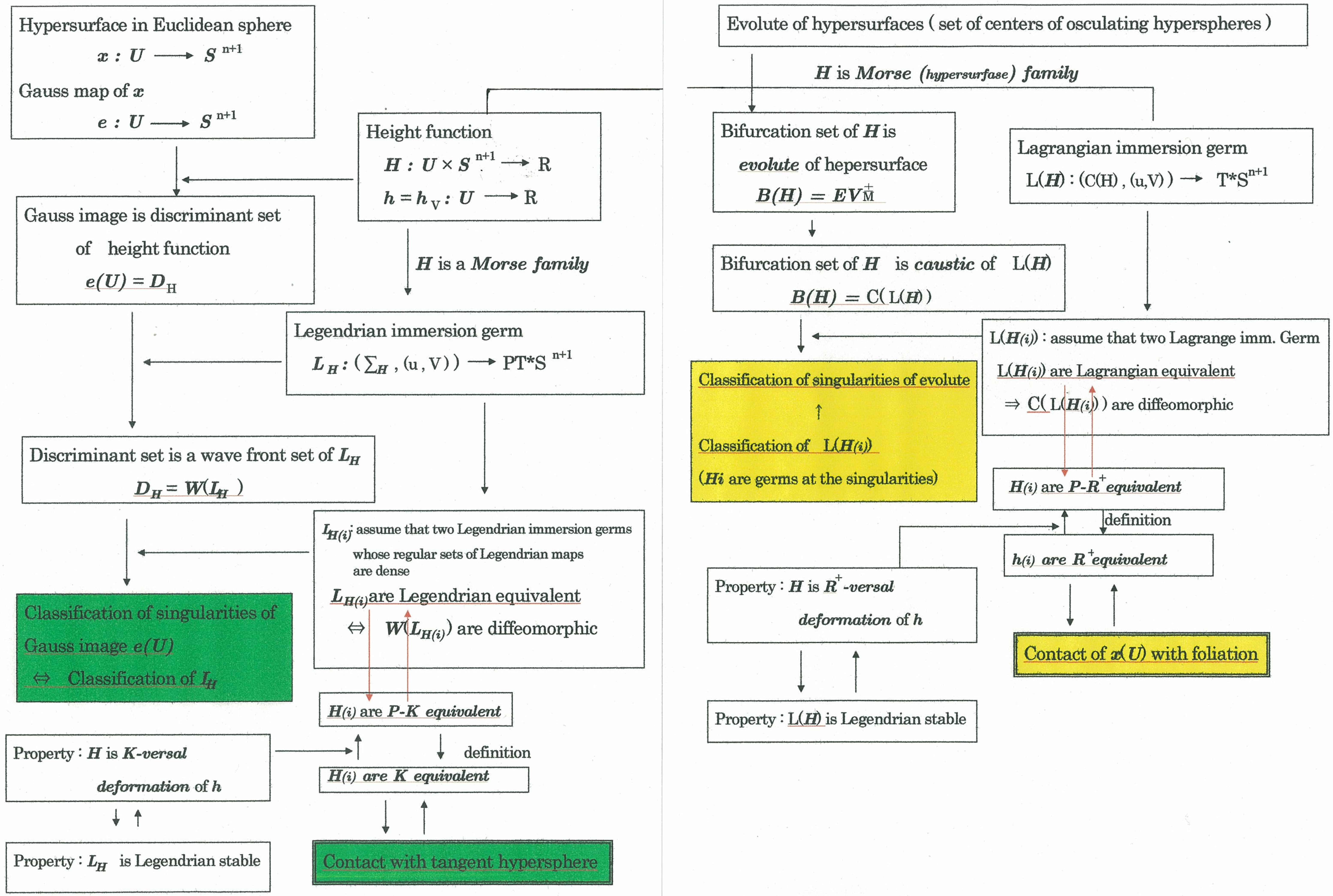
The singularities of Gauss maps and evolutes of hypersurfaces in Euclidean n-sphere

Reference

[1] SHYUICHI IZUMIYA, DONGHE PEI, TAKASHI SANO 'Singularities of hyperbolic Gauss maps' Proc. London Math. Soc. (3) 86 (2003) 485 – 512

[2] SHYUICHI IZUMIYA, DONGHE PEI, MASATOMO TAKAHASHI 'Singularities of evolutes of hypersurfaces in hyperbolic space' Proc. Edinburgh Math. Soc. (2004) **47** 131 – 153



Hypersurface in Euclidean sphere
$x : U \longrightarrow S^{n+1}$
Gauss map of x
$e : U \longrightarrow S^{n+1}$
Height function
$H : U \times S^{n+1} \longrightarrow R$
$h = h_V : U \longrightarrow R$
Gauss image is discriminant set of height function
$e(U) = D_H$
H is a Morse family
Legendrian immersion germ
$L_H : (\Sigma_H, (u, V)) \longrightarrow PT^*S^{n+1}$
Discriminant set is a wave front set of L_H
$D_H = W(L_H)$
$L_{H(i)}$: assume that two Legendrian immersion germs whose regular sets of Legendrian maps are dense
$L_{H(i)}$ are Legendrian equivalent
$\Leftrightarrow$ $W(L_{H(i)})$ are diffeomorphic
Classification of singularities of Gauss image $e(U)$
$\Leftrightarrow$ Classification of L_H
$H(i)$ are P-K equivalent
Property : H is K-versal deformation of h
definition
$H(i)$ are K equivalent
Property : L_H is Legendrian stable
Contact with tangent hypersphere
Evolute of hypersurfaces (set of centers of osculating hyperspheres)
H is Morse (hypersurfase) family
Bifurcation set of H is evolute of hepersurface
$B(H) = EV_M^{\pm}$
Lagrangian immersion germ
$L(H) : (C(H), (u,V)) \longrightarrow T^*S^{n+1}$
Bifurcation set of H is caustic of L(H)
$B(H) = C(L(H))$
L(H(i)) : assume that two Lagrange imm. Germ
L(H(i)) are Lagrangian equivalent
$\Rightarrow$ C(L(H(i))) are diffeomorphic
Classification of singularities of evolute
↑
Classification of L(H(i))
(Hi are germs at the singularities)
$H(i)$ are P-R^+ equivalent
definition
$h(i)$ are R^+ equivalent
Property : H is R^+-versal deformation of h
Contact of x(U) with foliation
Property : L(H) is Legendrian stable

1 Basic notations

Let $S^{n+1} := \{\, V = (v^0, v^1, \ldots, v^{n+1}) \in \mathbb{R}^{n+2} \mid V \cdot V = 1 \,\}$ be an $(n+1)$−dimensional Euclidean sphere , U be an open set in R^n and $(u^1, \ldots, u^n)$ is coordinate of U.

∘ **Hypersurface in Euclidean sphere**

$$\Longleftrightarrow \quad \boldsymbol{x} : U \to S^{n+1} \quad : \quad \text{embedding}$$

$$(\text{ or } \mathbf{M} := \boldsymbol{x}(U)\ )$$

∘ **Gauss map of hypersurface**

$$\boldsymbol{e} : U \to S^{n+1}$$

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_1(u) \wedge \boldsymbol{x}_2(u) \wedge \cdots \wedge \boldsymbol{x}_n(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_1(u) \wedge \boldsymbol{x}_2(u) \wedge \cdots \wedge \boldsymbol{x}_n(u)\|}$$

where $\boldsymbol{x}_i(u) := \frac{\partial \boldsymbol{x}}{\partial u^i}(u)$

∘ **Curvature**

$S_p = -d\boldsymbol{e}(u) \ : \ T_P M \to T_p M$ — *shape operator* of M at p

$\kappa_i(u)$: eigenvalue of S_p — *principal curvature* of M at p

$K_p := \det S_p$ — *Gauss − Kronecker curvature*

∘ **Umbilic**

p is an *umbilic point* $\Leftrightarrow \exists \kappa_p \in \mathbb{R}\ s.t.\ S_p = \kappa_p \cdot id_{T_p M}$

M is *totally umbilic* $\Leftrightarrow \forall p \in M$ *are umbilic*

∘ **Hypersphere**

$S^n(V, c) := \{V' \in S^{n+1} \mid V \cdot V' = c\}$

$c = 0$: *great hypersphere*

$c \neq 0$: *small hypersphere*

Remark

If M is totally umbilic , then $\kappa_p \equiv \exists K \in \mathbb{R}\ (\forall p \in M)$, furthermore

$K = 0 \Rightarrow \exists V \in S^{n+1}\ s.t.\ M \subset S^n(V, 0)$

$K \neq 0 \Rightarrow \exists V \in S^{n+1}$ *and* $\exists c \in \mathbb{R}\ s.t.\ M \subset S^n(V, c)$

Height function

$$H : U \times S^{n+1} \to \mathbb{R}$$

$$H(u, V) = \boldsymbol{x}(u) \cdot V$$

$$h = h_{V_0} : U \to \mathbb{R}$$

$$h(u) = H(u, V_0)$$

where $V_0 \in Sn + 1 : \textit{fixed}$

∘ **Discriminant set Bifurcation set**

$\Sigma(H) := \{(u, V) \in U \times S^{n+1} \,|\, H(u, V) = H_i(u, V) = 0\ (i = 1, \ldots, n)\}$

$(H_i = \frac{\partial H}{\partial u^i})$

$D_H := \{V \in S^{n+1} \mid \exists u \in U\ ,\ (u, V) \in \Sigma(H)\}$ — *discriminant set*

$$C(H) = \left\{(u, V) \in U \times S^{n+1} \,\middle|\, H_i(u, V) = 0\ (i = 1, \ldots, n)\right\}$$

$$B_H = \left\{V \in S^{n+1} \,\middle|\, \exists u \in U\ s.t.\ (u, V) \in C(H)\ and\ rk\left(\frac{\partial^2 H}{\partial u_i \partial u_j}(u, V)\right)_{i,j} < n\right\}$$

About Discriminant set

$H(u, V) = H_i(u, V) = 0 \Leftrightarrow \boldsymbol{x}(u) \cdot V = \boldsymbol{x}_i(u) \cdot V = 0$

hence

$$D_H = \{\pm \boldsymbol{e}(u) \mid u \in U\}$$

About Bifurcation set

$$B_H = \left\{ \pm \frac{\kappa(u)}{\sqrt{\kappa^2(u) + 1}} \left(\boldsymbol{x}(u) + \frac{1}{\kappa(u)} \boldsymbol{e}(u)\right) \,\middle|\, \kappa(u) \neq 0 \ :\ a\ principal\ curvature\ at\ \boldsymbol{x}(u) \right\}$$

We denote $EV_\kappa^{\pm}(u) := \pm \frac{\kappa(u)}{\sqrt{\kappa^2(u)+1}} \left(\boldsymbol{x}(u) + \frac{1}{\kappa(u)} \boldsymbol{e}(u)\right)$.

For any $u \in U$, let $V_0 = EV_\kappa^{\pm}(u)$, $c_0 = \pm \frac{\kappa(u)}{\sqrt{\kappa^2(u)+1}}$, then $\boldsymbol{x}(u)$ and $S^n(V_0, c_0)$ have degenerate contact.

Germs of the height function ***H*** are ***Morse family*** at any points. Then we can construct following immersion germs :

Legendrian immersion germ

$L_H : (\Sigma(H), (u_0, V_0)) \longrightarrow PT^*S.$ ($S = S^{n+1} \cap \{V \mid v_0 > 0\}$ without loss of generality)

$L_H(u, V) = (V, [x^1(u)e^0(u) - x^0(u)e^1(u) : \ldots : x^{n+1}(u)e^0(u) - x^0(u)e^{n+1}(u)]$)

Let $W(L_H)$ be a **wave front** of L_H ,then

$$D_H = W(L_H)$$

at the set germ. So classification of singularities of $e(U) \Leftrightarrow W(L_H)$.

Theorem

Let $u_i \in U$ (i=1,2) , $H_{(i)}$ and $L_{H(i)}$ be corresponding germs around u_i .

If $L_{H(i)}$ are Legendrian stable , then following conditions are equivalent ;

(1) $L_{H(1)}$ and $L_{H(2)}$ are legendrian equivalent;

(2) $W(L_{H(1)})$ and $W(L_{H(2)})$ are diffeomorphic as set germs;

(3) $H_{(1)}$ and $H_{(2)}$ are $P-\mathcal{K}$ equivalent;

(4) $h_{(1)}$ and $h_{(2)}$ are $\mathcal{K}$ equivalent;

To consider geometric meanings of above equivalence , we use the theory of contact due to Montaldi .

Contact with hyperspheres (due to Montaldi)

$X_i, Y_i \subset (\mathbb{R}^n, y_i)$ $(i = 1, 2)$: submanifold germs with $\dim X_1 = \dim X_2$, $\dim Y_1 = \dim Y_2$.

The *contact* of X_1 and Y_1 at y_1 is the same type as the *contact* of X_2 and Y_2 at y_2

$$\Leftrightarrow \exists \Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2) \quad \text{diffeomorphism germ}$$

$$s.t.\ \Phi(X_1, y_1) = (X_2, y_2)\ and\ \Phi(Y_1, y_1) = (Y_2, y_2)$$

In this case, we white $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$

Theorem (Montaldi)

Let $f_i : (\mathbb{R}^n, y_i) \to (\mathbb{R}^p, 0)$ submersion germs with $(f_i^{-1}(0), y_i) = (Y_i, y_i)$

$g_i : (X_i, x_i) \to (\mathbb{R}^n, y_i)$ immersion germs .

Then

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$$

$$\Longleftrightarrow \quad f_1 \circ g_1 \text{ and } f_2 \circ g_2 \text{ are } \mathcal{K} \text{ - equivalent.}$$

By applying it to our case , we have the following .

Theorem Following conditions are equivalent ;

- $W(L_{(1)})$ and $W(L_{(2)})$ are diffeomorphic as set germs
- $K(M_1\ ,\ S^n(\boldsymbol{e}(u_1), 0)\ ;\ p_1) = K(M_2\ ,\ S^n(\boldsymbol{e}(u_2), 0)\ ;\ p_2)$

  where $p_i = \boldsymbol{x}(u_i)$, M_i are set germs of the hypersurface around p_i .

Also , germs of the height function ***H*** are ***Morse (hypersurfase) family*** at any points. Then we can construct following immersion germs :

Lagrangian immersion germ

$$L(H) : \left(\, C(H).(u_0 V_0) \, \right) \to T^* S$$

$$L(H)(u,V) = \left(V \; , \; x^1(u) - x^0(u)\frac{V^1}{V^0} \; , \; \ldots \; , \; x^{n+1}(u) - x^0(u)\frac{V^{n+1}}{V^0}\right)$$

Let ***C(L(H))*** be a **caustic** of ***L(H)*** ,then

$$B^H = C(L(H))$$

at the set germ. So classification of singularities of evolute ⇔ $C(L^H)$.

If two Lagrangian immersion germs are Lagrangian equivalent , then their caustics are diffeomorphic.

Teorem

Let $u_i \in U$ (i=1,2) , $H_{(i)}$ and $L(H_{(i)})$ be corresponding germs around u_i .

If $L(H_{(i)})$ are Lagrangian stable , then following conditions are equivalent ;

(1) $L(H_{(1)})$ and $L(H_{(2)})$ are lagrangian equivalent

(2) $H_{(1)}$ and $H_{(2)}$ are $P-\mathcal{R}^+$ equivalent

(3) $h_{(1)}$ and $h_{(2)}$ are $\mathcal{R}^+$ equivalent

To consider geometric meanings of above equivalence , we use the theory of contact due to S.Izumiya , D.Pei and M.Takahashi .

$X_i \subset (\mathbb{R}^n, y_i)$ $(i=1,2)$ submanifold germs with $\dim X_1 = \dim X_2$

$f_i : (\mathbb{R}^n, y_i) \to (\mathbb{R}, 0)$ submersion germs

$g_i : (X_i, x_i) \to (\mathbb{R}^n, y_i)$ immersion germs

$\mathcal{F}_{f_i} = \{f_i^{-1}(c) \mid c \in (\mathbb{R}, 0)\}$ foliations

The *contact* of X_1 with $\mathcal{F}_{f_1}$ at y_1 is the **same type** as the *contact* of X_2 with $\mathcal{F}_{f_2}$ at y_2

$\iff \exists \Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$ diffeomorphism germ

$s.t.$ $\Phi(X_1, y_1) = (X_2, y_2)$ and $\Phi(f_1^{-1}(c)) = f_2^{-1}(c)$ for each $c \in (\mathbb{R}, 0)$

In this case we white $K(X_1, \mathcal{F}_{f_1}; y_1) = K(X_2, \mathcal{F}_{f_2}; y_2)$

Theorem (S.Izumiya , D.Pei and M.Takahashi)

As above , let $\dim X_1 = n-1$, and we assume that x_i are singularities of $f_i \circ g_i : (X_i, x_i) \to (\mathbb{R}, 0)$. Then

$$K(X_1, \mathcal{F}_{f_1}; y_1) = K(X_2, \mathcal{F}_{f_2}; y_2)$$
$$\iff f_1 \circ g_1 \text{ and } f_2 \circ g_2 \text{ are } \mathcal{R}^+ \text{ - equivalent.}$$

By applying it to our case, we have the following.

Theorem Following conditions are equivalent ;

- $L(H_{(1)})$ and $L(H_{(2)})$ are Lagrangian equivalent
- $K(M_1, \mathcal{F}_{\mathbf{h}_{V_1}}; p_1) = K(M_2, \mathcal{F}_{\mathbf{h}_{V_2}}; p_2)$

  where $V_i = EV_\kappa^\pm(u_i)$, $\mathbf{h}_{V_i} : S^{n+1} \ni V \longmapsto V \cdot V_i \in \mathbb{R}$