On stability of scale-critical circular flows in a two-dimensional exterior domain

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Mathematics for Nonlinear Phenomena: Analysis and Computation
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1-1. Two-dimensional Navier-Stokes equations

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Incompressible Navier-Stokes equations

\[
\begin{aligned}
\frac{\partial u}{\partial t} + u \cdot \nabla u &= \Delta u - \nabla p, & t > 0, & x \in \Omega, \\
\text{div} u &= 0, & t \geq 0, & x \in \Omega, \\
u &= 0, & t > 0, & x \in \partial \Omega, \\
u|_{t=0} &= u_0, & x \in \Omega.
\end{aligned}
\]

\[u = (u_1(t, x), u_2(t, x)):\text{ velocity field}\quad p = p(t, x):\text{ pressure}\]

\[\Omega = \mathbb{R}^2\text{ or exterior domain}\]
1-2. Scaling and self-similar solution

If \((u, p)\) solves (NS) in \(\mathbb{R}_+ \times \mathbb{R}^2\) then \((u_\lambda, p_\lambda)\) also solves (NS) in \(\mathbb{R}_+ \times \mathbb{R}^2\):

\[
    u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x), \quad \lambda > 0
\]

Lamb-Oseen vortex (forward self-similar solution, circular swirling flow)

\[
    U^G(t, x) = \frac{x^\perp}{2\pi|x|^2}(1 - e^{-\frac{|x|^2}{4t}}), \quad x^\perp = (-x_2, x_1)
\]

(i) For each \(\alpha \in \mathbb{R}\) the velocity \(\alpha U^G\) is a forward self-similar solution to (NS) (with \(\Omega = \mathbb{R}^2\)):

\[
    U^G(t, x) = U^G(t, x), \quad \lambda > 0.
\]

\[
    |U^G(t, x)| \leq C \min\{ |x|^{-1}, t^{-\frac{1}{2}} \}; \text{ infinite energy flow}
\]

(ii) The vorticity field is the two-dimensional Gaussian:

\[
    (\text{rot} U^G)(t, x) = (\partial_1 U^G_2 - \partial_2 U^G_1)(t, x) = G(t, x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}.
\]
\[ t^{\frac{1}{2}} u(t, x) = v(\tau, \xi), \quad \tau = \log(1 + t), \quad \xi = \frac{x}{t^{\frac{1}{2}}} \]

The following two are equivalent.

(i) Asymptotic stability of the Lamb-Oseen vortex \( \alpha U^G \) for the two-dimensional Navier-Stokes equations

(ii) Two-dimensional stability of the Burgers vortex with circulation \( \alpha \), which is a stationary solution to the three-dimensional Navier-Stokes equations


Two-dimensional vorticity equations: \( \omega = \text{rot} u \)

\[ \partial_t \omega + u \cdot \nabla \omega = \Delta \omega, \quad t > 0, \quad x \in \mathbb{R}^2. \]

Let $V(x) = x^\perp f(|x|)$, $x^\perp = (-x_2, x_1)$ for a scalar function $f$ in $\mathbb{R}^2$.

(i) $\text{div } V(x) = x^\perp \cdot \frac{x}{|x|} f'(|x|) = 0$

(ii) $V \cdot \nabla V = \frac{1}{2} \nabla |V|^2 + V^\perp \text{rot } V = \frac{1}{2} \nabla |V|^2 + \nabla P$, where

$$P(x) = \int_{|x|}^{\infty} f'(r)w(r) \, dr, \quad w(|x|) = \text{rot } V(x).$$

Therefore, we have $\mathbb{P}(V \cdot \nabla V) = 0$, where $\mathbb{P}$ is the Helmholtz projection.

(iii) For any radial function $\rho$ we have

$$\int_{\Omega} \rho g \, V \cdot \nabla g \, dx = -\frac{1}{2} \int_{\Omega} |g|^2 V \cdot \nabla \rho \, dx = 0, \quad g \in C_0^\infty(\Omega).$$
2-1. Stability of the Lamb-Oseen vortex in exterior problem

**Problem:** Large time behavior of solutions to (NS) for the initial velocity

\[ u_0(x) = \alpha U^G(1, x) + v_0(x), \quad |x| \gg 1, \quad v_0 \in L^2_\sigma(\Omega). \]

**Difficulty when \( \Omega \) in an exterior domain (even for \( 0 < |\alpha| \ll 1 \))**

(i) The vorticity equation is not useful to obtain the uniform bound in the scale-critical norms.

(ii) The Hardy inequality is not available in the two-dimensional case:

\[
\left\| \frac{1}{|x|} f \right\|_{L^2(\Omega)} \leq C \| \nabla f \|_{L^2(\Omega)}, \quad f \in \dot{W}^{1,2}_0(\Omega).
\]

In particular, one can not expect the coercive (positive) estimate such as

\[
\langle -\Delta v + \alpha (U^G \cdot \nabla v + v \cdot \nabla U^G) + v \cdot \nabla v, v \rangle_{L^2(\Omega)} \geq c \| \nabla v \|^2_{L^2(\Omega)}
\]

for \( v \in C^\infty_{0,\sigma}(\Omega) \), even when \( 0 < |\alpha| \ll 1 \).
Theorem 1 (Gallay-M. (2013), M. (2015)).

There is a constant $\delta > 0$ such that for any $u_0 \in L^2_{\sigma, \infty}(\Omega)$ of the form

$$u_0 = \alpha U^G|_{t=1} + v_0 , \quad |\alpha| \leq \delta , \quad v_0 \in L^2(\Omega)^2$$

there exists a unique solution $u$ to (NS) with initial data $u_0$ satisfying

$$\lim_{t \to \infty} t^{\frac{k}{2}} \| \nabla^k (u(t) - \alpha U^G(t)) \|_{L^2(\Omega)} = 0 , \quad k = 0, 1 .$$

- The local $L^2$ stability is proved by Iftimie-Karch-Lacave (2011).

- The similar result holds even when $\alpha U^G$ is replaced by the strong solution $U$ obtained in Kozono-Yamazaki (1995) which satisfies

$$\sup_{t > 0} \| U(t) \|_{L^2, \infty}(\Omega) + \sup_{t > 0} t^{\frac{1}{4}} \| U(t) \|_{L^4(\Omega)} \leq \delta \ll 1 .$$
(i) The logarithmic growth energy estimate for the perturbation $v(t) = u(t) - \alpha U^G(t)$:

$$\|v(t)\|_{L^2(\Omega)}^2 + \int_1^t \|\nabla v(s)\|_{L^2(\Omega)}^2 \, ds \leq C(\|v_0\|_{L^2(\Omega)}) + C_0 \alpha^2 \log(1 + t), \quad t > 1.$$ 

(ii) The analysis of the low frequency part of $v$ by using the argument in Borchers-Miyakawa (1992) and Kozono-Ogawa (1993).

Note. The argument essentially uses the scale-critical temporal decay of $U^G$ such as $\|U^G(t)\|_{L^4(\Omega)} \leq C t^{-1/4}$. 

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We set

\[ U(x) = \lim_{t \to 0} U^G(t, x) = \frac{x^\perp}{2\pi|x|^2}, \quad x \in \mathbb{R}^2 \setminus \{0\}. \]

- For each \( \alpha \in \mathbb{R} \) the velocity \( \alpha U \) is a stationary solution to the following Navier-Stokes system in \( \Omega = \{x \in \mathbb{R}^2 | |x| > 1\} \).

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u &= \Delta u - \nabla p, & t &> 0, \quad |x| > 1, \\
\text{div} \ u &= 0, & t &\geq 0, \quad |x| > 1, \\
\ u &= \frac{\alpha}{2\pi} x^\perp, & t &> 0, \quad |x| = 1, \\
\ u|_{t=0} &= u_0, & |x| &> 1.
\end{aligned}
\]
3-2. Two important aspects

Steady circular flow with a scale-critical decay

\[ \alpha U(x) = \frac{\alpha x^\perp}{2\pi|x|^2}, \quad \alpha^2 P(x) = -\frac{\alpha^2}{8\pi^2|x|^2}, \quad x^\perp = (-x_2, x_1) \]

(I) Simple model of the flow around a rotating obstacle

- The existence of two-dimensional periodic flows around a rotating obstacle is still open in general.

- Hishida (2015): the asymptotic estimates for the steady Stokes flows around a rotating obstacle.

- The unique existence and the stability for the three-dimensional problem: Borchers (1992), Galdi (2003), Silvestre (2004), Farwig-Hishida (2007), Hishida-Shibata (2009), ...
3-2. Two important aspects

Steady circular flow with a scale-critical decay

\[ \alpha U(x) = \frac{\alpha x^\perp}{2\pi|x|^2} \]
\[ \alpha^2 P(x) = -\frac{\alpha^2}{8\pi^2|x|^2} \]
\[ x^\perp = (-x_2, x_1) \]

(II) Stability of scale-critical flows

- The unique existence of stationary solutions having the spatial decay \( O(|x|^{-1}) \) is proved by Yamazaki (2011) under some symmetry conditions on both domains and given data.

- Hillairet and Wittwer (2013) proved the existence of the steady exterior flows in \( \Omega = \{x \in \mathbb{R}^2 \mid |x| > 1\} \) near \( \alpha U \) for large \( |\alpha| \).

The stability of these stationary solutions decaying in the order \( O(|x|^{-1}) \) is widely open even when the initial perturbation is small.
3-3. Local $L^2$ stability of $\alpha U$ for small $|\alpha|$ 

Theorem 2 (M.).

For any sufficiently small $|\alpha|$ there is a constant $\varepsilon = \varepsilon(\alpha) > 0$ such that if $\|u_0 - \alpha U\|_{L^2(\Omega)} \leq \varepsilon$ then there exists a unique solution $u$ to (NS$_\alpha$) satisfying

$$\lim_{t \to \infty} t^{\frac{k}{2}} \|\nabla^k (u(t) - \alpha U)\|_{L^2(\Omega)} = 0, \quad k = 0, 1.$$ 

The Helmholtz projection $\mathbb{P} : L^2(\Omega)^2 \to L^2_{\text{div}}(\Omega)$ satisfies $\mathbb{P}\nabla p = 0$.

The perturbed Stokes operator

$$D(A_\alpha) = W^{2,2}(\Omega)^2 \cap W^{1,2}_0(\Omega)^2 \cap L^2_{\text{div}}(\Omega)$$

$$A_\alpha v = -\mathbb{P}\Delta v + \alpha \mathbb{P}(U \cdot \nabla v + v \cdot \nabla U), \quad v \in D(A_\alpha)$$
3-4. Spectral analysis for the linearized operator

Key ingredient

Spectral analysis of the perturbed Stokes operator $A_\alpha$ by using the polar coordinates and the streamfunction-vorticity formulation

We set

$$F_n(z; \alpha) = \int_{1}^{\infty} s^{1-|n|} K_{\mu_n(\alpha)}(s z) \, ds, \quad \text{Re}(z) > 0, \quad n \in \mathbb{Z} \setminus \{0\},$$

where $K_\mu$ is the modified Bessel function of second kind of order $\mu$

$$K_\mu(z) = \frac{1}{2} \int_{0}^{\infty} e^{-\frac{z}{2}(t+\frac{1}{t})} t^{-\mu-1} \, dt, \quad \text{Re}(z) > 0$$

and

$$\mu_n(\alpha) = \left( n^2 + \frac{i\alpha}{2\pi} n \right)^{\frac{1}{2}}, \quad n \in \mathbb{Z} \setminus \{0\}$$
3-4. Spectral analysis for the linearized operator

\[ A_{\alpha}v = -\mathbb{P}\Delta v + \alpha\mathbb{P}(U \cdot \nabla v + v \cdot \nabla U), \quad U(x) = \frac{x^+}{2\pi|x|^2} \]

**Structure of the spectrum of \( A_{\alpha} \) (M.)**

Let \( \alpha \in \mathbb{R} \). Then the following statements hold.

1. \( \sigma(-A_{\alpha}) = \mathbb{R}_{\ominus} \cup \sigma_{disc}(-A_{\alpha}) \) and

   \[ \sigma_{disc}(-A_{\alpha}) = \{ \lambda \in \mathbb{C} \setminus \mathbb{R}_{\ominus} \mid F_n(\sqrt{\lambda}; \alpha) = 0 \text{ for some } n \in \mathbb{Z} \setminus \{0\} \} \]

2. For any \( \epsilon \in (0, \frac{\pi}{2}) \) there is a constant \( \delta_{\epsilon} > 0 \) such that if \( |\alpha| \leq \delta_{\epsilon} \) then the sector \( \Sigma_{\pi-\epsilon} \) is included in the resolvent set \( \rho(-A_{\alpha}) \).

\( \sigma_{disc}(-A_{\alpha}) \): isolated eigenvalues of \(-A_{\alpha}\) with finite algebraic multiplicities

\[ \Sigma_\phi = \{ z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \phi \} \]
3-5. Linear stability

Estimates for the perturbed Stokes semigroup (M.)

There is a constant $\delta > 0$ such that if $|\alpha| \leq \delta$ then the following statement holds. Let $1 < q \leq 2 \leq p < \infty$. Then it follows that

\[
\|e^{-tA_\alpha}f\|_{L^p(\Omega)} \leq Ct^{-\frac{1}{q} + \frac{1}{p}}\|f\|_{L^q(\Omega)}, \quad t > 0, \quad (1)
\]
\[
\|\nabla e^{-tA_\alpha}f\|_{L^2(\Omega)} \leq Ct^{-\frac{1}{q}}\|f\|_{L^q(\Omega)}, \quad t > 0, \quad (2)
\]

for $f \in L^2_\sigma(\Omega) \cap L^q(\Omega)^2$. Here the constant $C$ depends only on $\alpha$, $p$, and $q$. 

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Dear Professor Yoshikazu Giga,

I am deeply grateful for your guidance,
I wish you good health for many years to come.