Aharonov–Bohm effect in resonances of magnetic Schrödinger operators in two dimensions

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In quantum mechanics, a vector potential is said to have a direct significance to particles moving in a magnetic field. This quantum effect is known as the Aharonov–Bohm effect (AB effect) ([1]). We study the AB effect in resonances through a simple scattering system in two dimensions. The system consists of three scatterers, one bounded obstacle and two scalar potentials with compact supports at large separation, where the obstacle is placed between two supports and shields completely the support of a magnetic field. The field does not influence particles from a classical mechanical point of view, but quantum particles are influenced by the corresponding vector potential which does not necessarily vanish outside the obstacle. The resonances are shown to be generated near the real axis by the trajectories trapped between two supports of the scalar potentials as the distances between the three scatterers go to infinity. The location is described in terms of the backward amplitudes for scattering by the two scalar potentials, and it depends heavily on the magnetic flux of the field. We also discuss what happens in the case of two obstacles. This system yields a two dimensional model of scattering by toroidal solenoids in three dimensions. The result depends on the location of the obstacles as well as on the fluxes.

We write

\[ H(A, V) = (-i\nabla - A)^2 + V = \sum_{j=1}^{2} (-i\partial_j - a_j)^2 + V, \quad \partial_j = \partial/\partial x_j, \]

for the Schrödinger operator with the vector potential \( A = (a_1, a_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and the scalar potential \( V : \mathbb{R}^2 \rightarrow \mathbb{R} \). The magnetic field \( b : \mathbb{R}^2 \rightarrow \mathbb{R} \) associated with \( A \) is defined by

\[ b(x) = \nabla \times A(x) = \partial_1 a_2 - \partial_2 a_1 \]

and the quantity defined as the integral \( \alpha = (2\pi)^{-1} \int b(x) \, dx \) is called the magnetic flux of \( b \), where the integration with no domain attached is taken over the whole space.

Let \( \mathcal{O} \subset \mathbb{R}^2 \) be a simply connected bounded domain. We assume that \( \mathcal{O} \) contains the origin as an interior point and its boundary is smooth. For \( d \in \mathbb{R}^2 \), we set

\[ d_- = -\kappa_- d, \quad d_+ = \kappa_+ d, \quad \kappa_- > 0, \quad \kappa_+ + \kappa_- = 1, \]

so that \( d_+ - d_- = d \). The distance \( |d| \gg 1 \) is treated as a large parameter, but the direction \( \hat{d} = d/|d| \) is fixed. We consider the self–adjoint operator

\[ H_d = H(A, V_d) = (-i\nabla - A)^2 + V_d, \quad \mathcal{D}(H_d) = H^2(\Omega) \cap H^1_0(\Omega), \]
over the exterior domain $\Omega = \mathbb{R}^2 \setminus \mathcal{O}$ under the zero Dirichlet boundary conditions, where $V_d(x)$ takes the form

$$V_d(x) = V_{-d}(x) + V_{+d}(x) = V_-(x - d) + V_+(x - d)$$

with $V_\pm \in C_0^\infty(\mathbb{R}^2)$, and $A$ is defined as the Aharonov–Bohm potential

$$A(x) = \alpha \left(-x_2/|x|^2, x_1/|x|^2\right) = \alpha (-\partial_2 \log |x|, \partial_1 \log |x|)$$

over $\Omega$. The potential $A$ generates the solenoidal field

$$b = \nabla \times A = \alpha \left(\partial_1^2 + \partial_2^2\right) \log |x| = 2\pi \alpha \delta(x),$$

which has the support only at the origin and $\alpha$ as a magnetic flux. Hence the field $b = \nabla \times A$ is entirely shielded by the obstacle $\mathcal{O}$, although the corresponding vector potential $A$ does not necessarily vanish over $\Omega$. The resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2(\Omega) \to L^2(\Omega), \quad \text{Re} \, \zeta > 0, \; \text{Im} \, \zeta > 0,$$

is meromorphically continued from the upper half plane of the complex plane to the lower half plane across the positive real axis where the continuous spectrum of $H_d$ is located. Then $R(\zeta; H_d)$ with $\text{Im} \, \zeta \leq 0$ is well defined as an operator from $L^2_{\text{comp}}(\Omega)$ to $L^2_{\text{loc}}(\Omega)$, where $L^2_{\text{comp}}(W)$ denotes the space of square integrable functions with compact support in the closure $\overline{W}$ of a region $W \subset \mathbb{R}^2$ and $L^2_{\text{loc}}(W)$ denotes the space of locally square integrable functions over $W$. The resonances of $H_d$ are defined as the poles of $R(\zeta; H_d)$ in the lower half plane. Roughly speaking, the resonances near the real axis are almost regarded as positive eigenvalues in some sense, although $H_d$ has no positive eigenvalues.

One of the obtained results is stated as follows ([2]): The resonances are approximately determined as the solutions to the equation

$$\left(\frac{e^{2ik|d|}}{|d|}\right) \cos^2(\alpha \pi) f_-(\hat{d} \rightarrow \hat{d}; \zeta) f_+(\hat{d} \rightarrow -\hat{d}; \zeta) = 1, \quad \text{Im} \, k = \text{Im} \, \zeta^{1/2} < 0,$$

for $|d| \gg 1$, where $f_\pm(\hat{d} \rightarrow \mp \hat{d}; \zeta)$ is defined by analytic extension from the backward scattering amplitude $f_\pm(\pm \hat{d} \rightarrow \mp \hat{d}; E)$ at energy $E > 0$ for the Schrödinger operator

$$K_\pm = K_0 + V_\pm = -\Delta + V_\pm, \quad \mathcal{D}(K_\pm) = H^2(\mathbb{R}^2).$$

This relation makes sense only when the flux $\alpha$ is not a half–integer.
