The singularities of Gauss maps and evolutes of hypersurfaces in Euclidean n-sphere

Reference
1 Basic notations

Let $S^{n+1} := \{ V = (v^0, v^1, \ldots, v^{n+1}) \in \mathbb{R}^{n+2} \mid V \cdot V = 1 \}$ be an $(n+1)$-dimensional Euclidean sphere, $U$ be an open set in $\mathbb{R}^n$ and $(u^1, \ldots, u^n)$ is coordinate of $U$.

- Hypersurface in Euclidean sphere
  \[ \exists \mathbf{x} : U \to S^{n+1} \quad \text{embedding} \]
  \[ \text{(or } M := \mathbf{x}(U) \text{)} \]

- Gauss map of hypersurface
  \[ e(u) = \frac{\mathbf{x}(u) \wedge \mathbf{x}_1(u) \wedge \mathbf{x}_2(u) \wedge \cdots \wedge \mathbf{x}_n(u)}{\| \mathbf{x}(u) \wedge \mathbf{x}_1(u) \wedge \mathbf{x}_2(u) \wedge \cdots \wedge \mathbf{x}_n(u) \|} \]
  where $\mathbf{x}_i(u) := \frac{\partial \mathbf{x}}{\partial u^i}(u)$

- Curvature
  \[ S_p = -de(u) : T_pM \to T_pM \quad \text{shape operator of } M \text{ at } p \]
  \[ \kappa(u) : \text{eigenvalue of } S_p \]
  \[ K_p := \det S_p \quad \text{principal curvature of } M \text{ at } p \]

- Umbilic
  \[ p \text{ is an umbilic point } \iff \exists \kappa_p \in \mathbb{R} \text{ s.t. } S_p = \kappa_p \cdot \text{id}_{T_pM} \]
  \[ M \text{ is totally umbilic } \iff \forall p \in M \text{ are umbilic} \]

- Hypersphere
  \[ S^n(V, c) := \{ V' \in S^{n+1} \mid V \cdot V' = c \} \]
  \[ c = 0 : \text{great hypersphere} \]
  \[ c \neq 0 : \text{small hypersphere} \]

Remark
If $M$ is totally umbilic, then $\kappa_p \equiv \exists K \in \mathbb{R} \quad (\forall p \in M)$, furthermore
\[ K = 0 \implies \exists V \in S^{n+1} \quad \text{s.t. } M \subset S^n(V, 0) \]
\[ K \neq 0 \implies \exists V \in S^{n+1} \text{ and } \exists c \in \mathbb{R} \quad \text{s.t. } M \subset S^n(V, c) \]

Height function
\[ H : U \times S^{n+1} \to \mathbb{R} \]
\[ H(u, V) = \mathbf{x}(u) \cdot V \]
\[ h = h_0 : U \to \mathbb{R} \]
\[ h(u) = H(u, V_0) \]
where $V_0 \in S^{n+1}$ : fixed

- Discriminant set Bifurcation set
\[ \Sigma(H) := \{(u, V) \in U \times S^{n+1} \mid H(u, V) = H_i(u, V) = 0 \quad (i = 1, \ldots, n) \} \]
\[ (H_i = \frac{\partial H}{\partial u^i}) \]
\[ D_H := \{ V \in S^{n+1} \mid \forall u \in U \quad (u, V) \in \Sigma(H) \} \quad \text{discriminant set} \]
\[ C(H) = \left\{ (u, V) \in U \times S^{n+1} \mid H_i(u, V) = 0 \quad (i = 1, \ldots, n) \right\} \]
\[ B_H = \left\{ V \in S^{n+1} \mid \exists u \in U \text{ s.t. } (u, V) \in C(H) \text{ and } \operatorname{rk} \left( \frac{\partial^2 H}{\partial u^i \partial u^j}(u, V) \right)_{i,j} < n \right\} \]

About Discriminant set
\[ H(u, V) - H_i(u, V) = 0 \iff \mathbf{x}(u) \cdot V = \mathbf{x}_i(u) \cdot V = 0 \]

hence
\[ D_H = \{ \pm e(u) \mid u \in U \} \]

About Bifurcation set
\[ B_H = \left\{ \pm \frac{\kappa(u)}{\sqrt{\kappa^2(u) + 1}} \left( \mathbf{x}(u) + \frac{1}{\kappa(u)} e(u) \right) \middle| \kappa(u) \neq 0 : \text{a principal curvature at } \mathbf{x}(u) \right\} \]

We denote $EV_x^2(u) := \pm \frac{\kappa(u)}{\sqrt{\kappa^2(u) + 1}} \left( \mathbf{x}(u) + \frac{1}{\kappa(u)} e(u) \right)$.

For any $u \in U$, let $V_0 = EV_x^2(u) \quad c_0 = \pm \frac{\kappa(u)}{\sqrt{\kappa^2(u) + 1}}$, then $\mathbf{x}(u)$ and $S^n(V_0, c_0)$ have degenerate contact.
Germs of the height function $H$ are Morse family at any points. Then we can construct following immersion germs:

**Legendrian immersion germ**

$L_H : (\Sigma(H), (u_0, V_0)) \rightarrow PT^* S.$

for any $(u_0, V_0) \in S$ (without loss of generality)

$$L_H(u, V) = (V, [x^1(u)e^0(u) - x^0(u)e^1(u): \ldots: x^{n+1}(u)e^0(u) - x^0(u)e^{n+1}(u)]$$

Let $W(L^H)$ be a wave front of $L^H$, then

$$D^H = W(L^H)$$

at the set germ. So classification of singularities of $e(U) \Rightarrow W(L^H)$.

**Theorem**

Let $u_i \in U$ (i=1,2), $H_{(i)}$ and $L_{H_{(i)}}$ be corresponding germs around $u_i$.

If $L_{H_{(i)}}$ are Legendrian stable, then following conditions are equivalent:

1. $L_{H_{(1)}}$ and $L_{H_{(2)}}$ are legendrian equivalent;
2. $W(L_{H_{(1)}})$ and $W(L_{H_{(2)}})$ are diffeomorphic as set germs;
3. $H_{(1)}$ and $H_{(2)}$ are $P - \mathcal{K}$ equivalent;
4. $h_{(1)}$ and $h_{(2)}$ are $\mathcal{K}$ equivalent;
To consider geometric meanings of above equivalence, we use the theory of contact due to Montaldi.

**Contact with hyperspheres (due to Montaldi)**

\[ X_i, Y_i \subset (\mathbb{R}^n, y_i) \ (i = 1, 2) : \text{submanifold germs with } \dim X_1 = \dim X_2, \dim Y_1 = \dim Y_2. \]

The contact of \( X_1 \) and \( Y_1 \) at \( y_1 \) is the same type as the contact of \( X_2 \) and \( Y_2 \) at \( y_2 \)

\[ \iff \exists \Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2) \text{ diffeomorphism germ} \]

\[ \text{s.t. } \Phi(X_1, y_1) = (X_2, y_2) \text{ and } \Phi(Y_1, y_1) = (Y_2, y_2) \]

In this case, we write \( K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2) \)

**Theorem (Montaldi)**

Let \( f_i : (\mathbb{R}^n, y_i) \to (\mathbb{R}^p, 0) \) submersion germs with \( (f_i^{-1}(0), y_i) = (Y_i, y_i) \)

\[ g_i : (X_i, x_i) \to (\mathbb{R}^n, y_i) \text{ immersion germs.} \]

Then

\[ K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2) \]

\[ \iff f_1 \circ g_1 \text{ and } f_2 \circ g_2 \text{ are } k \text{- equivalent.} \]

By applying it to our case, we have the following.

**Theorem** Following conditions are equivalent;

- \( W(L_{(1)}) \) and \( W(L_{(2)}) \) are diffeomorphic as set germs
- \( K(M_1, S^n(e(u_1), 0); p_1) = K(M_2, S^n(e(u_2), 0); p_2) \)
  where \( p_i = x(u_i) \), \( M_i \) are set germs of the hypersurface around \( p_i \).
Also, germs of the height function $H$ are Morse \textit{(hypersurface) family} at any points. Then we can construct following immersion germs:

\textbf{Lagrangian immersion germ}

$L(H) : \left( C(H). (u_0 V_0) \right) \rightarrow T^* S$

$L(H)(u,V) = \left( V, x^1(u) - x^0(u) \frac{V^1}{V_0}, \ldots, x^{n+1}(u) - x^0(u) \frac{V^{n+1}}{V_0} \right)$

Let $C(L(H))$ be a \textbf{caustic} of $L(H)$, then

$$B^H = C(L(H))$$

at the set germ. So classification of singularities of evolute $\Rightarrow C(L^H)$. If two Lagrangian immersion germs are Lagrangian equivalent, then their caustics are diffeomorphic.

\textbf{Theorem}

Let $u_i \in U$ (i=1,2), $H_{(i)}$ and $L(H_{(i)})$ be corresponding germs around $u_i$. If $L(H_{(i)})$ are Lagrangian stable, then following conditions are equivalent:

1. $L(H_{(1)})$ and $L(H_{(2)})$ are lagrangian equivalent
2. $H_{(1)}$ and $H_{(2)}$ are $P - R^+$ equivalent
3. $h_{(1)}$ and $h_{(2)}$ are $R^+$ equivalent
To consider geometric meanings of above equivalence, we use the theory of contact due to S. Izumiya, D. Pei and M. Takahashi.

\[ X_i \subset (\mathbb{R}^n, y_i) \quad (i = 1, 2) \quad \text{submanifold germs with } \dim X_1 = \dim X_2 \]
\[ f_i : (\mathbb{R}^n, y_i) \to (\mathbb{R}, 0) \quad \text{submersion germs} \]
\[ g_i : (X_i, x_i) \to (\mathbb{R}^n, y_i) \quad \text{immersion germs} \]
\[ \mathcal{F}_{f_i} = \{ f_i^{-1}(c) \mid c \in (\mathbb{R}, 0) \} \quad \text{foliations} \]

The contact of \( X_1 \) with \( \mathcal{F}_{f_1} \) at \( y_1 \) is the same type as the contact of \( X_2 \) with \( \mathcal{F}_{f_2} \) at \( y_2 \)

\[ \iff \exists \Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2) \quad \text{diffeomorphism germ} \]
\[ \text{s.t. } \Phi(X_1, y_1) = (X_2, y_2) \text{ and } \Phi(f_1^{-1}(c)) = f_2^{-1}(c) \text{ for each } c \in (\mathbb{R}, 0) \]

In this case we write \[ K(X_1, \mathcal{F}_{f_1}; y_1) = K(X_2, \mathcal{F}_{f_2}; y_2) \]

**Theorem (S. Izumiya, D. Pei and M. Takahashi)**

As above, let \( \dim X_1 = n - 1 \), and we assume that \( x_i \) are singularities of \( f_i \circ g_i : (X_i, x_i) \to (\mathbb{R}, 0) \). Then

\[ K(X_1, \mathcal{F}_{f_1}; y_1) = K(X_2, \mathcal{F}_{f_2}; y_2) \]

\[ \iff f_1 \circ g_1 \text{ and } f_2 \circ g_2 \text{ are } \mathcal{R}^+ \text{- equivalent.} \]

By applying it to our case, we have the following.

**Theorem** Following conditions are equivalent:

- \( L(H_{(1)}) \) and \( L(H_{(2)}) \) are Lagrangian equivalent
- \( K(M_1, \mathcal{F}_{hV_1}; p_1) = K(M_2, \mathcal{F}_{hV_2}; p_2) \)
  \[ \text{where } V_i = EV_{w_i}^\pm(u_i), \ h_{V_i} : S^{n+1} \ni V \mapsto V \cdot V_i \in \mathbb{R} \]