Geometric Characterization of Monge-Ampere Equations

Atsushi YANO (Hokkaido University)
yano@math.sci.hokudai.ac.jp

Abstract

This paper discusses the geometric characterization of Monge-Ampere equations. Let $D$ be a differential system on a manifold $\mathcal{M}$, i.e., a subbundle of the tangent bundle of $\mathcal{M}$. The first order system $D$ is defined by $\mathcal{L}(\mathcal{D}) = \mathcal{D} \cap \mathcal{D}$, where $\mathcal{L}(\mathcal{D})$ is the characteristic system of the prolongation of $\mathcal{D}$.

1. Introduction

Let $D$ be a differential system on a manifold $\mathcal{M}$, i.e., a subbundle of the tangent bundle of $\mathcal{M}$. The first order system $D$ is defined by $\mathcal{L}(\mathcal{D}) = \mathcal{D} \cap \mathcal{D}$, where $\mathcal{L}(\mathcal{D})$ is the characteristic system of the prolongation of $\mathcal{D}$.

2. Lagrange-Grassmann bundle and single second-order PDEs

Let $(\mathcal{C}, \mathcal{D})$ be a contact manifold, i.e., a differential system of corank 1 on a manifold such that $\mathcal{C}$ is locally defined by a 1-form $\Theta$ satisfying $\Theta (\partial J)^n = 0$ at each point.

The canonical bundle $\mathcal{L}(\mathcal{T}(\mathcal{L}))$ of $\mathcal{L}(\mathcal{T}(\mathcal{L}))$ is defined by $\mathcal{L}(\mathcal{T}(\mathcal{L})) = \mathcal{C} \cap \mathcal{D}$.

We consider a second order system $\mathcal{D}(\mathcal{L}, \mathcal{D})$ of $\mathcal{L}$ and $\mathcal{D}$ with two independent variables. The dimension of $\mathcal{D}$ is $5$.

3. Monge-Ampere system

Monge-Ampere system on a 5-dimensional contact manifold $\mathcal{M}$ is an EDS (an ideal)

$\mathcal{D} = \{ \mathcal{D}, \partial \mathcal{D}, \mathcal{L}(\mathcal{D}), \mathcal{L}(\mathcal{D}) \}

4. Prolongation of Monge-Ampere system

If the equation is hyperbolic, it is a hyperbolic Monge-Ampere system around $\mathcal{M}$.

Theorem 1 (hyperbolic)

Let $D$ be a hyperbolic Monge-Ampere system on a manifold $\mathcal{M}$. Then $D$ has two decomposable 2-forms $\omega_1$ and $\omega_2$ around each point such that $\partial \mathcal{D} \supset \omega_1 \wedge \omega_2 \wedge \omega_3$ (mod $\mathcal{D}$).

Then Monge characteristic system $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{D}$ is defined by $\mathcal{N}_i = \{ \mathcal{D}, \partial \mathcal{D}, \mathcal{N}_i \}$ (for $i = 1, 2$).

Theorem 2 (hyperbolic)

Let $D$ be a parabolic Monge-Ampere system on a manifold $\mathcal{M}$. Then $D$ has an integrable 2-form $\omega$ around each point such that $\partial \mathcal{D} \supset \omega$ (mod $\mathcal{D}$).

Then Monge characteristic system $\mathcal{N}$ of $\mathcal{D}$ is defined by $\mathcal{N} = \{ \mathcal{D}, \partial \mathcal{D}, \mathcal{N} \}$.

Theorem 3 (parabolic)

Let $D$ be a parabolic Monge-Ampere system on a manifold $\mathcal{M}$. Then $D$ has an integrable 2-form $\omega$ around each point such that $\partial \mathcal{D} \supset \omega$ (mod $\mathcal{D}$).

Then Monge characteristic system $\mathcal{N}$ of $\mathcal{D}$ is defined by $\mathcal{N} = \{ \mathcal{D}, \partial \mathcal{D}, \mathcal{N} \}$.

Theorem 4 (parabolic)

Let $D$ be a parabolic Monge-Ampere system on a manifold $\mathcal{M}$. Then $D$ has an integrable 2-form $\omega$ around each point such that $\partial \mathcal{D} \supset \omega$ (mod $\mathcal{D}$).

Then Monge characteristic system $\mathcal{N}$ of $\mathcal{D}$ is defined by $\mathcal{N} = \{ \mathcal{D}, \partial \mathcal{D}, \mathcal{N} \}$.