Spacelike hypersurfaces and submanifolds with codim two in de Sitter space

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Introduction.

We investigate the singularities of lightcone Gauss indicatrices of spacelike hypersurfaces and lightlike hypersurfaces and Gauss maps of spacelike submanifolds in de Sitter space. Each singularities have geometrical meanings.

Spacelike hypersurfaces in de Sitter space

$$S_1^n = \{x \in \mathbb{R}_1^{n+1} \mid \langle x, x\rangle = 1\}.$$

$M^{n-1} = \mathbf{X}(U) \subset S_1^n$, embedding $\mathbf{X} : U \longrightarrow S_1^n$, $U \subset \mathbb{R}^{n-1}$ open.

- **lightcone Gauss indicatrix**

$\mathbb{L}^{\pm} : U \longrightarrow LC^*_{\pm}$, $\mathbb{L}^{\pm}(u) = \mathbf{X}(u) \pm \mathbf{e}(u)$,

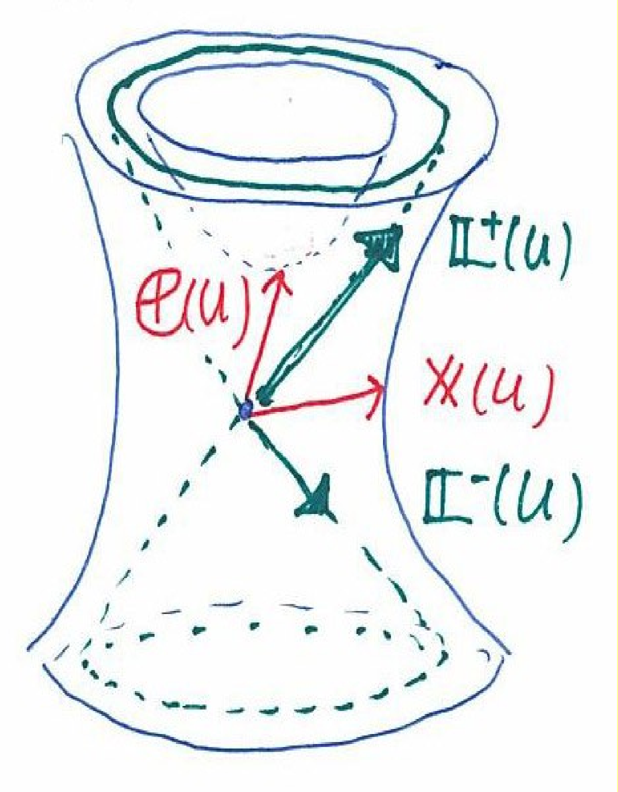
$LC^*_{\pm} = \{x \in \mathbb{R}_1^{n+1} \setminus \{0\} \mid \langle x, x\rangle = 0, x_0 > (or <) 0\}$,

$LC^* = LC^*_+ \cup LC^*_-$.

- **lightcone heightfunction**

$H : U \times LC^* \longrightarrow \mathbb{R}$, $\quad H(u, v) = \langle \mathbf{X}(u), v\rangle - 1$,

$h_{v_0} : U \longrightarrow \mathbb{R}$, $\quad h_{v_0}(u) = H(u, v_0) \quad (v_0 \in LC^*)$.



Remark.

(1) H is a Morse family. (2) Discriminant set of H is $\mathbb{L}^{\pm}(U)$.

(3) singular point u_0 of $\mathbb{L}^{\pm}$ corresponds to parabolic point (Gauss-Kronecker curvature w.r.t $\mathbb{L}^{\pm}$ is 0). Hess $h_{\mathbb{L}^{\pm}(u_0)}$ is degenerate

For generic spacelike hypersurfaces, corresponding maps have relations

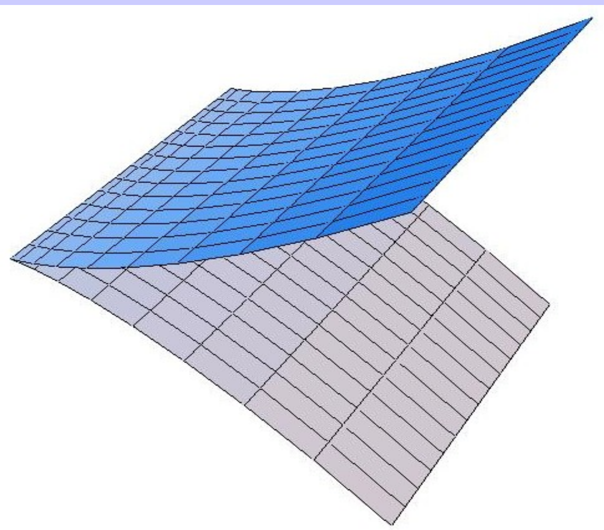
Theorem ($n \leq 6$) For generic $\mathbf{X}_1, \mathbf{X}_2$, the following conditions are equivalent:

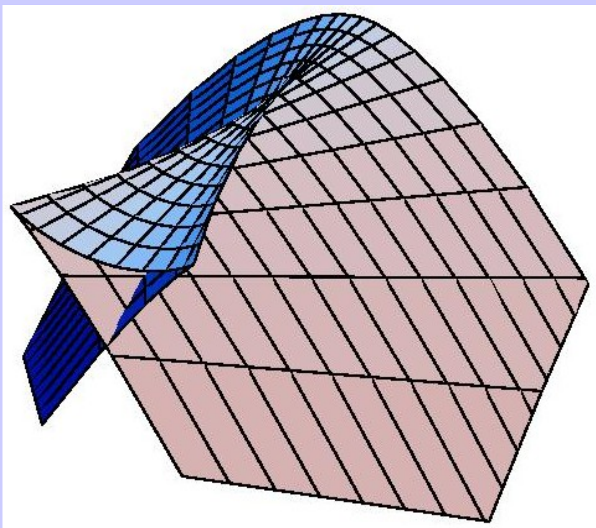
(1) $\mathbb{L}_1^{\pm}$ and $\mathbb{L}_2^{\pm}$ are $\mathcal{A}$-equivalent.

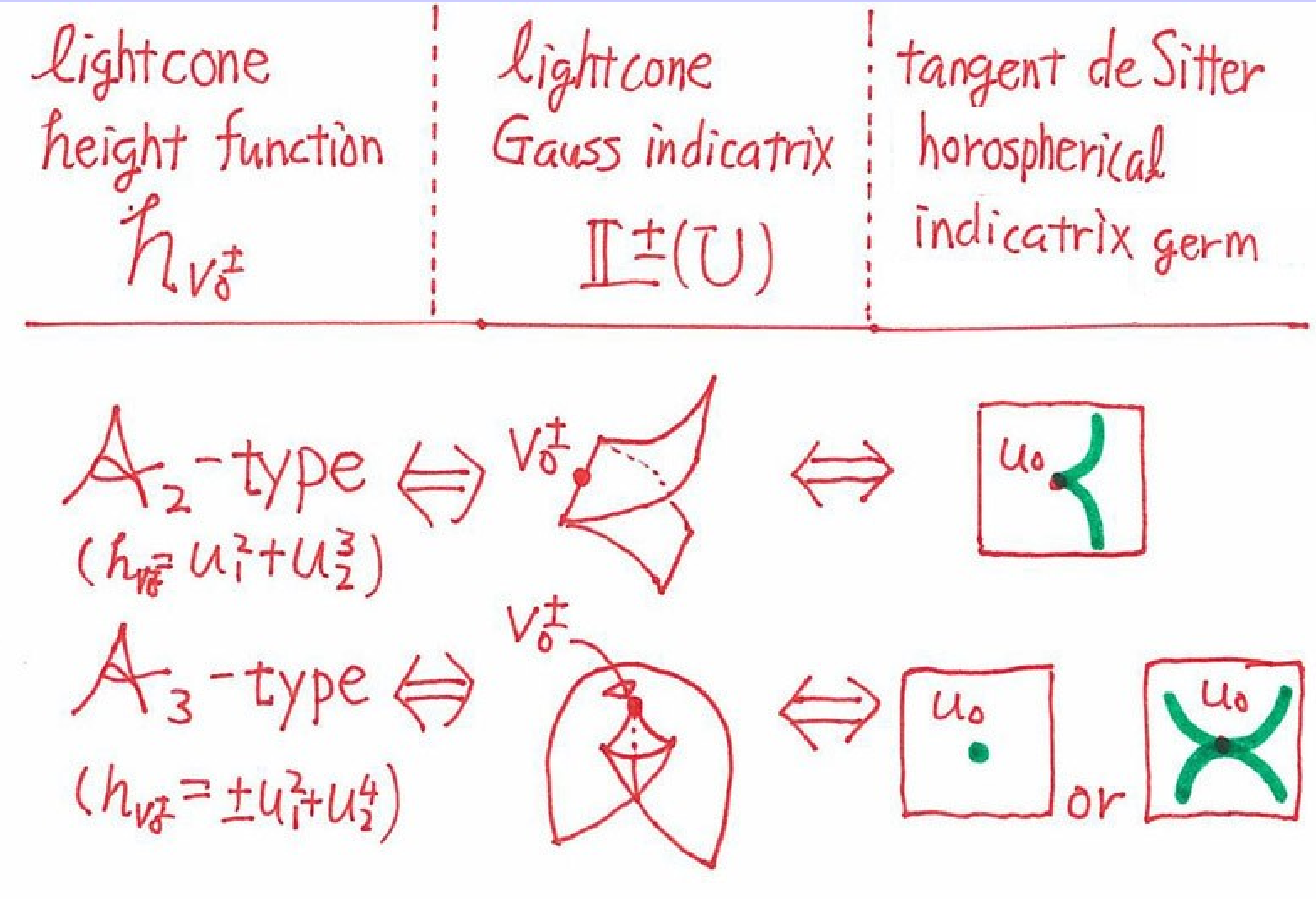
(2) $H_1$and H_2 are $\mathcal{PK}$-equivalent.

(3) $h_{v_1^{\pm}}$ and $h_{v_2^{\pm}}$ are $\mathcal{K}$-equivalent.

If n=3, we can classify two singular types of lightcone Gauss maps.







Spacelike submanifolds with cod=2

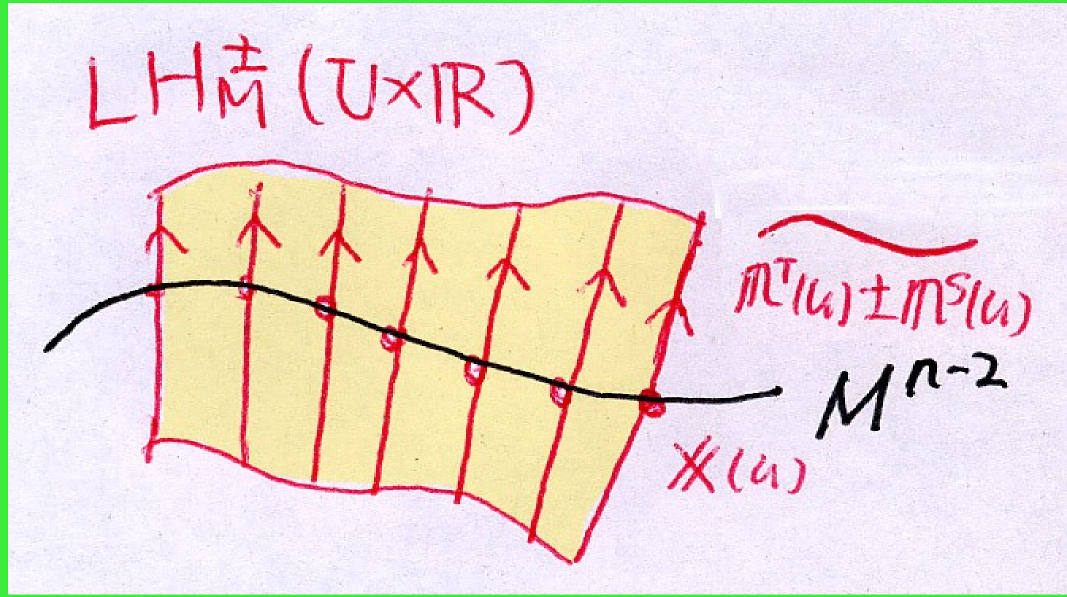
$M^{n-2} = \mathbf{X}(U) \subset S_1^n$, embedding map $X : U \longrightarrow S_1^n$, $U \subset \mathbb{R}^{n-2}$ open.

- **lightlike hypersurface** along M

$$LH_M^{\pm} : U \times \mathbb{R} \longrightarrow S_1^n,$$

$$LH_M^{\pm}(u,t) = \mathbf{X}(u) + t\,(\widetilde{\mathbf{n}^T(u) \pm \mathbf{n}^S(u)}),$$

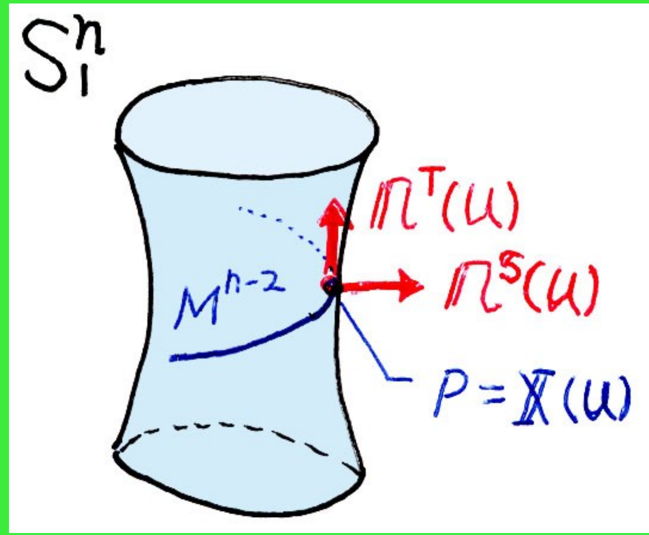
$\widetilde{v} = (1, v_1/v_0, \cdots, v_n/v_0)$ for $v = (v_0, \cdots, v_n) \in LC^*$.



- **Lorentzian distance squared function** of M

$$G : U \times S_1^n \longrightarrow \mathbb{R}, \qquad G(u,\lambda) = \langle \mathbf{X}(u) - \lambda, \mathbf{X}(u) - \lambda \rangle,$$

$$g_{\lambda_0} : U \longrightarrow \mathbb{R}, \qquad g_{\lambda_0}(u) = G(u, \lambda_0) \qquad (\lambda_0 \in S_1^n).$$



singular points of $LH_M^{\pm}$ correspond to **normalized lightcone principal curvatures** of M.

- **lightcone Gauss map**

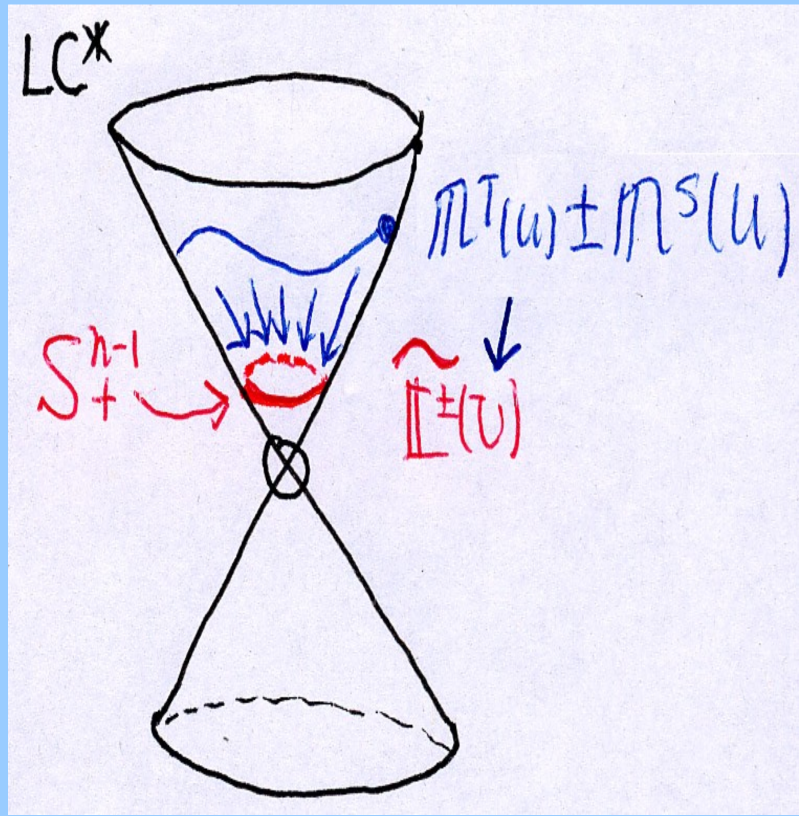
$$\widetilde{\mathbb{L}^{\pm}} : U \longrightarrow S_{+}^{n-1}, \qquad \widetilde{\mathbb{L}^{\pm}}(u) = \widetilde{\mathbf{n}^{T}(u) \pm \mathbf{n}^{S}(u)}$$

where $S_{+}^{n-1} = \{v = (1, v_1, \cdots, v_n) \mid \langle v, v \rangle = 0\}$
$= LC^{*} \cap \{v_0 = 1\}$

- **lightcone height function**

$$H : U \times S_{+}^{n-1} \longrightarrow \mathbb{R}, \qquad H(u, v) = \langle \mathbf{X}(u), v \rangle,$$

$$h_{v_0} : U \longrightarrow \mathbb{R} \qquad h_{v_0} = H(u, v_0) \qquad (v_0 \in S_{+}^{n-1})$$



singular points of $\widetilde{\mathbb{L}^{\pm}}$ correspond to **lightcone parabolic sets** on M.

Classification of Singularities

If we take $n = 4$ case, we have following classification.

- lightlike hypersurface $LH_M^{\pm}$ is $\mathcal{A}$-equivalent to the list of $f(u_1, u_2, u_3)$:

$$
\begin{array}{ll}
(\mathcal{A}_2\text{-type}) & (3u_1^2, 2u_1^3, u_1, u_2) \\
(\mathcal{A}_3\text{-type}) & (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3) \\
(\mathcal{A}_4\text{-type}) & (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3, u_2, u_3) \\
(\mathcal{D}_4^+\text{-type}) & (2(u_1^2 + u_2^2) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3) \\
(\mathcal{D}_4^-\text{-type}) & (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3)
\end{array}
$$

- lightcone Gauss map $\widetilde{\mathbb{L}}^{\pm}(u_1, u_2)$ is $\mathcal{A}$-equivalent to :

$$
\begin{array}{lll}
(\mathcal{A}_2\text{-type}) & (3u_1^2, 2u_1^3, u_1) & \text{(cuspidaledge)} \\
(\mathcal{A}_3\text{-type}) & (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2) & \text{(swallowtail)}
\end{array}
$$

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