Fluid-dynamic-type equations derived from the Boltzmann equation for small Knudsen numbers and their boundary conditions

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The purpose of the study of the Boltzmann equation in gas dynamics is to clarify the behavior of a gas with non-small mean free path (or Knudsen number) and to investigate the theoretical background of the conventional gas-dynamic equations, which are used to describe the behavior of the gas in the vanishing Knudsen number. For the second purpose, the asymptotic behavior of solutions of initial or boundary-value problems of the Boltzmann equation for small Knudsen numbers has been studied. Their behavior is described by fluid-dynamic-type equations in overall region outside initial layer, Knudsen layer, or shock layer. Here, the fluid-dynamic-type systems derived from the Boltzmann system for small Knudsen numbers are reviewed with the interest of gas dynamics in mind. Various kinds of fluid-dynamic-type equations are derived depending on the physical situations under interest. Among them, there is a system that is different from the Euler or Navier-Stokes system, which is used in conventional gas dynamics [incompleteness of the conventional fluid dynamics, Case (ii) below].

The velocity distribution function describing the overall behavior of the gas approaches a Maxwell distribution $f_e$, whose parameters (density, flow velocity, and temperature) depend on time and position, in the limit $K_n \to 0$ ($K_n$ : Knudsen number). The fluid-dynamic-type equations that determine the macroscopic variables in the limit differ depending on the character of the Maxwellian. In time-independent problems, the systems are classified by the size of $|f_e - f_{e0}|/f_{e0}$, where $f_{e0}$ is the reference stationary Maxwellian, as follows:

(i) $|f_e - f_{e0}|/f_{e0} = O(K_n)$: The limit $K_n \to 0$ is a uniform state at rest. The nonuniform state of the first order of Knudsen number is described by the “incompressible Navier-Stokes set” with the energy equation modified owing to the work done by (uniform) pressure, which shows that the gas is not incompressible.

(ii) $|f_e - f_{e0}|/f_{e0} = O(1)$ with $|v_i|/(2RT)^{1/2} = O(K_n)$ ($v_i$ and $T$ : the flow velocity and temperature in $f_e$ respectively): The temperature and density of the gas in the limit are determined together with the flow velocity of the first order of $K_n$ amplified by $1/K_n$ (the ghost effect of an infinitesimal flow), and the thermal stress of the order of $(K_n)^2$ must be retained in the momentum equations (the ghost effect of a non-Navier-Stokes stress). The thermal creep in the boundary condition must be taken into account (the ghost effect of a nonslip boundary condition). (SB system)

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(iii) $|f_e - f_{e0}|/f_{e0} = O(1)$ with $|v_i|/(2RT)^{1/2} = O(1)$:
(a) The behavior of the gas around a simple boundary, where $v_i n_i/(2RT)^{1/2} = 0$ ($n_i$: the normal to the boundary), is described by the combination of the Euler and viscous boundary-layer sets. (E+VB system)
(b) The behavior of the gas around the condensed phase of the gas, where evaporation or condensation with $v_i n_i/(2RT)^{1/2} = O(1)$, is taking place, is described by the Euler set. The Knudsen-layer correction is given by the nonlinear Boltzmann equation in contrast to the other cases, in which the Knudsen layer is governed by the linearized Boltzmann equation. (E system)

In time-dependent problems, there are two time scales of variation of variables, in addition to the mean free time: the time required for the sound wave to propagate over a gas-dynamic distance and a longer time given by the quotient of this time scale divided by the Knudsen number. The overall behavior is given by the Euler set for the first and by the set with the corresponding time-independent set being modified for the second.

The above discussion is for a general domain. A boundary of a ruled surface can be taken as the limit of a general surface with the generating straight lines being originally curves. The problem with such a boundary can be taken as various limiting cases where the Knudsen number and the curvature tend to zero simultaneously. The result depends on the relative speed of the parameters to the limit, and the effect of the curvature remains in the limiting equations (ghost effect of infinitesimal curvature). The bifurcation of the plane Couette flow at infinite Reynolds number and nonexistence of the Poiseuille flow with a parabolic profile through a circular straight pipe are shown in some cases.1