Variation formulas for principal functions (II)

Sachiko Hamano

In [1] and [2], we showed the variation formulas of second order for L_1 and L_0 principal functions with *logarithmic* poles on Riemann surfaces $R(t)$ moving smoothly with one complex parameter t in a disk B. Then we apply them to a simultaneous uniformization problem of Schottky coverings of compact Riemann surfaces and to the variation of harmonic span. In this talk, we shall show the variation formulas for principal functions with pole of $\mathfrak{R}\{\frac{1}{z}\}$. Then we apply them to the variation of *analytic* span from the several complex variables view point.

Let $\mathcal{R} : t \in B \to R(t)$ be a C^∞ smooth variation of unramified domains $R(t)$ over $\mathbb{C}_z$ with C^ω smooth boundary $C_1(t), \ldots, C_\nu(t)$ in $\tilde{R}(t)$ ($\Supset R(t)$). We set

$$\mathcal{R} = \bigcup_{t\in B}(t, R(t)) \quad \text{and} \quad \partial\mathcal{R} = \bigcup_{t\in B}(t, \partial R(t)).$$

We assume that $\mathcal{R}$ contains $B \times \{0\}$.

Definition 1 (Principal functions $p(t,z)$ and $q(t,z)$)**.**
For each $t \in B$, the *L_1-pricipal function* $p(t,z)$ for $(R(t), \{0\})$ is a harmonic function on $R(t) \setminus \{0\}$ such that

$$p(t,z) = \mathfrak{R}\left\{\frac{1}{z}\right\} + 0 + \mathfrak{R}\left\{\sum_{n=1}^{\infty} A_n(t) z^n\right\} \quad \text{near } z = 0,$$

$$p(t,z) = \begin{cases} \text{constant } \ a_j(t) \text{ on } C_j(t) \\ \displaystyle\int_{C_j(t)} \frac{\partial p_1(t,z)}{\partial n_z} ds_z = 0 \end{cases} \quad \text{for each } C_j(t),\ j = 1, \ldots, \nu.$$

For each $t \in B$, the *L_0-principal function* $q(t,z)$ for $(R(t), \{0\})$ is a harmonic function on $R(t) \setminus \{0\}$ such that

$$q(t,z) = \mathfrak{R}\left\{\frac{1}{z}\right\} + 0 + \mathfrak{R}\left\{\sum_{n=1}^{\infty} B_n(t) z^n\right\} \quad \text{near } z = 0,$$

$$\frac{\partial q(t,z)}{\partial n_z} = 0 \quad \text{on each } C_j(t),\ j = 1, \ldots, \nu.$$

Then we shall prove

Lemma 2. *Assume that each $R(t), t \in B$ is an unramified domain over $\mathbb{C}_z$. Then*

$$\frac{\partial^2 \mathfrak{R}\{A_1(t)\}}{\partial t \partial \bar{t}} = -\frac{1}{\pi} \int_{\partial D(t)} k_2(t,z) \left|\frac{\partial p(t,z)}{\partial z}\right|^2 ds_z - \frac{4}{\pi} \iint_{D(t)} \left|\frac{\partial^2 p(t,z)}{\partial \bar{t} \partial z}\right| dxdy. \tag{1}$$

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Lemma 3. *Assume that each* $R(t), t \in B$ *is an unramified planar domain over* $\mathbb{C}_z$*. Then*

$$\frac{\partial^2 \Re\{B_1(t)\}}{\partial t \partial \bar{t}} = \frac{1}{\pi} \int_{\partial D(t)} k_2(t,z) \left| \frac{\partial q(t,z)}{\partial z} \right|^2 ds_z + \frac{4}{\pi} \iint_{D(t)} \left| \frac{\partial^2 q(t,z)}{\partial \bar{t} \partial z} \right| dxdy. \tag{2}$$

In (1) and (2),

$$k_2(t,z) = \left(\frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \left| \frac{\partial \varphi}{\partial z} \right|^2 - 2\mathrm{Re} \left\{ \frac{\partial^2 \varphi}{\partial \bar{t} \partial z} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \right) / \left| \frac{\partial \varphi}{\partial z} \right|^3$$

on $\partial\mathcal{R}$, which does not depend on the choice of defining functions $\varphi(t,z)$ of $\partial\mathcal{R}$, and ds_z is the arc length element of $\partial R(t)$ at z. The function $k_2(t,z)$ on $\partial\mathcal{R}$ is due to Maitani-Yamaguchi in [3].

For each $t \in B$, we have the *analytic span* $S(t)$ with respect to $(R(t), \{0\})$ which is defined by

$$S(t) := B_1(t) - A_1(t) > 0.$$

We put $\mathcal{U}(R(t))$ the set of all univalent functions f on $R(t)$ such that

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n \quad \text{near } z = 0.$$

Then $\frac{\pi}{2} S(t)$ represents the maximum of the Euclidean area of the complement of $f(R(t))$ in $\mathbb{P}_w$ of all $f \in \mathcal{U}(R(t))$ (cf: p.45-46 in [4]).

Then we have the following main result in this talk which suggests that analytic span $S(t)$ with respect to $(R(t), \{0\})$ has a certain significant meaning in several complex variables as well as in one complex variable.

Theorem 4. *Let* $\mathcal{R} : t \in B \to R(t)$ *be a* C^ω *smooth variation of planar Riemann surfaces* $R(t)$ *with* C^ω *smooth boundary in* $\tilde{R}(t)$ $(\Supset R(t))$*. We set* $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$*. If* $\mathcal{R}$ *is a two-dimensional pseudoconvex domain in* $B \times \tilde{R}(t)$*, then* $\log S(t)$ *is a subharmonic function on* B*.*

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Sobolev homeomorphisms and nonnegativity of the jacobian

Stanislav Hencl

The talk is based on a joint result with Jan Malý [1]. Suppose that $\Omega \subset \mathbb{R}^n$ is a domain and that $f : \Omega \to \mathbb{R}^n$ is a homeomorphism of the Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. Is it true that the jacobian J_f (the determinant of the matrix of derivatives) is either non-negative almost everywhere or non-positive almost everywhere? We show this conjecture in dimension $n = 3$ and we also discuss what is known in higher dimensions.

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Upper and lower estimates for parabolic Green functions in Lipschitz domains

Kentaro Hirata

The Green functions play important roles in several areas. In this talk, we shall introduce known results about estimates for the Green functions for the Laplace operator and the heat operator (cf. [1]–[8]). The main aim is to present upper and lower estimates for the Green function for the heat operator in a bounded Lipschitz domain Ω. In this case, the boundary behavior of the Green function is complicated, so we need to introduce an auxiliary set: for $x \in \overline{\Omega}$ and $0 < t < T$,

$$\mathcal{B}_p(x,t) = \left\{ b \in \Omega : \frac{1}{\kappa}\|x-b\| \le \sqrt{t} \le \kappa \delta_\Omega(b) \right\}.$$

Let $g_\Omega(x) = G_\Omega(x,x_0) \wedge 1$, where $G_\Omega(x,x_0)$ is the Green function of Ω for the Laplace operator with a fixed pole x_0. Denote by Γ_Ω the Green function of $\Omega \times \mathbb{R}$ for the heat operator. Then we can obtain the following upper and lower estimates.

Theorem 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \ge 2)$ and let $T > 0$. Then there exists a constant $C > 1$ depending only on n, Ω and T such that the following upper and lower estimates hold for all $x, y \in \Omega$ and $0 < t < T$:*

$$\Gamma_\Omega(x,t;y,0) \le \frac{g_\Omega(x)g_\Omega(y)}{g_\Omega(b_x)g_\Omega(b_y)} \frac{C}{t^{n/2}} \exp\left\{-\frac{\|x-y\|^2}{Ct}\right\},$$
$$\Gamma_\Omega(x,t;y,0) \ge \frac{g_\Omega(x)g_\Omega(y)}{g_\Omega(b_x)g_\Omega(b_y)} \frac{1}{Ct^{n/2}} \exp\left\{-\frac{C\|x-y\|^2}{t}\right\},$$

where $b_x \in \mathcal{B}_p(x,t)$ and $b_y \in \mathcal{B}_p(y,t)$.

Let $y \in \partial\Omega$ and $T_0 = T+1$. We observe that a kernel function at $(y,0)$ normalized at (x_0,T_0) is given as a limit of the ratio $\Gamma_\Omega(x,t;y_j,0)/\Gamma_\Omega(x_0,T_0;y_j,0)$ for some $y_j \in \Omega$ converging to y. Therefore we can obtain the following.

Theorem 2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \ge 2)$ and let $T > 0$. Then there exists a constant $C > 1$ depending only on n, Ω and T such that the following upper and lower estimates hold for all $(x,t) \in \Omega \times (0,T)$ and $y \in \overline{\Omega}$:*

$$K_\Omega(x,t;y,0) \le C\frac{g_\Omega(x)}{g_\Omega(b_x)g_\Omega(b_y)} \frac{1}{t^{n/2}} \exp\left\{-\frac{\|x-y\|^2}{Ct}\right\},$$
$$K_\Omega(x,t;y,0) \ge \frac{1}{C}\frac{g_\Omega(x)}{g_\Omega(b_x)g_\Omega(b_y)} \frac{1}{t^{n/2}} \exp\left\{-\frac{C\|x-y\|^2}{t}\right\},$$

where $b_x \in \mathcal{B}_p(x,t)$ and $b_y \in \mathcal{B}_p(y,t)$.

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Atomic decompositions on parabolic Bergman spaces

Yôsuke Hishikawa

Let H be the upper half-space of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ $(n \geq 1)$, that is,

$$H = \mathbb{R}^n \times \mathbb{R}_+ = \{(x,t) \in \mathbb{R}^{n+1}; x \in \mathbb{R}^n, t > 0\}.$$

For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha,$$

where $\partial_t = \partial/\partial t$ and Δ_x is the Laplacian with respect to x. A continuous function u on H is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions. We denote by $W^{(\alpha)}$ the fundamental solution of $L^{(\alpha)}$.

For $1 \leq p < \infty$ and $\lambda > -1$, the parabolic Bergman space $\boldsymbol{b}^p_\alpha(\lambda)$ is defined by

$$\boldsymbol{b}^p_\alpha(\lambda) := \{u \in C(H); L^{(\alpha)}\text{-harmonic on } H, \|u\|_{L^p(\lambda)} := \left(\int_H |u(x,t)|^p t^\lambda dV(x,t)\right)^{1/p} < \infty\},$$

where dV is the Lebesgue volume measure on H. Here we remark that $\boldsymbol{b}^p_\alpha(\lambda)$ is the Banach space with the norm $\|\cdot\|_{L^p(\lambda)}$. When $\alpha = 1/2$, $\boldsymbol{b}^p_{1/2}(\lambda)$ is the harmonic Bergman space of Ramey and Yi [4].

Let ν be a real number. We denote by $\mathcal{D}^\nu_t = (-\partial_t)^\nu$ a fractional differential operator with respect to t (Riemann-Liouville operator). We remark that if $\nu > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$, then for each $u \in \boldsymbol{b}^p_\alpha(\lambda)$, $D^\nu_t u$ is well-defined.

Our aim is the study about atomic decompositions on parabolic Bergman spaces. When $\alpha = 1/2$, Choe and Yi [1] studied about atomic decompositions on harmonic Bergman spaces. According to [1], the Poisson kernel plays an important role for studying atomic decompositions on harmonic Bergman spaces. In this talk, we claim that atomic decompositions on parabolic Bergman spaces are given by fractional derivatives of the fundamental solution $W^{(\alpha)}$.

The following definition is the main tool for studying atomic decompositions on parabolic Bergman spaces.

Definition 1. Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $\kappa > 0$. For integers $j \geq 1$, suppose $\{(x_j, t_j)\}$ is points in H, and $\{\eta_j\}$ is a sequence of real numbers. We define a function u on H by the following:

$$u(x,t) = \sum_j \eta_j t_j^{\frac{n}{2\alpha}+\kappa-\left(\frac{n}{2\alpha}+\lambda+1\right)\frac{1}{p}} \mathcal{D}^\kappa_t W^{(\alpha)}(x - x_j, t + t_j). \tag{1.1}$$

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When $\alpha = 1/2$, $\lambda = 0$, and $\kappa = k \in \mathbb{N}$, (1.1) becomes

$$u(x,t) = \sum_j \eta_j t_j^{n+k-(n+1)\frac{1}{p}} \mathcal{D}_t^k P(x - x_j, t + t_j),$$

where $P(\cdot,\cdot) = W^{(1/2)}(\cdot,\cdot)$ is the Poisson kernel.

The following theorem is the main result in this talk, which contains the result of Choe and Yi [1].

Theorem 1. *Let* $0 < \alpha \leq 1$, $1 \leq p < \infty$, *and* $\lambda > -1$. *Suppose* $\kappa > \frac{\lambda+1}{p}$ *is a real number. Then there exist a sequence* $\{(x_j, t_j)\}$ *in* H *and a constant* $C > 0$ *with the following properties: for each* $\{\eta_j\} \in \ell^p$, *the function* u *defined by the series* (1.1) *belongs to* $\boldsymbol{b}_\alpha^p$ *with* $\|u\|_{L^p(\lambda)} \leq \|\{\eta_j\}\|_{\ell^p}$. *Conversely, given* $u \in \boldsymbol{b}_\alpha^p$, *there exists a sequence* $\{\eta_j\} \in \ell^p$ *such that* (1.1) *holds and* $\|\{\eta_j\}\|_{\ell^p} \leq C\|u\|_{L^p(\lambda)}$.

We remark that (1.1) is regarded as a discrete representation of parabolic Bergman functions. On the other hand, we gave the reproducing formula on parabolic Bergman spaces in [2]. It is regarded as a continuous representation of parabolic Bergman functions. Finally, we present the reproducing formula on parabolic Bergman spaces.

Theorem A ([2]). Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. And let ν and κ be real numbers such that $\nu > -\frac{\lambda+1}{p}$ and $\kappa > \frac{\lambda+1}{p}$. Then the reproducing formula

$$u(x,t) = C_{\nu+\kappa} \int_H \mathcal{D}_t^\nu u(y,s) \mathcal{D}_t^\kappa W^{(\alpha)}(x-y, t+s) s^{\nu+\kappa-1} dV(y,s) \tag{1.2}$$

holds for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ and $(x,t) \in H$, where $C_\kappa = 2^\kappa/\Gamma(\kappa)$. Moreover, (1.2) *also holds whenever* $p = 1$ *and* $\kappa = \lambda + 1$.

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Dirichlet forms on the Cantor sets as the traces on the Martin boundary of Random walks on trees

Jun Kigami

Let T is an contably infinite set and let $C : T \times T \to [0, \infty)$ which satisfies $C(x, y) = C(y, x)$ and $C(x, x) = 0$ for any $x \in T$. We call $(x_0, \ldots, x_n)$, where $x_i \in T$, a path between x_0 and x_n if and only if $C(x_i, x_{i+1}) > 0$ for any $i = 0, 1, \ldots, n-1$ and $x_i \neq x_j$ for $i \neq j$. Assume that there exists a unique path between x and y for any $x \neq y \in T$. Then (T, C) is called a **weighted tree**. Let

$$P(x, y) = \frac{C(x, y)}{\sum_{y \in T} C(x, y)}.$$

Then $\{P(x, y)\}_{x,y \in T}$ is the transition probability of the natural random walk $(\{X_n\}_{n \geq 0}, \{P_x\}_{x \in T})$ on T. The associted Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by

$$\mathcal{E}(u, v) = \frac{1}{2} \sum_{x,y \in T, C(x,y) > 0} \frac{(u(x) - u(y))(v(x) - v(y))}{C(x, y)}$$

$$\mathcal{F} = \{u | u : T \to \mathbb{R}, \mathcal{E}(u, u) < +\infty\}.$$

Assume that (T, C) is transitent. By the classical result of Cartier, it is known that the **Martin boundary** of the transient weighted tree coinsides with the set of infinite paths

$$\Sigma = \{(\phi, x_1, x_2, \ldots) | (\phi, x_1, \ldots, x_n) \text{ is the path between } \phi \text{ and } x_n \text{ for any } n \geq 1\},$$

where ϕ is a fixed reference point in T. Morevoer, Σ is a Cantor set, i.e. totally disconnected, uniformly perfect and compact.

Define the trace $(\mathcal{E}|_\Sigma, \mathcal{F}|_\Sigma)$ of $(\mathcal{E}, \mathcal{F})$ on the Martin boundary by

$$\mathcal{E}|_\Sigma(f, f) = \mathcal{E}(H(f), H(f)) \quad \text{and} \quad \mathcal{F}|_\Sigma = \{f | f : \Sigma \to \mathbb{R}, \mathcal{E}|_\Sigma(f, f) < +\infty\}$$

for $f : \Sigma \to \mathbb{R}$, where $H(f)$ is the **harmonic function** on T with the boundary value f on the Martin boundary Σ. In this talk, we will show that

$$\begin{aligned} \mathcal{E}_\Sigma(f, f) &= \sum_{x,y \in T, C(x,y) > 0} \alpha_{x,y} ((f)_{\nu,x} - (f)_{\nu,y})^2 \\ &= \int_{\Sigma \times \Sigma} J(\omega, \tau)(u(\omega) - u(\tau))^2 \nu(\omega) \nu(\tau), \end{aligned}$$

with the expicite formula of $\alpha_{x,y}$ and $J(\omega, \tau)$, where $(f)_{\nu,x}$ is the mean of f on Σ_x. Moreover, under the volume doubling condition, we obtain a full on- and off-diagonal estimate of the heat kernel assicated with the jump process derived from $(\mathcal{E}|_\Sigma, \mathcal{F}|_\Sigma)$. As an application, the Martin boundary of (the Sierpinski gasket) \ (the line segment between two boundary points) is shown to be the Cantor set.

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Some $L^{p(\cdot)}(\log L)^{q(\cdot)}$-norm inequalities involving convolution potentials

Fumi-Yuki Maeda

We consider two continuous variable exponents $p(x)$ and $q(x)$ on $\mathbf{R}^N$ such that

(P1) $1 < p^- := \inf p(\cdot) \le p^+ := \sup p(\cdot) < \infty$;

(Q1) $-\infty < q^- := \inf q(\cdot) \le q^+ := \sup q(\cdot) < \infty$.

Letting $c_0 \ge e$ satisfy $(1 + \log c_0)(p^- - 1) + q^- \ge 0$, we set

$$\Phi_{p(\cdot),q(\cdot)}(x, t) := t^{p(x)}(\log(c_0 + t))^{q(x)}$$

for $x \in \mathbf{R}^N$ and $t \in [0, \infty)$. Then $\Phi_{p(\cdot),q(\cdot)}(x, \cdot)$ is convex on $[0, \infty)$ for every $x \in \mathbf{R}^N$ (see [5, Proposition 5.2]) and

$$L^{p(\cdot)}(\log L)^{q(\cdot)} = \left\{ f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)\,;\ \int \Phi_{p(\cdot),q(\cdot)}(x, |f(x)|)\, dx < \infty \right\}$$

is a reflexive Banach space with respect to the norm

$$\|f\|_{p(\cdot),q(\cdot)} := \inf\left\{ \lambda > 0\,;\ \int_G \Phi_{p(\cdot),q(\cdot)}\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}$$

(cf. [6]).

We can prove the following extension of a result due to D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Pérez [1]:

Theorem 1. *Suppose $p(\cdot)$ is log-Hölder continuous in $\mathbf{R}^N$ as well as at ∞, and $q(\cdot)$ is log-log-Hölder continuous in $\mathbf{R}^N$. Let $\mathcal{F}$ be a family of ordered pairs (f, g) of nonnegative measurable functions on $\mathbf{R}^N$. Suppose that for some $0 < p_0 < p^-$,*

$$\int_{\mathbf{R}^N} f(x)^{p_0} w(x)\, dx \le C_0 \int_{\mathbf{R}^N} g(x)^{p_0} w(x)\, dx$$

for all $(f, g) \in \mathcal{F}$ and for all A_1-weights w, where C_0 depends only on p_0 and the A_1-constant of w. Then

$$\|f\|_{p(\cdot),q(\cdot)} \le C\|g\|_{p(\cdot),q(\cdot)}$$

for all $(f, g) \in \mathcal{F}$ such that $g \in L^{p(\cdot)}(\log L)^{q(\cdot)}$.

Applying this theorem to known results on weighted norm inequalities (see, [7], [8], [2]) we obtain, e.g.,

Theorem 2.

$$C^{-1}\|M_\alpha \mu\|_{p(\cdot),q(\cdot)} \le \|I_\alpha * \mu\|_{p(\cdot),q(\cdot)} \le C\|M_\alpha \mu\|_{p(\cdot),q(\cdot)}$$

and

$$C^{-1}\|M_{\alpha,R}\, \mu\|_{p(\cdot),q(\cdot)} \le \|G_\alpha * \mu\|_{p(\cdot),q(\cdot)} \le C\|M_{\alpha,R}\, \mu\|_{p(\cdot),q(\cdot)}$$

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for nonnegative measures μ *on* $\mathbf{R}^N$*, where* $0 < \alpha < N$*,* $I_\alpha(x)$ *is the Riesz kernel of order* α*,* $G_\alpha(x)$ *is the Bessel kernel of order* α*,*

$$M_\alpha \mu(x) = \sup_{r>0} r^{\alpha-N}\mu(B(x,r))$$

and, for $R > 0$

$$M_{\alpha,R}\,\mu(x) = \sup_{0<r<R} r^{\alpha-N}\mu(B(x,r)).$$

We can generalize this theorem to more general convolution potentials and related maximal functions (cf. [3], [4]).

Let $k(r)$ be a positive nonincreasing lower semicontinous function on $(0,\infty)$ such that $\int_0^1 k(r)r^{N-1}\,dr < \infty$. Set

$$\bar{k}(r) = \frac{1}{r^N}\int_0^r k(t)t^{N-1}\,dt,$$

$$M_k\,\mu(x) = \sup_{r>0} \bar{k}(r)\mu(B(x,r)) \quad \text{and} \quad M_{k,R}\,\mu(x) = \sup_{0<r<R} \bar{k}(r)\mu(B(x,r))$$

for nonnegative measures μ on $\mathbf{R}^N$ and $R > 0$. Let $k(x) = k(|x|)$ for $x \in \mathbf{R}^N$ (by abuse of notation). Then we have

Theorem 3. (1)

$$C^{-1}\|M_k\,\mu\|_{p(\cdot),q(\cdot)} \le \|k * \mu\|_{p(\cdot),q(\cdot)} \le C\|M_k\,\mu\|_{p(\cdot),q(\cdot)}$$

for nonnegative measures μ *on* $\mathbf{R}^N$*.*

(2) *If, in addition,* $\int_1^\infty k(r)r^{N-1}\,dr < \infty$*, then, for every* $R > 0$

$$C^{-1}\|M_{k,R}\,\mu\|_{p(\cdot),q(\cdot)} \le \|k * \mu\|_{p(\cdot),q(\cdot)} \le C\|M_{k,R}\,\mu\|_{p(\cdot),q(\cdot)}$$

for nonnegative measures μ *on* $\mathbf{R}^N$*.*

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Hardy-Orlicz space for harmonic functions and the class of bounded harmonic funcions on a Riemann surface

Hiroaki Masaoka

Let R be a Riemann surface which admit a Green's function. Let $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a convex strictly increasing function sush that (i) $\lim_{t\to+0}\Phi(t) = 0$ and $\lim_{t\to\infty}\frac{\Phi(t)}{t} = \infty$. Set $h_\Phi(R) = \{u|u$ is harmonic on R and $\Phi\circ|u|$ has a harmonic majorant on $R\}$. We call $h_\Phi(R)$ the Hardy-Orlicz space for harmonic hunctions on R. Let $HB(R)$ be the class of bounded harmonic funcions on R.

In this talk, using the Martin Theory, we give a necessity and sufficient condition for $HB(R) = h_\Phi(R)$.

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Hardy's inequality in Orlicz-Sobolev spaces of variable exponent

Yoshihiro Mizuta

Our aim in this talk is to deal with a norm version of Hardy's inequality for Orlicz–Sobolev functions with $|\nabla u| \in L^{p(\cdot)} \log L^{p(\cdot)q(\cdot)}(\Omega)$ for an open set $\Omega \subset R^n$. Here $p(\cdot)$ and $q(\cdot)$ are variable exponents satisfying the log-Hölder and loglog-Hölder conditions, respectively. We are also concerned with the case when p attains the value 1 in some parts of the domain is included in the results.

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Carleson inequality on α-parabolic Hardy spaces

Hayato Nakagawa and Noriaki Suzuki

Let $\mathbb{R}^{n+1}_+ = \{(x,t) \mid x \in \mathbb{R}^n, t > 0\}$ be the upper half-space of the $(n+1)$-dimensional Euclidean space ($n \geq 1$). For $0 < \alpha \leq 1$, we denote by $L^{(\alpha)}$ the parabolic operator of order α on $\mathbb{R}^{n+1}_+$, defined by

$$L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha,$$

where Δ_x is the Laplace operator with respect to $x \in \mathbb{R}^n$ ([NSS]). The fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ is given by

$$W^{(\alpha)}(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t|\xi|^{2\alpha}} e^{ix\cdot\xi}\, d\xi$$

where $x \cdot \xi$ is the inner product of x and ξ, and $|\xi| = (\xi \cdot \xi)^{1/2}$. Note that $W^{(1/2)}(x,t)$ is the Poisson kernel and $W^{(1)}(x,t)$ is the Gauss kernel.

For $1 < p \leq \infty$, we set

$$h^p_\alpha = \{u \in C(\mathbb{R}^{n+1}_+) \mid L^{(\alpha)}u = 0, \|u\|_{h^p_\alpha} < \infty\},$$

where

$$\|u\|_{h^p_\alpha} := \begin{cases} \sup\limits_{t>0} \left(\displaystyle\int_{\mathbb{R}^n} |u(x,t)|^p dx \right)^{\frac{1}{p}} & (1 < p < \infty), \\ \sup\limits_{(x,t)\in\mathbb{R}^{n+1}_+} |u(x,t)| & (p = \infty). \end{cases}$$

We discuss a Carleson inequality on α-parabolic Hardy space h^p_α.

We know some results concerning to parabolic Bergman spaces. For $1 \leq p < \infty$, the parabolic Bergman space b^p_α is given by

$$b^p_\alpha = \{u \in C(\mathbb{R}^{n+1}_+) \mid L^{(\alpha)}u = 0, \|u\|_{L^p(V)} < \infty\},$$

where V is the Lebesgue measure on $\mathbb{R}^{n+1}_+$. The nesessary and sufficient condition for the Carleson inequality on the parabolic Bergman space b^p_α holds is characterlized by the τ-Carleson measure ([NSY]). For a positive measure μ, we say that μ is a τ-Carleson measure (with respect to $L^{(\alpha)}$) if there exists a constant $C > 0$ such that

$$\mu(Q^{(\alpha)}(x,t)) \leq Ct^{(\frac{n}{2\alpha}+1)\tau}$$

holds for all $(x,t) \in \mathbb{R}^{n+1}_+$, where $\tau > 0$ and

$$Q^{(\alpha)}(x,t) = \{(y_1,\ldots,y_n,s) \mid t \leq s \leq 2t, x = (x_1,\ldots,x_n), |y_j - x_j| \leq t^{1/2\alpha}/2, j = 1,\ldots,n\}.$$

Theorem 1 (Theorem 1 of [NSY]). *Let* $1 \leq p \leq q < \infty$ *and* μ *be a positive Borel measure on* $\mathbb{R}^{n+1}_+$. *Then* μ *is a* q/p*-Carleson measure if and only if there exists a constant* $C > 0$ *such that the inequality*

$$\|u\|_{L^q(\mathbb{R}^{n+1}_+, d\mu)} \leq C\|u\|_{L^p(V)}$$

holds for all $u \in b^p_\alpha$.

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This time we study the Carleson inequality on the α-parabolic Hardy space. For a positive measure μ, we say that μ is a T_τ-Carleson measure (with respect to $L^{(\alpha)}$) if there exists a constant $C > 0$ such that

$$\mu(T^{(\alpha)}(x,t)) \leq Ct^{(\frac{n}{2\alpha}+1)\tau}$$

where $\tau > 0$ and $T^{(\alpha)}(x,t) := \{(y,s) \in \mathbb{R}^{n+1}_+ \mid |x-y|^{2\alpha} + s \leq t\}$.

Proposition 2. *When* $\frac{n}{2\alpha}/(\frac{n}{2\alpha}+1) < \tau$*,* μ *is the* τ*-Carleson measure if and only if* μ *is the* T_τ*-Carleson measure. When* $\frac{n}{2\alpha}/(\frac{n}{2\alpha}+1) \geq \tau$*,* μ *is the* τ*-Carleson measure if* μ *is the* T_τ*-Carleson measure, but* μ *is not always the* T_τ*-Carleson measure even if* μ *is the* τ*-Carleson measure.*

We show a theorem for the Carleson inequality on the α-parabolic Hardy space. Here we consider the case $p \leq q$.

Theorem 3. *Let* $1 < p \leq q < \infty$ *and* μ *be a positive Borel measure on* $\mathbb{R}^{n+1}_+$*. Then* μ *is a* T_τ*-Carleson measure with* $\tau = \frac{q}{p} \cdot \frac{n}{2\alpha}/(\frac{n}{2\alpha}+1)$ *if and only if there exists a constant* $C > 0$ *such that the inequality*

$$||u||_{L^q(\mathbb{R}^{n+1}_+, d\mu)} \leq C||u||_{h^p_\alpha}$$

holds for all $u \in h^p_\alpha$*.*

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Capacity versus harmonic measure on Royden harmonic boundary

Mitsuru Nakai

The purpose of this talk is to exhibit an instructive example W of a Riemann surface with various interesting but pathological and abnormal properties cropprd up in our recent collaborative study on the inverse inclusion problem in the classification theory of Riemann surfaces with Professor Hiroaki Masaoka at Kyoto Sangyo University. Among the above mentioned properties our central concern in this talk is the validity of the following inequalities

$$(\mathrm{hm}(K))^2 \leqq C_1\mathrm{cap}(K) \leqq C_2\mathrm{hm}(K) \tag{1}$$

for compact subsets K in the Royden harmonic boundary $\delta_{\mathcal{R}}W$ of W carrying infinite dimensional spaces of bounded and Dirichlet finite harmonic functions on W, where $\mathrm{hm}(K)$ and $\mathrm{cap}(K)$ are the harmonic measure and the capacity of K, respectively, calculated for W less a compact parametric disc and C_1 and C_2 are constants depending only upon W. Here the first inequality of (1) is generally true (cf. [3]) and the second of (1) is the special one only true for the present particular W.

Recall that an afforested surface $R := \langle P, (T_n)_{n\in\mathbb{N}}, (\sigma_n)_{n\in\mathbb{N}}\rangle$ introduced in e.g. [4] (cf. also [5] and [6]) consists of three ingredients: a Riemann urface P called a plantation, a sequence $(T_n)_{n\in\mathbb{N}}$ of Riemann surfaces T_n called trees, and a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of slits σ_n commonly included in P and in each T_n called the roots of T_n or root holes in P. Then an afforested surface R is the Riemann surface given by the following formula

$$R := \cdots\Big(\Big((\mathbb{C}\setminus\cup_{i\in\mathbb{N}}\sigma_i)\mathop{\bigcup\!\!\!\!\!\times}_{\sigma_1}(T_1\setminus\sigma_1)\Big)\mathop{\bigcup\!\!\!\!\!\times}_{\sigma_2}(T_2\setminus\sigma_2)\Big)\cdots . \tag{2}$$

Here we have denoted by $(X\setminus\gamma)\mathop{\bigcup\!\!\!\!\!\times}_{\gamma}(Y\setminus\gamma)$ the Riemann surface obtained from two Riemann surfaces X and Y by pasting $X\setminus\gamma$ and $Y\setminus\gamma$ crosswise along the common slit γ contained in X and Y. Specifically the root sequence $(\sigma_n)_{n\in\mathbb{N}}$ is given as follows: let $V_n := \{|z| < 1\}$ be parametric discs commonly contained in P and T_n for each $n \in \mathbb{N}$ such that $\overline{V}_n \cap \overline{V}_m = \emptyset$ $(n \neq m)$ fixed in advance and $\sigma_n \subset V_n$ such that $\sigma_n = [-s_n, s_n]$ $(0 < s_n < 1/2)$ for each $n \in \mathbb{N}$. Afforested surfaces R are convenient and usefull tools to construct various examples of Riemann surfaces with certain required properties.

We now give the example W mentioned at the begining as a suitable afforested surface $W := \langle P, (T_n)_{n\in\mathbb{N}}, (\sigma_n)_{n\in\mathbb{N}}\rangle$, where the plantation $P = \mathbb{C}$, the complex plane, the trees $T_n = \hat{\mathbb{D}}$ for all $n \in \mathbb{N}$ which is the Sario-Tôki disc (cf. e.g. [1], [7]) so that $\hat{\mathbb{D}} := \mathbb{D}/Q$ is the quotient space of an equivalece relation Q for $\mathbb{D}$ with Q = id. on $\{|z| < a\}$ for some $0 < a < 1$, with $|z|$ which is univalent on $\hat{\mathbb{D}}$, and with only constant positive harmonic functions on $\hat{\mathbb{D}}$. As V_n we take $V_n := \{|\zeta| < 1\}$ with $\zeta := z/a$ in $T_n \equiv \hat{\mathbb{D}}$ and also $V_n := \{|\zeta| < 1\}$ with $\zeta = z - 4n$ in $P \equiv \mathbb{C}$. We denote by M_j the Harnack constant of the compact subset $\{0\} \cup \partial V_j$ considered in $\mathbb{C}$ for each $j \in \mathbb{N}$ with respect to positive harmonic functions on $\mathbb{C} \setminus \cup_{k\in\mathbb{N}}(1/2)\overline{V}_k$ in $\mathbb{C}$. Let

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$\sigma_n := [-s_n, s_n] + 4n \subset V_n \subset P \equiv \mathbb{C}$ and $\sigma_n := [-s_n, s_n] \subset V_n \subset T_n \equiv \hat{\mathbb{D}}$ $(n \in \mathbb{N})$. Then the required surface W is given as an afforested surface by $W := \langle \mathbb{C}, (\hat{\mathbb{D}})_{n\in\mathbb{N}}, (\sigma_n)_{n\in\mathbb{N}} \rangle$ with

$$s_n := \exp\left(-2^n \sum_{1\leqq j\leqq n} M_j\right) \quad (n \in \mathbb{N}). \tag{3}$$

We use the following standard notation in the classification theory of Riemann surfaces (cf. e.g. [7]): H (harmonic), P (essentially positive), B (bounded), B' (quasibounded), D (Dirichlet finite), BD (B and D simultaneously), M_2 (Hardy square mean bounded); $Hm(W)$ (the set of harmonic measure functions w on W characterized by the greatest harmonic minorant $w \wedge (1 - w) = 0$); $\delta_{\mathcal{W}} W$ and $\delta_{\mathcal{R}} W$ (the Wiener and the Royden harmonic boundary of W, respectively) (cf. e.g. [2], [7]); cap (capacity), hm (harmonic measure) calculated for W less a compact parametric disc (cf. [3]). We maintain the following result:

Theorem. *The afforested surface W with (3) enjoys the following properties:*

(a) $HBD(W) = HB(W) < HD(W) = HM_2(W) < HB'(W) < HP(W)$, *where* $<$ *indicates the strict inclusion relation, and the linear dimension of* $HBD(W)$ *is infinite;*

(b) $\delta_{\mathcal{W}} W = \delta_{\mathcal{R}} W$ *although the latter is a quotient space of the former in general;*

(c) $\sup_{w\in Hm(W)} D(w; W) < +\infty$, *where* $D(w; W) := \int_W dw \wedge *dw$ *is the Dirichlet integral of* w *taken over* W*;*

(d) $(\mathrm{hm}(K))^2 \leqq C_1 \mathrm{cap}(K) \leqq C_2 \mathrm{hm}(K)$ *for every compact subset* $K \subset \delta_{\mathcal{R}} W$ *with constants* C_1 *and* C_2 *depending only upon* W.

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Toeplitz operators of Schatten class of small exponent on the parabolic Bergman space

Masaharu Nishio, Noriaki Suzuki, and Masahiro Yamada

We consider a parabolic operator

$$L^{(\alpha)} := \frac{\partial}{\partial t} + (-\Delta_x)^\alpha$$

on the upper half space $\boldsymbol{R}^{n+1}_+ = \boldsymbol{R}^n \times (0, \infty)$, where $\Delta_x := \partial^2_{x_1} + \cdots + \partial^2_{x_n}$ denotes the Laplacian on the x-space $\boldsymbol{R}^n$ and $0 < \alpha \le 1$. The parabolic Bergman space is a Hilbert space defined by

$$\boldsymbol{b}^2_\alpha := \{u \in C(\boldsymbol{R}^{n+1}_+); L^{(\alpha)}u = 0 \text{ in the sense of distribution}, u \in L^2(\boldsymbol{R}^{n+1}_+, V)\},$$

where V denotes the $(n+1)$-dimensional Lebesgue measure on $\boldsymbol{R}^{n+1}_+$. The parabolic Bergman space has a reproducing kernel R_α By using the kernel, we define the Toeplitz operators by

$$(T_\mu u)(X) := \int R_\alpha(X, Y)\, u(Y)\, d\mu(Y)$$

with symbol μ, which are positive Radon measures on $\boldsymbol{R}^{n+1}_+$. In the study of the Toeplitz operators, the averaging functions and the Berezin transformations play important roles (cf. [1], [2], [4], [5], [9]).

Definition 1. *For a Radon measure $\mu \ge 0$ on $\boldsymbol{R}^{n+1}_+$, we put*

$$\hat{\mu}^{(\alpha)}(Y) := \mu(Q^{(\alpha)}(Y)) \Big/ V(Q^{(\alpha)}(Y)),$$
$$\tilde{\mu}^{(\alpha)}(Y) := \int R_\alpha(X, Y)^2 d\mu(X) \Big/ \int R_\alpha(X, Y)^2 dV(X),$$

where $Q^{(\alpha)}(Y)$ is an α-parabolic Carleson box, defined by

$$Q^{(\alpha)}(Y) := \{(x_1, \cdots, x_n, t);\ s \le t \le 2s, |x_j - y_j| \le 2^{-1}s^{1/2\alpha}, j = 1, \cdots, n\}.$$

We call them the averaging function and the Berezin transformation of μ, respectively.

Compact Toeplitz operators are characterized as follows.

Theorem A (cf. [5, Theorem 1]). *Let $\mu \ge 0$ be a Radon measure on $\boldsymbol{R}^{n+1}_+$ satisfying*

$$\int (1 + t + |x|^{2\alpha})^{-\tau}\, d\mu(x, t) < \infty \tag{1}$$

for some $\tau \in \boldsymbol{R}$. Then the following statements are equivalent:

(i) *The Toeplitz operator T_μ is compact on $\boldsymbol{b}^2_\alpha$;*
(ii) $\lim_{Y \to \mathcal{A}} \hat{\mu}^{(\alpha)}(Y) = 0$*;*
(iii) $\lim_{Y \to \mathcal{A}} \tilde{\mu}^{(\alpha)}(Y) = 0$,

where $\boldsymbol{R}^{n+1}_+ \cup \{\mathcal{A}\}$ denotes the one point compactification of $\boldsymbol{R}^{n+1}_+$.

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Here, we recall the definition of the Schatten calsses.

Definition 2. *Let* $0 < \sigma < \infty$. *A compact operator* T *on a Hilbert space* $\mathcal{H}$ *is said to be of Schatten* σ*-class if the sequence of all singular values* $(\lambda_j)_j$ *of* T *belongs to the sequence space* l^σ*, where the singular values* λ_j *of* T *mean the eigenvalue of* $|T| := \sqrt{T^*T}$. *We denote by* $\mathcal{S}^\sigma(\mathcal{H})$ *the totality of compact operators on* $\mathcal{H}$ *of Schatten* σ*-class.*

In this talk, we give a characterization for the Toeplitz operator to be of Schatten class. In [7], we obtain the following result.

Theorem B. *Let* $1 \leq \sigma < \infty$. *For a Radon measure* $\mu \geq 0$ *on* $\boldsymbol{R}^{n+1}_+$ *satisfying* (1)*, the following three statements are equivalent:*

(i) $T_\mu \in \mathcal{S}^\sigma$;
(ii) $\hat{\mu}^{(\alpha)} \in L^\sigma(\boldsymbol{R}^{n+1}_+, V^*)$;
(iii) $\tilde{\mu}^{(\alpha)} \in L^\sigma(\boldsymbol{R}^{n+1}_+, V^*)$,

where $dV^*(X) := t^{-(\frac{n}{2\alpha}+1)}dV(X)$.

We mainly treat the remainder case when $0 < \sigma < 1$.

Theorem 1. *Let* $0 < \sigma < 1$. *For a Radon measure* $\mu \geq 0$ *on* $\boldsymbol{R}^{n+1}_+$ *satisfying* (1)*, the Toeplitz operator* T_μ *on* $\boldsymbol{b}^2_\alpha$ *is in the Schatten* σ*-class* $\mathcal{S}^\sigma(\boldsymbol{b}^2_\alpha)$ *if and only if* $\hat{\mu}^{(\alpha)} \in L^\sigma(V^*)$. *Moreover, both norms are comparable.*

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Regularity and Free Boundary Regularity for the p-Laplace Operator in Reifenberg Flat and Ahlfors Regular Domains

Kaj Nyström

A classical result concerning the harmonic measure ω, due to Lavrentiev, states that if $\Omega \subset \mathbf{R}^2$ is a chord arc domain, then ω is mutually absolutely continuous with respect to the surface measure σ, i.e., $d\omega = kd\sigma$ where k is the associated Poisson kernel. Moreover, Lavrentiev proved that $\log k$ is in the space of functions of bounded mean oscillation, defined with respect to σ, on $\partial\Omega$. Furthermore, later Pommerenke proved that $\Omega \subset \mathbf{R}^2$ is vanishing chord arc if and only if $\log k$ is in the space of functions of vanishing mean oscillation, defined with respect to σ, on $\partial\Omega$. Concerning higher dimensional analogues of the results of Lavrentiev and Pommerenke, such results have been established by Carlos Kenig and Tatiana Toro in a sequence of papers. Furthermore, recently John Lewis and I have established appropriate versions, valid for the p-Laplace equation, $1 < p < \infty$, of the results proved by Kenig and Toro. While the results of Kenig and Toro concern harmonic functions and harmonic measure, i.e., the case $p = 2$, our results are valid for the whole range $1 < p < \infty$ and our results are completely new in the case $p \neq 2$, $1 < p < \infty$. Consequently we have also establish versions, valid in all dimensions, for the p-Laplace equation, $1 < p < \infty$, of the classical results of Lavrentiev and Pommerenke mentioned above. The purpose of the talk is to discuss these recent results.

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Riesz potentials and Sobolev embeddings on Morrey spaces of variable exponent

Yoshihiro Mizuta, Eiichi Nakai, Takao Ohno, and Tetsu Shimomura

Let G be a bounded open set in $\mathbf{R}^n$. We denote by d_G the diameter of G. For a measurable function $\alpha(\cdot) : \mathbf{R}^n \to (0, n)$, we define the Riesz potential of order $\alpha(\cdot)$ for an integrable function f on G by

$$I_{\alpha(x)} f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha(x)-n} f(y) dy.$$

In what follows we assume that $f = 0$ outside G.

For $0 \le \nu \le n$ and $p \ge 1$, we define the Morrey space $L^{p,\nu}(G)$ to be the family of all $f \in L^p_{loc}(G)$ for which there is a positive constant C such that

$$\fint_{B(z,r)} |f(x)|^p \, dx \le C^p r^{-\nu} \quad \text{whenever } z \in G \text{ and } 0 < r \le d_G,$$

where $\fint_{B(z,r)}$ is the integral mean over $B(z, r)$. The norm of $f \in L^{p,\nu}(G)$ is defined by the infimum of the constants C satisfying the inequality.

For this spaces, we know the following Morrey version of Sobolev's type inequality for Riesz potentials of functions in $L^{p,\nu}(G)$.

Theorem A([2, Theorem 1.2], [3, Theorem 3.2], [1, Theorem 3.1]). *Let* $0 < \alpha < \nu \le n, \varepsilon > 0$ *and* $p \ge 1$. *Let* f *be a nonnegative measurable function on* G *with the norm of* $f \in L^{p,\nu}(G)$ *is less than* 1. *Suppose that* $p < \nu/\alpha$. *Then there exists a constant* $C > 0$ *such that*

(1) *in case* $p = 1$,

$$\fint_{B(z,r)} I_\alpha f(x)^{p^*} (\log(e + I_\alpha f(x)))^{-(1+\varepsilon)} dx \le C r^{-\nu} (\log(e + 1/r))^{-\varepsilon}$$

for all $z \in G$ *and* $0 < r < d_G$, *where* $1/p^* = 1/p - \alpha/\nu$;

(2) *in case* $p > 1$,

$$\fint_{B(z,r)} I_\alpha f(x)^{p^*} dx \le C r^{-\nu}$$

for all $z \in G$ *and* $0 < r < d_G$.

Our aim in this talk is to show the following Sobolev's type inequality for Riesz potentials of functions in Morrey spaces of variable exponent. For this purpose, we consider continuous exponents $p(\cdot)$ and $q(\cdot)$ on $\mathbf{R}^n$ such that

(P1) $1 \le p_- \equiv \inf_{x\in\mathbf{R}^n} p(x) \le \sup_{x\in\mathbf{R}^n} p(x) \equiv p_+ < \infty$;
(P2) $|p(x) - p(y)| \le C/\log(e + 1/|x-y|)$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$;
(P3) $-\infty < q_- \equiv \inf_{x\in\mathbf{R}^n} q(x) \le \sup_{x\in\mathbf{R}^n} q(x) \equiv q_+ < \infty$;
(P4) $|q(x) - q(y)| \le C/\log(e + (\log(e + 1/|x-y|)))$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$.

We consider a measurable function f satisfying the Morrey condition :

$$\fint_{B(x,r)} |f(y)|^{p(y)} (\log(e + |f(y)|))^{q(y)} \, dy \le r^{-\nu(x)} (\log(e + 1/r))^{-\beta(x)} \tag{1}$$

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for all $x \in G$ and $0 < r < d_G$, where $\nu(\cdot) : \mathbf{R}^n \to (0, n]$ and $\beta(\cdot) : \mathbf{R}^n \to \mathbf{R}$ are bounded measurable functions. We denote by $L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)$ the family of all measurable functions f on G such that f/λ satisfies the condition (1) for some constant $\lambda > 0$. If $1/p(x) - \alpha(x)/\nu(x) > 0$, then we set

$$\Psi(x,t) = t^{p^*(x)}(\log(e+t))^{p^*(x)(q(x)/p(x)+\alpha(x)\beta(x)/\nu(x))},$$

where $1/p^*(x) = 1/p(x) - \alpha(x)/\nu(x)$.

Theorem 1([4, Theorems 2.1, 3.1 and 4.5]). *Let f be a nonnegative measurable function on G satisfying* (1). *Suppose $\alpha_- > 0$ and* $\operatorname{ess\,inf}_{x \in G}(\nu(x)/p(x) - \alpha(x)) > 0$.

(1) *Futher suppose that $t^{p(x)-1}(\log(e+t))^{q(x)}$ is almost increasing and $p_- = 1$. Then there exists a constant $C > 0$ such that*

$$\fint_{B(z,r)} \Psi(x, I_{\alpha(x)}f(x))(\log(e + I_{\alpha(x)}f(x)))^{-(1+\varepsilon)}\,dx \le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)-\varepsilon}$$

for all $z \in G$ and $0 < r < d_G$;

(2) *Futher suppose that there exists a constant $q_0 > 0$ such that $t^{p(x)-1}(\log(e+t))^{q(x)-q_0}$ is almost increasing and $p_- = 1$. Then there exists a constant $C > 0$ such that*

$$\fint_{B(z,r)} \Psi(x, I_{\alpha(x)}f(x))(\log(e + I_{\alpha(x)}f(x)))^{-1}\,dx \le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)}$$

for all $z \in G$ and $0 < r < d_G$;

(3) *Futher suppose that $p_- > 1$. Then there exists a constant $C > 0$ such that*

$$\fint_{B(z,r)} \Psi(x, I_{\alpha(x)}f(x))\,dx \le Cr^{-\nu(z)}(\log(e+1/r))^{-\beta(z)}$$

for all $z \in G$ and $0 < r < d_G$.

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Bergman kernel and asymptotic geometry

Takeo Ohsawa

It is widely recognized that reproducing kernels often encode substantial information on geometric or physical objects. The Bergman kernels, which naturally arise in complex analysis, complex geometry and mathematical physics, are known as such examples. At first we shall review several important breakthroughs in the study of the asymptotics of the Bergman kernels, including the following achievements.

1. Quantitative solutions of the Levi problem on pseudoconvex domains and related asymptotics of the Bergman kernels. (1965, 79, 87, 92)

2. Solution to S.Kobayashi's completeness question for the Bergman metric on hyperconvex domains. (1998, 2004)

3. Asymptotic expansions of the weighted Bergman kernels for positive line bundles over compact complex manifolds. (1990, 98, 99, 2008)

Next, we shall present quite recent results on a conjecture of Kazhdan-Mumford-Yau for towers of ”Bergman metrized” complex manifolds. (2009, preprints)

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Removable sets for continuous solutions of quasilinear elliptic equations with lower order terms

Takayori Ono

We consider quasi-linear second order elliptic differential equations with lower order terms of the form

$$-\operatorname{div}\mathcal{A}(x,\ \nabla u)+\mathcal{B}(x,\ u)=0,$$

where $\mathcal{A}(x,\xi):\mathbf{R}^N\times\mathbf{R}^N\to\mathbf{R}^N$ satisfies structure conditions of p-th order and $\mathcal{B}(x,t):\mathbf{R}^N\times\mathbf{R}\to\mathbf{R}$ is nondecreasing in t. In this talk, we investigate removable sets for continuous solutions of such equations.

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On Hamilton-Jacobi flows and solutions of Aronsson equations

Eero Saksman

The talk, based on a joint work with Petri Juutinen (Jyväskylä), gives a characterization of viscosity solutions for certain Aronsson type equations (e.g. the Δ_∞-equation) in terms of the induced Hamilton-Jacobi flows. The result answers positively a conjecture of E.N. Barron, L. C. Evans and R. Jensen.

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A two phase free boundary problem with applications in Potential Theory

Henrik Shahgholian

In this talk I will present some recent directions, still to be developed, in potential theory, that are connected to a two-phase free boundary problems. The potential theoretic topic that I will discuss is the so called Quadrature Domains.

The most simple free boundary/potential problem that we can present is the following. Given constants $a_\pm, \lambda_\pm > 0$ and two points $x^\pm$ in $\mathbf{R}^n$. Find a function u such that

$$\Delta u = (\lambda_+ \chi_{\{u>0\}} - a_+ \delta_{x^+}) - (\lambda_- \chi_{\{u<0\}} - a_- \delta_{x^-}),$$

where δ is the Dirac mass.

In general this problem is solvable for two Dirac masses. The requirement, somehow implicit in the above equation, is that the support of the measures (in this case the Dirac masses) is to be in included in the positivity and the negativity set (respectively).

In general this problem does not have a solution, and there some strong restrictions on the measures, in order to have some partial results.

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Liouville type theorem associate with the wave equation

Katsunori Shimomura

The well known Liouville's theorem states that every conformal mapping in the n-dimensional Euclidean space ($n \geq 3$) is a similarity or an inversion with respect to a sphere. The conformal mapping associate with the Laplace equation in the following sense : let $U, V \subset \mathbb{R}^{n+1}$ are domains and $f = (f_0, f_1, \ldots, f_n) : U \to V$ a C^2-mapping and φ a positive C^2 function on U. Assume that $\varphi(x) \cdot (u \circ f)(x)$ satisfies the Laplace equation on U for every solution u of the Laplace equation on V. This is possible only if f is a conformal mapping. In the talk, we consider same type of theorem associate with the wave equation instead of the Laplace equation.

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space ($n \geq 2$), and denote the point by $x = (x_0, x_1, \ldots, x_n)$, inner product of x and y by (x, y). We define the quadratic form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n+1}$ by

$$\langle x, y\rangle = (x, Jy) = -x_0y_0 + x_1y_1 + \cdots + x_ny_n, \quad \text{where } J = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

We consider the wave equation

$$Wu(x) := \Big(\frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\Big)u(x) = 0$$

on $\mathbb{R}^{n+1}$. Let $U, V \subset \mathbb{R}^{n+1}$ are domains and $f = (f_0, f_1, \ldots, f_n) : U \to V$ a C^2-mapping and φ a positive C^2 function on U. Assume that $\varphi(x) \cdot (u \circ f)(x)$ satisfies the wave equation on U for every solution u of the wave equation on V. This is possible if and only if f and φ satisfy the following equations on U:

$$\begin{array}{lll} (1) & W\varphi = 0, & \\ (2) & \varphi\, W f_j - 2\langle \nabla\varphi, \nabla f_j\rangle = 0, & (j = 0, 1, \ldots, n) \\ (3) & \langle \nabla f_j, \nabla f_k\rangle = 0, & (0 \leq j < k \leq n) \\ (4) & \langle \nabla f_j, \nabla f_j\rangle = -\langle \nabla f_0, \nabla f_0\rangle, & (1 \leq j \leq n) \end{array}$$

where $\nabla f_j = (\frac{\partial f_j}{\partial x_0}, \frac{\partial f_j}{\partial x_1}, \ldots \frac{\partial f_j}{\partial x_n})$.

Let $\frac{\partial f}{\partial x}(x) := \Big(\frac{\partial f_i}{\partial x_j}\Big)_{i,j=0}^{n}$ be the Jacobian matrix of the mapping f. Then the combination of conditions (3) and (4) is equivalent to the following condition : there exists a function $\lambda(x) \geq 0$ such that

$$\langle \frac{\partial f}{\partial x}(x)u, \frac{\partial f}{\partial x}(x)v\rangle = \lambda(x)^2\langle u, v\rangle \quad \forall x \in U,\ \forall u, v \in \mathbb{R}^{n+1}.$$

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We call the mapping f satisfying this condition J-conformal. And we call J-conformal affine mapping J-similarity.

We can obtain a non-trivial J-conformal mapping J-inversion $\frac{x}{\langle x, x\rangle}$.

There is another non-trivial J-conformal mapping :

$$B(x) = (\frac{\langle x, x\rangle + 1}{2(x_0 + x_1)}, \frac{\langle x, x\rangle - 1}{2(x_0 + x_1)}, \frac{x_2}{x_0 + x_1}, \ldots, \frac{x_n}{x_0 + x_1}).$$

Our main result is the following.

Theorem. Let f be a J-conformal mapping. Then f is a J-similarity, or a composition of a J-similarity with a J-inversion, or a composition of a J-similarity with B in the above.

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Sobolev inequalities in variable exponent Orlicz spaces

Tetsu Shimomura

In this talk, we discuss Sobolev's embeddings for Sobolev–Orlicz functions with $\nabla u \in L^{p(\cdot)} \log L^{q(\cdot)}(\Omega)$ for $\Omega \subset \mathbf{R}^n$.

Following Cruz-Uribe and Fiorenza [1], let $p : \mathbf{R}^n \to [1, \infty)$ and $q : \mathbf{R}^n \to \mathbf{R}$ be continuous functions. We will be considering spaces of type $L^{p(\cdot)} \log L^{q(\cdot)}(\Omega)$. For simplicity we denote the function defining the space by Φ, i.e. $\Phi(x, t) = t^{p(x)}(\log(c_0 + t))^{q(x)}$.

We assume that our variable exponents p and q are continuous functions on $\mathbf{R}^n$ satisfying:

(p1) $1 \le p^- := \inf_{x\in\mathbf{R}^n} p(x) \le \sup_{x\in\mathbf{R}^n} p(x) =: p^+ < \infty$;

(p2) $|p(x) - p(y)| \le \dfrac{C}{\log(e + 1/|x - y|)}$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$;

(p3) $|p(x) - p(y)| \le \dfrac{C}{\log(e + |x|)}$ whenever $|y| \ge |x|/2$;

(q1) $-\infty < q^- := \inf_{x\in\mathbf{R}^n} q(x) \le \sup_{x\in\mathbf{R}^n} q(x) =: q^+ < \infty$;

(q2) $|q(x) - q(y)| \le \dfrac{C}{\log(e + \log(e + 1/|x - y|))}$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$.

Moreover, we assume that

(Φ_1) there exists $c_0 \in [e, \infty)$ such that $\Phi(x, \cdot)$ is convex on $[0, \infty)$ for every $x \in \mathbf{R}^n$.

Note that (Φ_1) implies the following condition:

(Φ_2) $t \mapsto t^{-1}\Phi(x, t)$ is nondecreasing on $(0, \infty)$ for fixed $x \in \mathbf{R}^n$.

We define the space $L^\Phi(\Omega)$ to consist of all measurable functions f on the open set $\Omega \subset \mathbf{R}^n$ with

$$\int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx < \infty$$

for some $\lambda > 0$. We define the norm

$$\|f\|_{\Phi(\cdot,\cdot)(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}$$

for $f \in L^\Phi(\Omega)$. These spaces have been studied in [1, 5]. In case $q \equiv 0$, $L^\Phi(\Omega)$ reduces to the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$. The Sobolev space $W^{1,\Phi}(\Omega)$ consists of those functions $u \in L^\Phi(\Omega)$ with a distributional gradient satisfying $|\nabla u| \in L^\Phi(\Omega)$. Further we denote by $W_0^{1,\Phi}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the space $W^{1,\Phi}(\Omega)$.

Let

$$1/p^*(x) = 1/p(x) - 1/n.$$

Theorem. *Let p and q satisfy the above conditions. If $p^+ < n$, then*

$$\|u\|_{\Psi(\cdot,\cdot)(\Omega)} \le c_1\|\nabla u\|_{\Phi(\cdot,\cdot)(\Omega)}$$

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for every $u \in W_0^{1,\Phi}(\Omega)$, *where* $\Phi(x,t) := (t\log(c_0+t)^{q(x)/p(x)})^{p(x)}$ *and* $\Psi(x,t) := (t\log(c_0+t)^{q(x)/p(x)})^{p^*(x)}$.

This extends [2, Proposition 4.2(1)] and [3, Theorem 3.4] which dealt with the case $q \equiv 0$.

This is a joint work with P. Hästö, Y. Mizuta and T. Ohno.

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Dimension estimates for exceptional sets of Euclidean and sub-Riemannian fractals

Jeremy T. Tyson

Potential theory is one of the primary tools useful for the computation or estimation of dimensions of fractals and more general nonsmooth sets and measures. Falconer (1988) used energy estimates to compute almost sure dimensions of generic representatives in parameterized families of self-affine Euclidean fractals. Falconer and Miao (2008) gave more precise estimates on the size of the exceptional set. We describe a heuristic principle relating self-affine fractal geometry in Euclidean space and self-similar fractal geometry in sub-Riemannian Carnot (nilpotent stratified Lie) groups. We then adapt Falconer's arguments to the sub-Riemannian setting. Self-similar sets in Carnot groups turn out to be generically horizontal: they exhibit the smallest possible dichotomy between sub-Riemannian and Euclidean dimensions.

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Sets of determination for harmonic functions in an abstract harmonic space

Hidenobu Yoshida

The term “set of determination” was introduced by Gardiner [4] in 1993. Let B be the unit ball in the Euclidean space $\mathbf{R}^n$ $(n \geq 2)$. Let $E \subseteq B$ and let u be a positive harmonic function on B with the correspondng measure μ_u:

$$u(x) = \int_{\partial B} P(x, y) d\mu_u(y) \quad (x \in B),$$

where $P(x, y)$ is the Poisson kernel for B with pole y in the boundary ∂B of B. He characterized the sets E of determination for which

$$\inf_{X \in E} \frac{h(x)}{u(x)} = \inf_{x \in B} \frac{h(x)}{u(x)}$$

for all positive harmonic functions h on B, i.e. if there is a positive constant c such that $h(x) \geq cu(x)$ on E, then $h(x) \geq cu(x)$ on B.

Before him, the special cases of $u(x) = P(x, y)$ were originally considered by Beurling $(n = 2)$ [2] and Dahlberg (higher dimension) [3].

Soon after him, Aikawa [1] extended his results to subsets E of an NTA domain D. His caracterizations which we are specially interested in are

Characterization (1): for all ϱ $(0 < \varrho < 1)$, the “bubble” set of E

$$E_\varrho = \cup_{x \in E} B(x, \varrho\delta(x))$$

is not minimally thin at y for μ_u-almost every boundary point y, where $B(x, r)$ denotes the ball with center x and radius r, $\delta(x) = dist(x, \partial D)$ and μ_u is the Martin representing measure of u,

Characterization (2): E includes a nontangential sequence converging to y for ω-almost every boundary point $y \in \partial D$, where ω is the harmonic measure at a point in D (if D is a Lipschitz domain, then ω can be replaced by the surface measure on ∂D).

In his paper, Aikawa raised

Question. Can one extend these results to a general Martin space?

When the following Ranosová’s results are seen, this presentation is seemed to be natural. From the same view points as them, Ranosová also characterized the sets of determination for positive parabolic functions h satisfying the heat equation $\Delta h = \partial h/\partial t$ on a slab $\mathbf{R}^n \times (0, T)$ $(0 < T \leq +\infty)$ [8] and for positive solutions h of the Helmholtz equation $\Delta h = 2\alpha h$ $(\alpha > 0)$ on $\mathbf{R}^n$ [7].

Characterization (1) will be connected with Fatou-Naïm-Doob theorem which was extended in the axiomatic system of Brelot by Gowrisankaran [5] and also extended to a very general setting by Sibony [9] (also see Taylor [10]). Let (X, χ) be a measurable space and H be convex cones of functons on X valued in $(0, +\infty)$ which satisfy several hypotheses. A function h on X is said to be “positive harmonic” if $h \in H$. In addition there is a mesurable boundary space (B, β)

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such that each harmonic function $h \in H$ has the integral representation by a unique positive measure μ_h on B as

$$h(x) = \int_B K_b(x) d\mu_h$$

where K_b is a minimal harmonic function for $b \in B$. The fine limit of a function f on X at b is definied by an usual way and denoted by $(fine - lim\ f)(b)$.

The Fatou-Naïm-Doob theorem. *If $u, h \in H$ and are μ_u, μ_h are their representing measures, then*

$$(fine - lim\ h/u)(b) \ = \ (d\mu_h/d\mu_u)(b) \quad \mu_u - a.e.$$

Characterization (2) will be connected with the following Fatou type theorem due to Korányi and Taylor [6]. Assume that $\hat{X} = X \cup B$ is a topological space. An "admissible system" A is a functon on $\hat{X} \backslash X = B$ with $A(b) \subset X$ and $b \in B$ is a limit point of $A(b)$. A function f on X "converges A-admissibly" to λ at b if for any $\varepsilon > 0$ there exists a neighbourhood U of b such that $|f(x) - \lambda| < \varepsilon$ for all $x \in U \cap A(b)$. This is indicated by writing $\lambda = (A - lim\ f)(b)$.

The Fatou theorem. *Let A be an admissible system and let u be a positive harmonic function with the representing measure μ_u. If for any positive harmonic function h*

$$(fine - lim\ h/u)(b) = 0 \quad \Rightarrow \quad (A - lim\ h/u)(b) = 0 \quad \mu_u - a.e.,$$

then

$$(A - lim\ h/u)(b) \ = \ (d\mu_h/d\mu_u)(b) \quad \mu_u - a.e.$$

In this talk, we shall show the relations between sets of determination and these theorems.

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