

Geometric Characterization of Monge-Ampère Equations

Atsushi YANO (Hokkaido University)
yano@math.sci.hokudai.ac.jp

Abstract

I'll explain about geometric characterization of Monge-Ampère equation

$$Az_{xx} + 2Bz_{xy} + Cz_{yy} + D + E(z_{xx}z_{yy} - z_{xy}^2) = 0,$$

where each capital letter indicates a function of variables x, y, z, z_x, z_y . A single second order PDE $F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0$ of one unknown function with two independent variables corresponds to a hypersurface R in 2-jet space $J^2(\mathbb{R}^2, \mathbb{R})$ with coordinates (x, y, z, p, q, r, s, t) ($p = z_x, q = z_y, r = z_{xx}, s = z_{xy}, t = z_{yy}$).

On the other hand, it is well-known that a Monge-Ampère equation can be expressed in terms of exterior differential system (EDS)—*Monge-Ampère system* \mathcal{I} . I studied the relation between *Monge characteristic systems* of Monge-Ampère equation R and those of Monge-Ampère system \mathcal{I} . From this observation, I characterized Monge-Ampère equations as hypersurfaces R in $J^2(\mathbb{R}^2, \mathbb{R})$.

1 notation

Let D be a **differential system** on a manifold Σ , i.e. a subbundle of the tangent bundle of Σ . The **first derived system** ∂D of D is defined by, in terms of sections,

$$\partial D = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$$

where \mathcal{D} is the space of sections of D and $[\cdot, \cdot]$ is Lie bracket for vector fields. Furthermore, the **k -th derived system** $\partial^k D$ is defined inductively by

$$\partial^k D = \partial(\partial^{k-1} D).$$

Set $D = \{\omega^1 = \dots = \omega^n = 0\}$ locally. **Cauchy characteristic system** $\text{Ch}(D)$ of D is defined as, for $x \in \Sigma$,

$$\text{Ch}(D)(x) = \{v \in D(x) \mid v \lrcorner d\omega^i \equiv 0 \pmod{\omega^1, \dots, \omega^n}, 1 \leq i \leq n\}$$

2 Lagrange-Grassmann bundle and single second-order PDEs

Let (J, C) be a **contact manifold**, i.e. C is a differential system of corank 1 on a manifold J such that C is locally defined by a 1-form θ satisfying $\theta \wedge (d\theta)^n \neq 0$ at each point.

Lagrange-Grassmann bundle $L(J)$ (**geometric 2-jet space**) over the contact manifold (J, C)

$$L(J) = \bigcup_{u \in J} L(J)_u \xrightarrow{\pi} J,$$

$$L(J)_u = \{v \subset T_u J : n\text{-dim. subspace such that } \theta|_v = d\theta|_v = 0\},$$

where $L(J)_u$ is the Grassmannian of all Lagrangian subspaces of the symplectic vector space $(C(u), d\theta_u)$.

The **canonical system** $E \subset T(L(J))$ is defined by

$$E(v) := \pi_*^{-1}(v) \subset T_v(L(J)) \quad \text{for } v \in L(J).$$

We consider a single second order PDE (R, D) of one unknown function with two independent variables: Set the dimension of J is 5.

$R \subset L(J)$: a hypersurface such that $\pi|_R$ is submersion

D : the restriction of E to R

Let us fix a point $v_o \in R$. It is well-known that the structure equation of $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ is expressed as follows:

If the equation R is **hyperbolic** around v_o , the structure equation is

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\text{mod } \varpi_0), \\ d\varpi_1 \equiv \omega^1 \wedge \pi_{11} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \omega^2 \wedge \pi_{22} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

where $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{11}, \pi_{22}\}$ is a coframe around $v_o \in R$.

Then the **Monge characteristic systems** \mathcal{M}_i of (R, D) are defined as

$$\mathcal{M}_1 = \{\varpi_0 = \varpi_1 = \varpi_2 = \omega^1 = \pi_{11} = 0\}$$

and

$$\mathcal{M}_2 = \{\varpi_0 = \varpi_1 = \varpi_2 = \omega^2 = \pi_{22} = 0\}.$$

If the equation R is **parabolic** around v_o , the structure equation is

$$\begin{cases} d\varpi_0 \equiv \omega^1 \wedge \varpi_1 + \omega^2 \wedge \varpi_2 & (\text{mod } \varpi_0), \\ d\varpi_1 \equiv \omega^2 \wedge \pi_{12} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \\ d\varpi_2 \equiv \omega^1 \wedge \pi_{12} + \omega^2 \wedge \pi_{22} & (\text{mod } \varpi_0, \varpi_1, \varpi_2), \end{cases}$$

where $\{\varpi_0, \varpi_1, \varpi_2, \omega^1, \omega^2, \pi_{12}, \pi_{22}\}$ is a coframe around $v_o \in R$.

Then the **Monge characteristic system** \mathcal{M} of (R, D) is defined as

$$\mathcal{M} = \{\varpi_0 = \varpi_1 = \varpi_2 = \omega^2 = \pi_{12} = 0\}.$$

3 Monge-Ampère system

Monge-Ampère system on a 5-dimensional contact manifold J is an EDS (an ideal)

$$\mathcal{I} = \{\theta, d\theta, \Psi\}_{\text{alg}} \subset \Omega^*(J)$$

where θ is the contact form and Ψ is a 2-form that is linearly independent from $d\theta$, modulo θ .

$$\mathcal{I} \text{ is } \begin{cases} \text{hyperbolic} \\ \text{parabolic} \\ \text{elliptic} \end{cases} \text{ at } u \in J$$

$$\stackrel{\text{def}}{\iff} \mathcal{I}_u \text{ has } \begin{cases} \text{two} \\ \text{one} \\ \text{no} \end{cases} \text{ decomposable 2-covector(s)}$$

Proposition (hyperbolic)

Let \mathcal{I} be a hyperbolic Monge-Ampère system around $u \in J$. Then \mathcal{I} has two decomposable 2-forms $\omega^1 \wedge \pi_1, \omega^2 \wedge \pi_2$ around u such that $d\theta \equiv \omega^1 \wedge \pi_1 + \omega^2 \wedge \pi_2 \pmod{\theta}$. Hence

$$\mathcal{I} = \{\theta, \omega^1 \wedge \pi_1, \omega^2 \wedge \pi_2\}_{\text{alg}}.$$

Then **Monge characteristic systems** \mathcal{H}_1 and \mathcal{H}_2 of \mathcal{I} is defined by

$$\mathcal{H}_i = \{\theta = \omega^i = \pi_i = 0\} \quad \text{for } i = 1, 2.$$

Proposition (parabolic)

Let \mathcal{I} be a parabolic Monge-Ampère system around $u \in J$. Then \mathcal{I} has one decomposable 2-form $\omega \wedge \pi$ around u such that $d\theta \equiv 0 \pmod{\theta, \omega, \pi}$. Hence

$$\mathcal{I} = \{\theta, d\theta, \omega \wedge \pi\}_{\text{alg}}.$$

Then the **Monge characteristic system** \mathcal{H} of \mathcal{I} is defined by

$$\mathcal{H} = \{\theta = \omega = \pi = 0\}.$$

4 Prolongation of Monge-Ampère system

Let $\mathcal{I} = \{\theta, d\theta, \Psi\}_{\text{alg}}$ be a Monge-Ampère system on J . The **prolongation** $(\mathcal{V}_2(\mathcal{I}), D)$ of \mathcal{I} is defined as follows:

$$\mathcal{V}_2(\mathcal{I}) = \bigcup_{u \in J} \mathcal{V}_2(\mathcal{I})_u \xrightarrow{\rho} J$$

$$\mathcal{V}_2(\mathcal{I})_u = \{v \subset T_u J : 2\text{-dim. integral element}\}$$

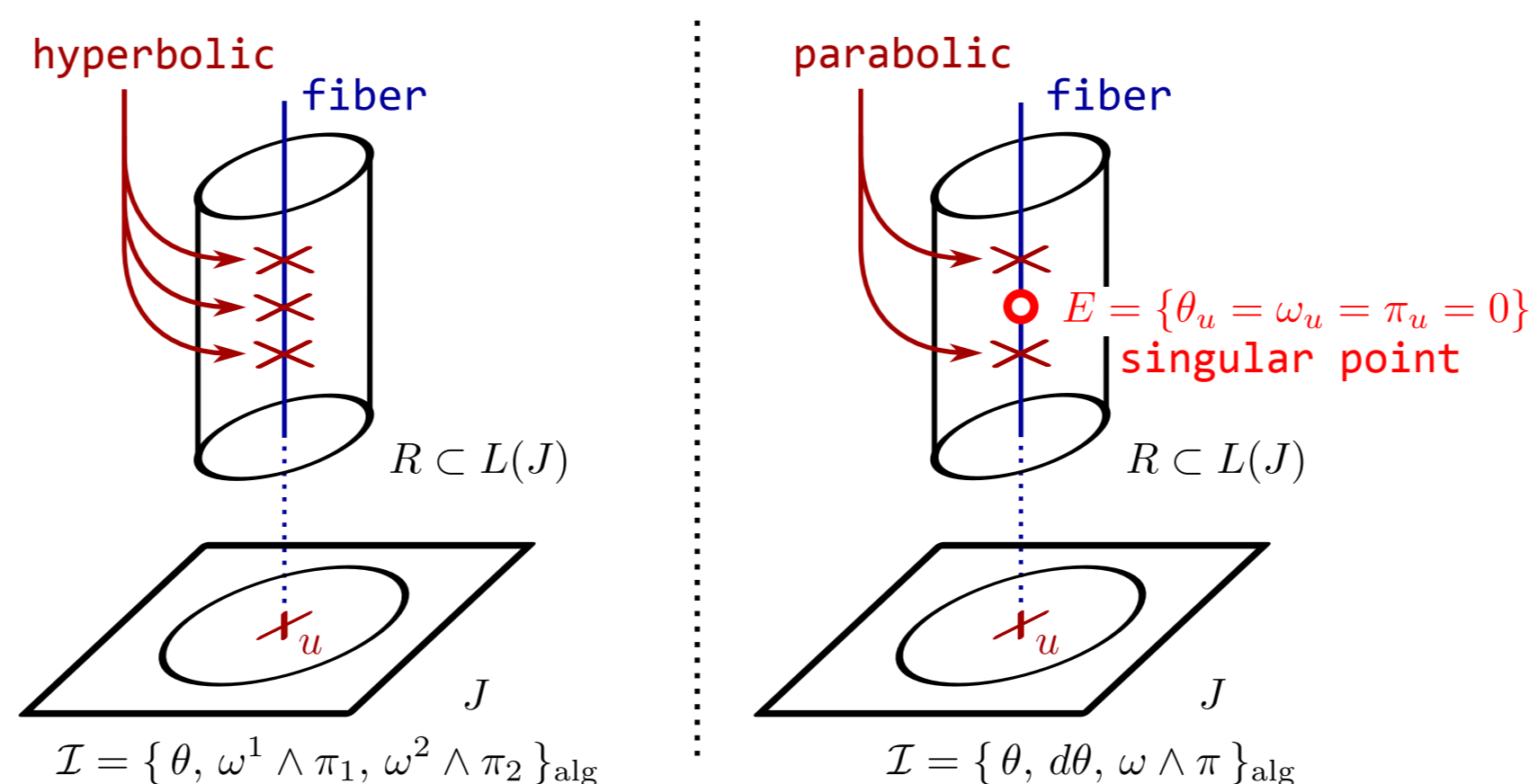
$$\stackrel{\text{i.e.}}{=} \{v \subset T_u J : 2\text{-dim. subspace s.t. } \theta|_v = d\theta|_v = \Psi|_v = 0\}$$

$$\mathcal{V}_2(\mathcal{I}) = \{v \in L(J) \mid \Psi|_v = 0\} \subset L(J) : \text{subvariety}$$

as $\mathcal{V}_2(\mathcal{I})$ is a submanifold,

$$D(v) := \rho_*^{-1}(v) \quad (v \in \mathcal{V}_2(\mathcal{I}))$$

Let \mathcal{I} be a hyperbolic or parabolic Monge-Ampère system and (R, D) the prolongation of \mathcal{I} .



hyperbolic: there is no singular points in R . Namely, R is a submanifold of $L(J)$.

parabolic: there is only one singular point E in each fiber $\rho^{-1}(u)$ ($u \in J$).

5 Relations between \mathcal{H}_i and \mathcal{M}_i

\mathcal{I} : a hyperbolic Monge-Ampère system

$\mathcal{H}_1, \mathcal{H}_2$: Monge characteristic systems of \mathcal{I}

(R, D) : the prolongation of \mathcal{I}

Then, for each $i = 1, 2$, there exists a Monge characteristic system \mathcal{M}_i of (R, D) such that

$$\mathcal{M}_i \subset \rho_*^{-1}(\mathcal{H}_i).$$

Furthermore, we have the following theorem:

Theorem (hyperbolic)

$\partial \mathcal{M}_i$ and $\partial^2 \mathcal{M}_i$ are (regular) differential systems of rank 3 and 4 respectively and we have

$$\partial^2 \mathcal{M}_i \subset \rho_*^{-1}(\partial \mathcal{H}_i), \quad \partial \mathcal{M}_i + \text{Ch}(\partial D) = \rho_*^{-1}(\mathcal{H}_i).$$

\mathcal{I} : a parabolic Monge-Ampère system

\mathcal{H} : the Monge characteristic system of \mathcal{I}

(R, D) : the prolongation of \mathcal{I}

Then the Monge characteristic system \mathcal{M} of (R, D) is contained in the pullback of \mathcal{H} . Namely,

$$\mathcal{M} \subset \rho_*^{-1}(\mathcal{H}).$$

Furthermore, we have the following theorem:

Theorem (parabolic)

$$\partial(\mathcal{M} + \text{Ch}(\partial D)) = \rho_*^{-1}(\mathcal{H})$$

and \mathcal{H} is completely integrable if and only if \mathcal{M} is completely integrable.

On the other hand, if $\partial \mathcal{M}$ has constant rank and does not coincide with \mathcal{M} , then

$$\partial^2 \mathcal{M} = \rho_*^{-1}(\mathcal{H}).$$

6 Characterization of Monge-Ampère equations

The results in the previous section guide us to consider the geometric characterization of Monge-Ampère equations. In fact, for Monge characteristic systems \mathcal{M}_i (resp. \mathcal{M}) of a hyperbolic (resp. parabolic) PDE (R, D) , the corank 3 differential systems $\partial \mathcal{M}_i + \text{Ch}(\partial D)$ (resp. $\partial(\mathcal{M} + \text{Ch}(\partial D))$) is candidate for Monge characteristic systems of a Monge-Ampère system:

Theorem 1 (hyperbolic)

Let (R, D) be a hyperbolic PDE and let \mathcal{M}_1 and \mathcal{M}_2 denote the Monge characteristic systems of (R, D) . If $\partial \mathcal{M}_i + \text{Ch}(\partial D)$ drops down to J for each $i = 1, 2$, there exists a Monge-Ampère system \mathcal{I} whose prolongation coincides with (R, D) locally.

Moreover, $\partial \mathcal{M}_1 + \text{Ch}(\partial D)$ and $\partial \mathcal{M}_2 + \text{Ch}(\partial D)$ are then pullbacks of the Monge characteristic systems of the system \mathcal{I} .

$$\begin{array}{ccc} L(J) \supset R & D \supset \partial \mathcal{M}_i + \text{Ch}(\partial D) = \rho_*^{-1}(\mathcal{H}_i) & \\ \pi \downarrow \swarrow \rho = \pi|_R & & \downarrow \text{drops down} \\ J & & TJ \supset \mathcal{H}_i \quad (\text{corank } 3) \end{array}$$

Theorem 2 (hyperbolic)

Let (R, D) be a hyperbolic PDE and let \mathcal{M}_1 and \mathcal{M}_2 denote the Monge characteristic systems of (R, D) . If $\text{Ch}(\partial \mathcal{M}_1 + \text{Ch}(\partial D)) = \text{Ch}(\partial D)$ (resp. $\text{Ch}(\partial \mathcal{M}_2 + \text{Ch}(\partial D)) = \text{Ch}(\partial D)$), $\partial \mathcal{M}_1 + \text{Ch}(\partial D)$ (resp. $\partial \mathcal{M}_2 + \text{Ch}(\partial D)$) drops down to $J = R/\text{Ch}(\partial D)$. Thus this theorem implies that if one drops down, the other also does.

Hence it is understood that each Monge characteristic system has the complete information whether R is a Monge-Ampère equation or not.

On the other hand, in parabolic case, we have the following theorems:

Theorem 3 (parabolic)

Let (R, D) be a parabolic PDE. Let \mathcal{M} denote the Monge characteristic system of (R, D) . If $\partial(\mathcal{M} + \text{Ch}(\partial D))$ drops down to J , there exists a Monge-Ampère system \mathcal{I} whose prolongation coincides with (R, D) locally.

Moreover, $\partial(\mathcal{M} + \text{Ch}(\partial D))$ is then the pullback of the Monge characteristic system of the system \mathcal{I} .

$$\begin{array}{ccc} L(J) \supset R & D \supset \partial(\mathcal{M} + \text{Ch}(\partial D)) = \rho_*^{-1}(\mathcal{H}) & \\ \pi \downarrow \swarrow \rho = \pi|_R & & \downarrow \text{drops down} \\ J & & TJ \supset \mathcal{H} \quad (\text{corank } 3) \end{array}$$

Theorem 4 (parabolic)

Let (R, D) be a parabolic PDE. Let \mathcal{M} denote the Monge characteristic system of (R, D) and **assume $\partial \mathcal{M}$ has constant rank**. If $\text{Ch}(\partial(\mathcal{M} + \text{Ch}(\partial D)))(v)$ contains $\text{Ch}(\partial D)(v)$ at each point $v \in R$, there exists a Monge-Ampère system \mathcal{I} whose prolongation coincides with (R, D) locally.

7 Example

Through the following two examples, we'll see differences between a Monge-Ampère equation and a non Monge-Ampère equation.

Example (Monge-Ampère equation)

$$r + s = 0 \quad (z_{xx} + z_{xy} = 0) \quad \text{Monge-Ampère equation}$$

$$(J^2(\mathbb{R}^2, \mathbb{R}) \rightarrow \mathbb{R}^5 : (x, y, z, p, q, r, s, t) \mapsto (x, y, z, p, q))$$

$$\mathcal{M}_1 = \{\varpi_0 = \varpi_1 = \varpi_2 = dx - dy = ds = 0\}$$

$$\mathcal{M}_2 = \{\varpi_0 = \varpi_1 = \varpi_2 = dy = dt + ds = 0\}$$

where $\varpi_0 = dz - pdx - qdy$, $\varpi_1 = dp + sdx - sdy$, $\varpi_2 = dq - sdx - tdy$

$$\partial \mathcal{M}_1 = \{\varpi_0 = \varpi_1 = dx - dy = ds = 0\}$$

$$\partial \mathcal{M}_2 = \{\varpi_0 = \varpi_2 + \varpi_1 = dy = dt + ds = 0\}$$

$$\partial \mathcal{M}_1 + \text{Ch}(\partial D) = \{\varpi_0 = \varpi_1 = dx - dy = 0\}$$

$$= \{dz - pdx - qdy = dp = dx - dy = 0\}$$

$$\partial \mathcal{M}_2 + \text{Ch}(\partial D) = \{\varpi_0 = \varpi_2 + \varpi_1 = dy = 0\}$$

$$= \{dz - pdx - qdy = dp + dq = dy = 0\}$$

Example (non Monge-Ampère equation)

$$r + \frac{1}{2}s^2 = 0, s \neq 0 \quad (z_{xx} + \frac{1}{2}(z_{xy})^2 = 0) \quad \text{non Monge-Ampère equation}$$

$$(J^2(\mathbb{R}^2, \mathbb{R}) \rightarrow \mathbb{R}^5 : (x, y, z, p, q, r, s, t) \mapsto (x, y, z, p, q))$$

$$\mathcal{M}_1 = \{\varpi_0 = \varpi_1 = \varpi_2 = dx - \frac{1}{s}dy = ds = 0\}$$

$$\mathcal{M}_2 = \{\varpi_0 = \varpi_1 = \varpi_2 = dy = dt + \frac{1}{s}ds = 0\}$$

where $\varpi_0 = dz - pdx - qdy$, $\varpi_1 = dp + \frac{1}{2}s^2 dx - sdy$, $\varpi_2 = dq - sdx - tdy$

$$\partial \mathcal{M}_1 = \{\varpi_0 = \varpi_1 = dx - \frac{1}{s}dy = ds = 0\}$$

$$\partial \mathcal{M}_2 = \{\varpi_0 = \varpi_2 + \frac{1}{s}\varpi_1 = dy = dt + \frac{1}{s}ds = 0\}$$

$$\partial \mathcal{M}_1 + \text{Ch}(\partial D) = \{\varpi_0 = \varpi_1 = dx - \frac{1}{s}dy = 0\}$$

$$= \{dz - pdx - qdy = dp - \frac{1}{2}sdy = dx - \frac{1}{s}dy = 0\}$$

$$\partial \mathcal{M}_2 + \text{Ch}(\partial D) = \{\varpi_0 = \varpi_2 + \frac{1}{s}\varpi_1 = dy = 0\}$$

$$= \{dz - pdx - qdy = dq + \frac{1}{s}dp - \frac{1}{2}dx = dy = 0\}$$