\[
\begin{cases}
  r \geq N(p - 1) & \text{if } p > p_*, \\
  r > 1 & \text{if } p = p_*, \\
  r \geq 1 & \text{if } 1 < p < p_*,
\end{cases}
\]

there exists a positive constant \( \gamma_1 \) such that, for any \( \varphi \in \mathcal{L}^r_{uloc}(\Omega) \) with

\[
\rho^\frac{1}{p-1} - \frac{N}{r} \| \varphi \|_{r, \rho} \leq \gamma_1
\]

for some \( \rho \in (0, p_*/2) \), problem (1) possesses a \( \mathcal{L}^r_{uloc}(\Omega) \)-solution \( u \) in \( \Omega \times [0, \mu \rho^2] \) satisfying

\[
\sup_{0 < t < \mu \rho^2} \| u(t) \|_{r, \rho} \leq C \| \varphi \|_{r, \rho},
\]

\[
\sup_{0 < t < \mu \rho^2} t^\frac{N}{r} \| u(t) \|_{L^\infty(\Omega)} \leq C \| \varphi \|_{r, \rho}.
\]

Here \( C \) and \( \mu \) are constants depending only on \( N, \Omega, p \) and \( r \).

**Theorem 2** Assume the same conditions as in Theorem 1. Let \( v \) and \( w \) be \( \mathcal{L}^r_{uloc}(\Omega) \)-solutions of (1) in \( \Omega \times [0, T) \) such that \( v(x, 0) \leq w(x, 0) \) for almost all \( x \in \Omega \), where \( T > 0 \) and \( r \) is as in (6). Assume, if \( r = 1 \), that

\[
\limsup_{t \to 0} t^{\frac{1}{p-1}} \left[ \| v(t) \|_{L^\infty(\Omega)} + \| w(t) \|_{L^\infty(\Omega)} \right] < \infty.
\]

Then there exists a positive constant \( \gamma_2 \) such that, if

\[
\rho^\frac{1}{p-1} - \frac{N}{r} \left[ \| v(0) \|_{r, \rho} + \| w(0) \|_{r, \rho} \right] \leq \gamma_2
\]

for some \( \rho \in (0, p_*/2) \), then

\[
v(x, t) \leq w(x, t) \quad \text{in} \quad \Omega \times (0, T).
\]

We give some comments related to Theorems 1 and 2.

(i) Let \( u \) be a \( \mathcal{L}^r_{uloc}(\Omega) \)-solution of (1) in \( \Omega \times [0, T) \). It follows from Definition 1 that \( u \in L^\infty(\tau, \sigma : L^\infty(\Omega)) \) for any \( 0 < \tau < \sigma < T \). This together with Theorem 6.2 of [5] implies that \( u(t) \in BUC(\Omega) \) for any \( t \in (0, T) \). This means that \( u(0) \in \mathcal{L}^r_{uloc, \rho}(\Omega) \) for any \( \rho > 0 \).

(iii) Let \( 1 \leq r < \infty \). If, either

(a) \( f \in \mathcal{L}^r_{uloc, 1}(\Omega), \quad r > N(p - 1) \) or

(b) \( f \in \mathcal{L}^r(\Omega), \quad r \geq N(p - 1), \)

then, for any \( \gamma > 0 \), we can find a constant \( \rho > 0 \) such that \( \rho^\frac{1}{p-1} - \frac{N}{r} \| f \|_{r, \rho} \leq \gamma. \)
As a corollary of Theorem 1, we have:

**Corollary 1** Assume the same conditions as in Theorem 1 and $p > p_*$. 

(i) For any $\varphi \in L^N(p-1)(\Omega)$, problem (1) has a unique $L^N_{uloc}(\Omega)$-solution in $\Omega \times [0, T]$ for some $T > 0$.

(ii) Assume $p_* = \infty$. Then there exists a constant $\gamma$ such that, if

$$\|\varphi\|_{L^N(p-1)(\Omega)} \leq \gamma;$$

then problem (1) has a unique $L^N_{uloc}(\Omega)$-solution $u$ such that

$$\sup_{0 < t < \infty} \|u(t)\|_{L^N(p-1)(\Omega)} + \sup_{0 < t < \infty} t^{\frac{1}{2(p-1)}} \|u(t)\|_{L^\infty(\Omega)} < \infty.$$

Furthermore, as an application of our theorems, we give a lower blow-up estimate of the solution $u$ of (1).

**Corollary 2** Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a uniformly regular domain of class $C^1$. Let $u$ be a solution of (1) blowing up at $t = T < \infty$. Then

$$\liminf_{t \to T} (T - t)^{\frac{1}{2(p-1)}} \frac{N}{\pi} \|u(t)\|_{L^r(\Omega)} > 0,$$

where

$$\begin{cases} 
N(p-1) \leq r \leq \infty & \text{if } p > 1 + 1/N, \\
1 < r \leq \infty & \text{if } p = 1 + 1/N, \\
1 \leq r \leq \infty & \text{if } 1 < p < 1 + 1/N.
\end{cases}$$

**References**


