Physical motivation. I will talk about a mathematical interpretation of D-branes in Type II string theory using B-L Wang’s twisted geometric cycles [Wa].

In string theory D-branes were proposed as a mechanism for providing boundary conditions for the dynamics of open strings moving in space-time. Initially they were thought of as submanifolds. As D-branes themselves can evolve over time one needs to study equivalence relations on the set of D-branes. An invariant of the equivalence class is the topological charge of the D-brane which should be thought of as an analogue of the Dirac monopole charge as these D-brane charges are associated with gauge fields (connections) on vector bundles over the D-brane. These vector bundles are known as Chan-Paton bundles.

In [MM] Minasian and Moore made the proposal that D-brane charges should take values in $K$-groups and not in the cohomology of the space-time or the D-brane. In string theory there is an additional field on space-time known as the $H$-flux which may be thought of as a global closed three form. Locally it is given by a family of ‘two-form potentials’ known as the $B$-field. Mathematically we think of these $B$-fields as defining a degree three integral Čech class on the space-time, called a ‘twist’. Witten [Wit], extending [MM], gave a physical argument for the idea that D-brane charges in the presence of a twist should take values in twisted $K$-theory (at least in the case where the twist is a torsion class). Subsequently Bouwknegt and Mathai [BouMat] extended Witten’s proposal to the non-torsion case using ideas from operator algebra theory.

In this talk I will review the mathematical background. Then I will move to a discussion of the resolution of Witten’s proposal.

Mathematical issues. From a mathematical perspective some immediate questions arise from the physical input summarised above. When there is no twist it is well known that $K$-theory provides the main topological tool for the index theory of elliptic operators. One version of the Atiyah-Singer index theorem establishes a relationship between the analytic viewpoint provided by elliptic differential operators and the geometric viewpoint provided by the notion of geometric cycle introduced in the fundamental paper of Baum and Douglas [BD2]. The viewpoint that geometric cycles in the sense of [BD2] are a model for D-branes in the untwisted case is surveyed in [Sz].

It is thus tempting to conjecture that there is an analogous picture of D-branes as a type of geometric cycle in the twisted case as well. More precisely we ask the question of whether there is a way to formulate the notion of ‘twisted geometric cycle’ (generalising[BD2]) so that it is adapted to the application to string theory. This question was answered in the affirmative by B-L. Wang [Wa]. It is important to emphasise that string geometry ideas from [FreWit] played a key role.

In Type II superstring theory on a manifold $X$, a string worldsheet is an oriented Riemann surface $\Sigma$, mapped into $X$ with $\partial \Sigma$ mapped to an oriented submanifold $M$ (called a D-brane world-volume). The theory also has a Neveu-Schwarz $B$-field, a system of local 2-forms on...
$X$ classified by a characteristic class $[\alpha] \in H^3(X, \mathbb{Z})$ (Čech cohomology) referred to as a ‘twist’. In physics, the D-brane world volume $M$ carries a connection on a complex vector bundle (called the Chan-Paton bundle), and thus a D-brane is given by a submanifold $M$ of $X$ with a complex bundle $E$ and a connection.

For a D-brane $M$ to define a class in the $K$-theory of $X$, its normal bundle $\nu_M$ must be endowed with a $Spin^c$ structure. Equivalently, the embedding $\iota : M \rightarrow X$ is such that the classical push-forward map in $K$-theory ([AH])

$$\iota'_K : K^0(M) \rightarrow K^{ev/odd}(X)$$

is well-defined, (it takes values in even or odd $K$-groups depending on the dimension of $M$). So the D-brane charge of $(\iota : M \rightarrow X, E)$ is given by pushing forward the $K$-theory class $[E]$ of $E$ which we write as:

$$\iota'_K([E]) \in K^{ev/odd}(X).$$

When the $B$-field is not topologically trivial, that is $[\alpha] \neq 0$, in order to have a well-defined worldsheet path integral, Freed and Witten in [FreWit] showed that the pull back under the imbedding $\iota$ of the twist class in $H^3(X, \mathbb{Z})$ should equal the third Stiefel-Whitney class of $M$, denoted $W_3$, which in general has to be allowed to be non-trivial so that $M$ is not necessarily $Spin^c$.

This latter situation is summarised as

$$(0.1) \quad \iota^*[\alpha] + W_3(\nu_M) = 0.$$  

When $\iota^*[\alpha] \neq 0$, then the push-forward map in $K$-theory ([AH])

$$\iota'_K : K^0(M) \rightarrow K^*(X)$$

is not well-defined.

In [Wa], the mathematical meaning of (0.1) was discovered. First of all we need to allow more general maps $\iota : M \rightarrow X$ than just imbeddings. He uses a generalisation of the notion of $Spin^c$ manifolds for a continuous map $\alpha$ into the classifying space for principal projective unitary group bundles that represent $[\alpha] \in H^3(X, \mathbb{Z})$.

In Wang’s formulation we see that when $M$ is $Spin^c$, the datum to describe a D-brane is exactly a Baum-Douglas geometric cycle. For a general manifold $X$ with a twisting $\alpha$, and a continuous map $\iota : M \rightarrow X$ with $M$ having non-zero Stieffel-Whitney class, we need a new concept, which we call an $\alpha$-twisted $Spin^c$ structure on $M$. Loosely speaking what Wang shows is that given a twisting $\alpha$ on a smooth manifold $X$, every twisted $K$-class is represented by a twisted geometric cycle supported on a twisted $Spin^c$ manifold $M$ and an ordinary $K$-class $[E] \in K^0(M)$. It remains open as to whether this theory can be extended beyond targets $X$ that are manifolds, a question of interest to topologists.

**References**


