Time quasi-periodic solutions to the nonlinear Klein-Gordon equations

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In this talk, we shall consider the nonlinear Klein-Gordon equations (NLKG):

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u + u + u^{2p+1} + h.o.t. = 0
\]

where \( p \in \mathbb{N} \) is arbitrary.
And also the nonlinear Schrödinger (NLS):

\[ i \frac{\partial u}{\partial t} + \Delta u + |u|^{2p} u + h.o.t. = 0, \]

since the method applies to both equations.
The dynamics of the linear Klein-Gordon equations (Schrödinger equations) are completely described by the corresponding eigenfunctions and eigenvalues – the eigenfunctions form a spanning set and all solutions are their linear superpositions.

In the presence of the nonlinear term, there is no longer this linear superposition. Simplest generalizations of eigenfunctions are the time-periodic solutions (to the nonlinear equations). The next generalizations are the quasi-periodic solutions, solutions which are periodic with several frequencies.
Example. When $\alpha$ is irrational, the function

$$f = \cos t + \cos \alpha t$$

is quasi-periodic in time with two frequencies.
Quasi-periodic solutions can be seen as bifurcations of linear superpositions of eigenfunctions. (If there is only one eigenfunction, then bifurcation leads to periodic solutions, as mentioned earlier.) They are hence quite natural. Clearly not all linear solutions will bifurcate to nonlinear solutions.
So one of our tasks is to impose reasonable conditions on the linear solutions, which are often algebro-geometric and arithmetic, as we shall explain, so that they could become solutions to the nonlinear equations, after small deformation. This bifurcation analysis, will be the main topic of this talk.

(The algebraic aspect could also be understood from a variational point of view, more on this later.)
Generally speaking, in order to have solutions which are quasi-periodic, the equations need to be posed on a compact manifold. The most basic is perhaps the flat torus $\mathbb{T}^d$, which leads to time quasi-periodic but space periodic solutions.

The plan of the talk is to give an idea of the proof of the following:
Theorem. There is a set of global solutions to the NLKG (and NLS) in arbitrary dimension $d$ and for arbitrary nonlinearity $p$. These are quasi-periodic solutions, which are Gevrey functions in space and time.

- Along the proof, we establish a new method, which is based on multiscale analysis (Fröhlich and Spencer), harmonic analysis (Bourgain, Bourgain-Goldstein-Schlag) and algebraic analysis (W).
NLS with external parameters $\omega$ (spectrally defined Laplacian):
- Bourgain (Multiscale Analysis), 1998;
- Eliasson-Kuksin (KAM $\Rightarrow$ infinite sequence of change of coordinates), 2010

The role of the external parameters is to control the resonances. Controlling the resonances is fundamental to the subject, as resonances lead to delocalization – the opposite of KAM solutions, which are localized.
More recently, there are the following two results for the (original) NLS mentioned on the first page:

- Wang (Multiscale Analysis), 2016
- Procesi-Procesi (KAM method), 2016;

Since the (original) NLS do not depend on parameters, one needs to extract parameters from the nonlinear term, to deal with resonances. This is the key difference with the previous works of Bourgain and Eliasson-Kuksin.
NLKG
• NLKG with external parameters $\omega$ (spectrally defined Laplacian): Bourgain (sketch of proof), 2005
• NLKG: Wang, 2017

As for NLS, one of the key differences is that [W] treats the original NLKG by extracting parameters from the nonlinear term; while Bourgain, a modified, parameter dependent NLKG. Another essential difference is that [W] uses directly the second-order equation, which avoids non self-adjoint issues.
For NLKG, which is a second-order in time equation, Multiscale Analysis is, so far, the only method that works. As we will see, one of the main differences between NLKG and NLS is that the linear wave operator has dense spectrum, in other words, there is no spectral gap, which makes the problem much more difficult. Moreover, following a theorem of Duistermaat and Guillemin, NLKG could represent a generic case.
Coming back to the problem at hand, we start from the linear equations posed on the torus $\mathbb{T}^d$: 

$$i \frac{\partial u}{\partial t} + \Delta u = 0;$$

and

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u = 0,$$

where the mass term 1 is added, in order to avoid the singularity at 0.
These linear equations are solved using Fourier series and spectral theory. The spectrum of the Laplacian:

$$\sigma(\Delta) = \{ j_1^2 + j_2^2 + \ldots + j_d^2 | j_k \in \mathbb{Z}, k = 1, 2, \ldots, d \};$$

while that of the wave operator $D$, $D = \sqrt{-\Delta} + 1$:

$$\sigma(D) = \{ \sqrt{j_1^2 + j_2^2 + \ldots + j_d^2 + 1} | j_k \in \mathbb{Z}, k = 1, 2, \ldots, d \}.$$

They play an essential role in our analysis. We note that in $d \geq 2$, they are highly **degenerate**. The spectrum of the wave operator $D$ is moreover **dense**, which makes NLKG more difficult, as mentioned earlier.
Below we explain our method using NLKG. The linear equation admits solutions of the form

$$\cos(-(\sqrt{j^2 + 1})t + j \cdot x), \ j \in \mathbb{Z}^d.$$ 

More generally, let $u^{(0)}$ be a solution of finite number of frequencies, $b$ frequencies, to the linear equation:

$$u^{(0)}(t, x) = \sum_{k=1}^{b} a_k \cos(-(\sqrt{j_k^2 + 1})t + j_k \cdot x).$$
Note that at this stage, the frequencies $\omega := \omega^{(0)}$ are fixed, because $u^{(0)}$ is a solution to the linear equation.

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u = 0.$$ 

Time quasi-periodic solutions, maybe viewed as “periodic on higher dimensional torus”: to each frequency, we add a dimension; so as an ansatz, we may seek solutions to the nonlinear equations in the form:

$$u(t, x) = \sum_{(n,j) \in \mathbb{Z}^{b+d}} \hat{u}(n,j) \cos(n \cdot \omega t + j \cdot x)$$

where $b$ is the number of basic frequencies in time, with the frequency $\omega \in \mathbb{R}^b$ to be determined.
In this formulation, $u^{(0)}$ may be written in the form:

$$u^{(0)}(t, x) = \sum_{k=1}^{b} a_k \cos\left(-\left(\sqrt{j_k^2 + 1}\right)t + j_k \cdot x\right)$$

$$= \sum_{k=1}^{b} \hat{u}^{(0)}(\mp e_k, \pm j_k) \cos\left(\mp e_k \cdot \omega^{(0)} t \pm j_k \cdot x\right),$$

where $e_k = (0, 0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^b$ is a unit vector, with the only non-zero component in the $k$th direction, $\omega_k^{(0)} = \sqrt{j_k^2 + 1}$ and

$$\hat{u}(-e_k, j_k) = \hat{u}(e_k, -j_k) = a_k/2.$$
Remark. The time part of the Fourier series can be understood using the substitution:

$$\frac{\partial}{\partial t} \mapsto \sum_{i=1}^{b} \omega_i \frac{\partial}{\partial y_i},$$

where $y_i$ are the variables on the higher dimensional torus in time.
Using the cosine series ansatz, the linear Klein-Gordon operator:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u,$$

becomes the diagonal matrix operator

$$\text{diag} [(n \cdot \omega)^2 - j^2 - 1],$$

on $\ell^2(\mathbb{Z}^{b+d})$. 
As mentioned earlier, $\omega$ is to be determined. Since we perturb from the linear solution $u^{(0)}$. As an initial approximation $\omega = \omega^{(0)}$ are the linear frequencies $\sqrt{j_k^2 + 1}, k = 1, 2, ..., b$.

So the diagonals can be 0. The off-diagonal matrix is a convolution matrix, since the nonlinear term $u^{2p+1}$ is multiplicative.
Since we seek small solutions, we may use a Newton scheme and linearize about $u^{(0)}$, leading to the linear self-adjoint matrix operator:

$$T = D + P = \text{diag}[(n \cdot \omega^{(0)})^2 - j^2 - 1] + \hat{u}^{(0)} \ast \hat{u}^{(0)} \ast,$$

where for illustration, we have set $p = 1$ and taken the cubic nonlinearity.

To illustrate the convolution matrix $P$, let us take a simplified example and assume $u := u^{(0)} = \cos x, x \in \mathbb{R}$. Then since

$$u^2 = \cos^2 x = \frac{\cos 2x + 1}{2},$$

$P$ has diagonal entries $1/2$ and two off-off diagonal entries $1/2$. 
In the nonlinear construction to prove the Theorem, the key is the invertibility of the (truncated) linearized operators, i.e., the Green’s function estimates. The linearized operator is akin to:

\[ H = -\Delta + V(n\omega), \]

where \(-\Delta\) is a discrete Laplacian and \(V\) is a quasi-periodic potential and the frequency \(\omega\) is a parameter.

Analysis of quasi-periodic operators is a subject founded by Sinai (see the paper Dinaburg-Sinai). One of the goals is to prove that \(H\) has pure point spectrum with localized eigenfunctions, the so called Anderson localization (AL).
In 1d, the first results on quasi-periodic AL were proven by Sinai (1987), and by Fröhlich-Spencer-Wittwer (1990) for the cosine potential:

\[ \cos(n\omega + \theta). \]

In 2d, there is the paper of Bourgain-Goldstein-Schlag for general analytic potentials. (Cf. also the generalization by Bourgain to arbitrary dimensions.) The methods in [BGS] and [B] for \( d > 1 \) are essentially different from that in \( d = 1 \).
The main difference of our operator $T$ and the $H$ in AL is that the diagonals can be 0. This is because NLKG is a deterministic equation and there is no parameter at this stage. So we need to analyze the geometry of the zero-set:

$$C := \{(n, j) \in \mathbb{Z}^{b+d} | (n \cdot \omega)^2 - j^2 - 1 = 0\},$$

in order that it does not percolate, leading to delocalization. We call the above set, the characteristic. Its geometry will play an essential role in estimating $T^{-1}$. (This step is non-perturbative.)
For the usual quasi-periodic Anderson models mentioned above:

\[ H = -\Delta + V(n\omega), \]

the characteristic set \( C = \emptyset \), after excision in \( \omega \). This is not possible for \( T \), as explained earlier. The next best thing is then trying to achieve some separation property (clustering property) on \( C \).

The key question becomes: separated, but relative to what ??

Answer: relative to a precise notion of connected, which we define below.
Recall that

$$T = D + P$$

and $P$ is a convolution matrix. So there is a natural notion of connected, namely, we say two points $x$ and $y \in \mathbb{Z}^{b+d}$ are connected if

$$(x - y) \in \text{supp } P.$$

We want then that the connected sets on the characteristic $C$ are small. Now the control on the size needs to be quantitatively optimal.
VIII. The hyperplanes and the hyperboloids

We obtain this quantitative control by making an algebraic description of the connected sets and variable reductions as follows. (Below we write $\omega$ for $\omega^{(0)}$.)

- The algebraic description: by definition, if $(n, j) \in \mathcal{C}$ and $(n', j') \in \mathcal{C}$ are connected, then for some $(\nu, \eta) \in \text{supp } P$,

  \[ n' = n + \nu \]

  and

  \[ j' = j + \eta, \]
and the following two equations are satisfied:

\[
(n \cdot \omega)^2 - j^2 - 1 = 0,
\]

\[
[(n + \nu) \cdot \omega]^2 - (j + \eta)^2 - 1 = 0,
\]

for some \((\nu, \eta) \in \text{supp } P\).
In fact, it is easier to illustrate the idea on Schrödinger, so let us first do that. The characteristics for Schrödinger is:

\[ \mathcal{C} := \{(n, j) \in \mathbb{Z}^{b+d} | n \cdot \omega + j^2 = 0\} \].

So if \((n, j) \in \mathcal{C}\) and \((n', j') \in \mathcal{C}\) are connected, then the following two equations are satisfied:

\[ n \cdot \omega + j^2 = 0, \]
\[ (n + \nu) \cdot \omega + (j + \eta)^2 = 0. \]
Subtracting the two equations leads to a linear equation in $j$, i.e., a hyperplane, parametrized by $(\nu, \eta)$, which are determined by $u^{(0)}$, the solution to the linear equation.

So a block, a connected set can be described by a system of linear equations; the size of a block, the size of a connected set is then bounded above by the size of the corresponding compatible linear system.

Under appropriate geometric conditions on $u^{(0)}$, variable reductions give that the size is at most $2d$, which is essentially optimal. (Recall that $j$ is the unknown and is a $d$-vector, and the characteristic has two branches.)
For Wave, the same reasoning leads to hyperboloids. The variable reductions are much more difficult, because the hyperboloids

\[(n \cdot \omega)^2 - j^2 - 1 = 0,\]

have an essentially flat direction away from the origin, compatible with convolution (translation invariance). So aside from geometric conditions akin to that for Schrödinger, arithmetic conditions also come into play.
Recall that the frequencies

\[ \omega_k = \sqrt{j_k^2 + 1}, \]

where

\[ j_k^2 := j_{k,1}^2 + j_{k,2}^2 + \ldots + j_{k,d}^2. \]

- We impose the condition that \((j_k^2 + 1) (j_k \neq 0)\) are distinct and **square-free**, for \(k = 1, 2, \ldots, b\).

As a **consequence**, there is the usual (linear) independence:

\[ \|n \cdot \omega\|_T \neq 0, \]

where \(n \neq 0;\)
as well as the **quadratic** non-equality:

$$\| \sum_{k<\ell} n_k n_{\ell} \omega_k \omega_{\ell} \|_T \neq 0,$$

where $\sum_{k<\ell} |n_k n_{\ell}| \neq 0$.

The linear independence is familiar – it takes care of the (usual) small-divisors arising from multi-scale analysis.
The quadratic non-equality is new and takes care of hyperbolicity; moreover it **doubles** as a small-divisor lower bound for the dense linear flow:

If

\[ n \cdot \omega + \sqrt{j^2 + 1} \neq 0, \]

then

\[ |n \cdot \omega + \sqrt{j^2 + 1}| > c |n|^{-\alpha}, \quad |n| \neq 0, \]

where \( c, \alpha > 0 \), cf. [W. Schmidt].
It is worth noting that these are new types of small-divisors, which are geometric in origin and do not appear in NLS. The quadratic Diophantine property arises because NLKG is a second order in time equation. This appears to be new.
Once the separation is achieved, one can extract parameters from the nonlinearity – the parameters are the amplitudes $a$, the Fourier coefficients of the linear solution $u^{(0)}$. Bourgain’s non-resonant method becomes available to deal with small-divisors.

There is, however, a new analysis point in our application, namely for measure estimates, it is more convenient to lower regularity and work in the Lipschitz setting. (Recall that the Theorem is in $C^\infty$-Gevrey setting.) With this addition, one may then proceed to prove the Theorem.
Using variational method, Rabinowitz constructed time-periodic solutions for the wave equation in one dimension. There is, however, no variational constructions for time quasi-periodic solutions – because the critical points are very degenerate.

So the bifurcation analysis described above, the algebraic part of the argument could be seen as sorting out this degeneracy, although our method, particularly for NLKG, is really not variational.

**Remark.** Our method is closer to a Lagrangian approach.