Effective nonvanishing of pluriadjoint line bundles

Tomoki Arakawa (Sophia University)

1. Introduction

Let $X$ be a smooth projective variety defined over $\mathbb{C}$ and $L$ an ample line bundle over $X$. Then the pair $(X,L)$ is called a polarized manifold.

In the classification theory of polarized manifolds, it is important to study a condition on the integer $m$ for which $|K_X + mL|$ is free. Fujita’s freeness conjecture predicts that $|K_X + mL|$ is free for any $m \geq \dim X + 1$. It is known that the above conjecture is true when $\dim X \leq 4$. In higher dimensional case, it is proved that $|K_X + mL|$ is free for every integer $m \geq \dim X (\dim X + 1)/2 + 1$ (see [1], [9]).

On the other hand when $K_X + L$ is nef, by the nonvanishing theorem due to V. Shokurov, we see that $|m(K_X + L)| \neq \emptyset$ holds for $m \gg 0$. Then it is important to find an integer $m$ with $|m(K_X + L)| \neq \emptyset$. Concerning this, Y. Kawamata ([7]) proposed the following conjecture:

**Conjecture 1.1.** Let $X$ be a normal projective variety and let $B$ be a $\mathbb{Q}$-effective divisor on $X$ such that $(X,B)$ is a KLT pair. Let $D$ be a nef Cartier divisor on $X$ such that $D - (K_X + B)$ is nef and big. Then $H^0(X, O_X(D)) \neq 0$ holds.

When $X$ is smooth, $B = 0$ and $D := K_X + L$ is nef, this implies that $|K_X + L| \neq \emptyset$ holds for any polarized manifold $(X,L)$ with $K_X + L$ nef. In [7], Kawamata solved the conjecture above when $X$ is 2-dimensional and when $X$ is a minimal 3-fold. A. Höring ([6, Theorem 1.5]) solved it when $X$ is a normal projective 3-fold with at most $\mathbb{Q}$-factorial canonical singularities, $B = 0$, and $D - K_X$ is a nef and big Cartier divisor. These results are immediate consequences of the Hirzebruch-Riemann-Roch theorem and some classical results on surfaces and 3-folds. In higher dimensional case, it is rather difficult to calculate $\dim H^0(X, O_X(D))$. Indeed, Conjecture 1.1 is still widely open for the case of $\dim X \geq 4$.

Concerning the effective nonvanishing of global sections of pluri-adjoint line bundles, Y. Fukuma proposed the following problem:

**Problem 1.2([4, Problem 3.2]).** For any fixed positive integer $n$, find the smallest positive integer $m_n$ depending only on $n$ such that $H^0(X, O_X(m(K_X + L))) \neq 0$ for every $m \geq m_n$ and for every polarized manifold $(X,L)$ of dimension $n$ with $s(K_X + L) \geq 0$.

It is known that $m_1 = 1$, $m_2 = 1$ (cf. [4, Theorem 2.8]) and $m_3 = 1$ ([6]). Recently, Fukuma also treated the case of $\dim X = 4$ ([5]).

Our main result is the following:

**Theorem 1.3.** Let $(X,L)$ be a polarized manifold of dimension $n$ with $K_X + L$ nef. Then $H^0(X, O_X(m(K_X + L))) \neq 0$ holds for every positive integer $m \geq n(n + 1)/2 + 2$.

The above theorem gives a partial answer to Problem 1.2 in higher dimensional case. We give the proof in Section 3; our basic tool is singular hermitian metrics, which will be reviewed in the next section.

2. Preliminaries

We introduce the notions of singular hermitian metrics and multiplier ideal sheaves.

**Definition 2.1.** Let $L$ be a holomorphic line bundle over a complex manifold $X$. A singular hermitian metric $h$ on $L$ is given by $h = h_0 \cdot e^{-\varphi}$, where $h_0$ is a $C^\infty$-hermitian metric on $L$. The zero set of $\varphi$, denoted by $\varphi^{-1}(0)$, is called the singular set of $h$.
Let \( L \) and \( \varphi \in L^1_{\text{loc}}(X) \). The **curvature current** \( \Theta_h \) of \( h \) is defined by

\[
\Theta_h := \Theta_{h_0} + \sqrt{-1} \partial \bar{\partial} \varphi,
\]

where \( \Theta_{h_0} \) denotes the curvature form of \( h_0 \), and \( \partial \bar{\partial} \varphi \) is taken in the sense of currents.

**Example 2.2.** Let \( L \) be a holomorphic line bundle over a complex manifold \( X \). Suppose that there exists a positive integer \( m \) such that \( \Gamma(X, \mathcal{O}_X(mL)) \neq 0 \). Let \( \sigma \in \Gamma(X, \mathcal{O}_X(mL)) \) be a nontrivial section. Then

\[
h := \frac{1}{|\sigma|^2/m} = \frac{h_0}{h_0^{\otimes m}(\sigma, \sigma)^{1/m}}
\]

is a singular hermitian metric on \( L \), where \( h_0 \) is an arbitrary \( C^\infty \)-hermitian metric on \( L \). By Poincaré-Lelong’s formula, we have \( \Theta_h = 2\pi/m(\sigma) \), where \( (\sigma) \) denotes the current of integration over the divisor of \( \sigma \). In particular, we see that \( \Theta_h \) is a positive current.

**Definition 2.3.** Let \( L \) be a line bundle over a complex manifold \( X \) and \( h \) a singular hermitian metric on \( L \). We shall write \( h \) as \( h = h_0 \cdot e^{-\varphi} \), where \( h_0 \) is a \( C^\infty \)-hermitian metric on \( L \) and \( \varphi \in L^1_{\text{loc}}(X) \). Then we define the **multiplier ideal sheaf** \( \mathcal{I}(h) \) of \( (L, h) \) by

\[
\Gamma(U, \mathcal{I}(h)) := \{ f \in \Gamma(U, \mathcal{O}_X) \mid |f|^2 \cdot e^{-\varphi} \in L^1_{\text{loc}}(U) \},
\]

where \( U \) runs over the open subsets of \( X \).

The following vanishing theorem due to A. Nadel ([8]) plays a crucial role in the proof of Theorem 1.3 (cf. Remark 2.4.1).

**Theorem 2.4.** Let \( L \) be a line bundle over a compact Kähler manifold \( (X, \omega) \), and \( h \) a singular hermitian metric on \( L \). Suppose that the curvature current \( \Theta_h \) of \( h \) is strictly positive, i.e., there exists a constant \( \varepsilon > 0 \) such that \( \Theta_h - \varepsilon \omega \) is a positive \((1,1)\)-current. Then \( \mathcal{I}(h) \) is a coherent sheaf on \( X \), and

\[
H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h)) = 0
\]

holds for every \( q \geq 1 \).

**Remark 2.4.1.** We shall explain how to establish the effective nonvanishing of global sections of (multi-)adjoint line bundles by using the above theorem. Suppose that there exists a singular hermitian metric \( h \) on a line bundle \( L \) such that

1. \( \Theta_h \) is strictly positive;
2. \( \mathcal{O}_X/\mathcal{I}(h) \) has isolated support at a point \( x \) in \( X \).

Then by Theorem 2.4, we have \( H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h)) = 0 \). This implies that the map:

\[
H^0(X, \mathcal{O}_X(K_X + L)) \to H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{O}_X/\mathcal{I}(h))
\]

is surjective. Therefore, since the support of \( \mathcal{O}_X/\mathcal{I}(h) \) is isolated at \( x \), we can take a global section \( \sigma \in H^0(X, \mathcal{O}_X(K_X + L)) \) with \( \sigma(x) \neq 0 \). In particular, we conclude \( H^0(X, \mathcal{O}_X(K_X + L)) \neq 0 \).

### 3. Sketch of the proof of Theorem 1.3

We shall prove Theorem 1.3. Let \( \Phi_m : X \to \mathbb{P} H^0(X, \mathcal{O}_X(m(K_X + L)))^* \) denote the rational map associated with \( |m(K_X + L)| \). By the base point free theorem and by taking an integer \( m \gg 1 \), we obtain a surjective morphism \( f := \Phi_m : X \to Y \), where \( Y \) denotes the image of \( X \). We may assume that \( \kappa(X, K_X + L) = \dim Y \) and \( \kappa(F, K_F + L|_F) = 0 \) for the general fiber \( F \) of \( f \). Taking a suitable modification, we may also assume that \( Y \) is smooth. Now we define the reflexive sheaf \( B \) on \( Y \) by \( B := f_! \mathcal{O}_X(K_Y + L)^* \). Since \( K_F + L|_F \) is trivial, \( B \) is an invertible sheaf on \( Y \). Moreover we have the following:
Lemma 3.1. B is big, and $K_Y + B$ is nef and big.

Proof. Let $h_L$ be a $C^\infty$-hermitian metric on $L$ with strictly positive curvature. Then we define the singular hermitian metric $h_B$ on $B$ by

$$h_B(\sigma, \sigma) := \int_{X/Y} h_L \cdot \sigma \wedge \overline{\sigma},$$

where $\sigma \in \Gamma(Y, B)$ is a global section of $B$. Then by [3, Theorem 0.1], we see that $h_B$ has strictly positive curvature current. This implies that $B$ is big. On the other hand, by the construction of $B$, it follows immediately that $K_Y + B$ is big. (For the nefness of $K_Y + B$, see [2, Lemma 4.3].) □

So it suffices to show the following:

Lemma 3.2. $H^0(Y, \mathcal{O}_Y(m(K_Y + B))) \neq 0$ holds for every integer $m \geq d(d + 1)/2 + 2$, where $d := \text{dim} Y$.

Sketch of the proof of Lemma 3.2. We use the technique adopted by Angehrn and Siu ([1]) and Tsuji ([9]) in their study of Fujita’s freeness conjecture. We set $\mu_0 := N^d$ and fix a point $y_0$ on $Y$. First, by a dimension counting argument, we have the following:

Lemma 3.3. $H^0(Y, \mathcal{O}_Y(m(K_Y + B))) \otimes \mathcal{O}_{y_0}((\sqrt{\mu_0(1-\varepsilon)m})) \neq 0$ holds for every $0 < \varepsilon < 1$ and every $m \gg 0$.

Fix $0 < \varepsilon < 1$ and a positive integer $m_0$ as in the above lemma, and take a nontrivial global section:

$$\sigma_0 \in H^0(Y, \mathcal{O}_Y(m_0(K_Y + B))) \otimes \mathcal{O}_{y_0}\left[\sqrt{\mu_0(1-\varepsilon)m_0}\right].$$

We define the singular hermitian metric $h_0$ on $K_Y + B$ by $h_0 = |\sigma_0|^{-2/m_0}$. Let $\alpha_0$ be the positive number defined by $\alpha_0 := \inf \{\alpha > 0 \mid I(h_0^\alpha)_{y_0} \neq \mathcal{O}_{y_0}\}$. Then by the fact that $(\sum_{i=1}^n |z_i|^2)^{-n}$ is not locally integrable around the origin of $\mathbb{C}^n$, we get $\alpha_0 \leq (d/\sqrt{\mu_0})(1 - \varepsilon)^{-1}$. Fix an arbitrary positive number $\lambda \ll 1/d$. Since $\mu_0 \geq 1$ holds, by taking $\varepsilon$ sufficiently small, we may assume that $\alpha_0 \leq d + \lambda$ holds.

Let $V_1$ be the analytic set whose structure sheaf is $\mathcal{O}_Y/I(h_0^{\alpha_0})$, and $Y_1$ an irreducible component of $V_1$ which passes through $y_0$. Here, for simplicity, we suppose that $\text{dim} Y_1 = 0$. Then we have the following:

Lemma 3.4. $H^0(Y, \mathcal{O}_Y(m(K_Y + B))) \neq 0$ holds for every $m \geq d + 2$.

Proof. Fix an integer $m \geq \alpha_0$. Then by $\alpha_0 \leq d + \lambda$, we have $m \geq d + 1$.

Since $K_Y + B$ is big, by Kodaira’s lemma, we have an effective $\mathbb{Q}$-divisor $G$ on $Y$ such that $K_Y + B - G$ is ample. We may assume that the support of $G$ does not contain $y_0$. Let $0 < \delta \ll 1$ be a rational number, and we set $A := (m - \alpha_0)(K_Y + B) - \delta G$. Note that $A$ is ample, because $K_Y + B$ is nef. Fix a $C^\infty$-hermitian metric $h_A$ on $A$ with strictly positive curvature. Let $G = \sum e_iE_i$ be the irreducible decomposition of $G$ and $\sigma_i \in \Gamma(Y, E_i)$ a global section with $(\sigma_i) = E_i$. Then we define the singular hermitian metric $h$ on $\mathcal{O}_Y(m(K_Y + B))$ by

$$h := h_0^{\alpha_0} \cdot h_A \prod_i |\sigma_i|^{2\delta e_i}.$$

Since $h \cdot h_B$ has strictly positive curvature current, by virtue of Theorem 2.4 (cf. Remark 2.4.1), we see that the restriction map:

$$H^0(Y, \mathcal{O}_Y((m + 1)(K_Y + B))) \longrightarrow H^0(Y, \mathcal{O}_Y((m + 1)(K_Y + B)) \otimes \mathcal{O}_Y/I(h \cdot h_B))$$

is surjective. Now we may assume that $y_0$ is not on the singular locus of $h_B$, and hence $\mathcal{O}_Y/I(h \cdot h_B)$ has isolated support at $y_0$. Therefore by the surjectivity of (3.2), there exists a global section $\tau \in H^0(Y, \mathcal{O}_Y((m + 1)(K_Y + B))$ with $\tau(y_0) \neq 0$. We have thus proved the lemma. □
When \( \dim Y = 1 > 0 \), we need to cut down the support of \( \mathcal{O}_Y/\mathcal{I}(h \cdot h_B) \) in order to construct a singular hermitian metric as in Remark 2.4.1; by Angehrn-Siu's argument, we obtain the following lemma (see [1], [2, Section 3] for details).

**Lemma 3.5.** Let \( m \) be an integer with \( m \geq d(d + 1)/2 + 1 \). Then there exists a singular hermitian metric \( h_{y_0} \) on \( \mathcal{O}_Y(m(K_Y + B)) \) such that \( h_{y_0} \) has strictly positive curvature current, and \( \mathcal{O}_Y/\mathcal{I}(h_{y_0}) \) has isolated support at \( y_0 \).

Then by an similar argument to that in the proof of Lemma 3.4, we see that there exists a global section \( \tau \in H^0(Y, \mathcal{O}_Y(m(K_Y + B))) \) with \( \tau(y_0) \neq 0 \) for every \( m \geq d(d + 1)/2 + 2 \). □

**References**


*Present Address:*
Department of Science and Technology, Sophia University, Kioicho, Chiyoda-ku, Tokyo, 102-8554 Japan.
e-mail: tomoki-a@sophia.ac.jp