

Centroaffine space curves with constant curvatures and homogeneous surfaces

Na Hu

Department of Mathematics,
Hokkaido University

hu@math.sci.hokudai.ac.jp

Abstract

We classify the centroaffine space curves with constant centroaffine curvatures, which are centroaffine homogeneous curves in \mathbb{R}^3 . Moreover, we can find a centroaffine homogeneous surface on which such a space curve lies. This surface gives rise to a centroaffine minimal surface if the centroaffine second curvature of the curve vanishes.

1. Introduction

THE fundamental theorem of curves in centroaffine geometry is obtained by Gardner and Wilkens [2]. In that paper they used the classical Cartan approach to moving frames in order to find the formulation of the local rigidity theorem for curves that is amenable to direct application to problems in control theory.

2. Basic notions of Centroaffine space curves

Definition 2.1 Let $I = (a, b)$ be an interval in \mathbb{R} , $\varphi : I \rightarrow \mathbb{R}^3$ a C^∞ map. $\varphi(t)$ is called a centroaffine curve in \mathbb{R}^3 if $[\varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t)] \neq 0$, for all $t \in I$, where $[\cdot, \cdot, \cdot]$ is the standard volume form of \mathbb{R}^3 , that is, the determinant.

Definition 2.2 A centroaffine curve φ is called nondegenerate if $[\dot{\varphi}(t), \ddot{\varphi}(t), \ddot{\ddot{\varphi}}(t)] \neq 0$, for all $t \in I$.

Definition 2.3 A nondegenerate centroaffine curve is said to be parameterized by centroaffine arc-length parameter if $\varepsilon(s) := \frac{[\dot{\varphi}(s), \ddot{\varphi}(s), \ddot{\ddot{\varphi}}(s)]}{[\varphi(s), \dot{\varphi}(s), \ddot{\varphi}(s)]} = \pm 1$, for all $s \in I$, where ε is called the signature of φ , s is called the centroaffine arc-length parameter of φ .

Remark 2.4 Any nondegenerate centroaffine curve has a reparametrization by centroaffine arc-length parameter.

We use " \cdot " as the differential operator with respect to the general parameter and " $'$ " as the one with respect to the centroaffine arc-length parameter.

Definition 2.5 For a nondegenerate centroaffine curve parameterized by centroaffine arc-length parameter, we define the centroaffine first curvature by

$$\kappa_1(s) := -\frac{[\varphi(s), \varphi''(s), \varphi'''(s)]}{[\varphi(s), \varphi'(s), \varphi''(s)]}$$

and the centroaffine second curvature by

$$\kappa_2(s) := \frac{[\varphi(s), \varphi'(s), \varphi'''(s)]}{[\varphi(s), \varphi'(s), \varphi''(s)]}.$$

Remark 2.6 The centroaffine curvatures $\kappa_1(s)$ and $\kappa_2(s)$ are centroaffine invariants and from the definition of $\kappa_1(s)$ and $\kappa_2(s)$ we get $\varphi'''(s) = \varepsilon(s)\varphi(s) + \kappa_1(s)\varphi'(s) + \kappa_2(s)\varphi''(s)$, that is

$$\frac{d}{ds} \begin{pmatrix} \varphi(s) \\ \varphi'(s) \\ \varphi''(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon(s) & \kappa_1(s) & \kappa_2(s) \end{pmatrix} \begin{pmatrix} \varphi(s) \\ \varphi'(s) \\ \varphi''(s) \end{pmatrix} =: \Omega \begin{pmatrix} \varphi(s) \\ \varphi'(s) \\ \varphi''(s) \end{pmatrix} \quad (1)$$

3. Method

Without loss of generality, we assume $\varepsilon = -1$, the case $\varepsilon = 1$ can be studied similarly. From the fundamental theorem for centroaffine space curves, the centroaffine space curves are uniquely determined by the centroaffine curvatures up to a centroaffine transformation in \mathbb{R}^3 . In the following, we obtain the curve by solving the ODE: $\varphi''' = -\varphi + \kappa_1\varphi' + \kappa_2\varphi''$ with constant coefficients. First, using Shengjin's formula in [1] we get the eigenvalues of the coefficient matrix Ω in (1) which is a one variable cubic equation:

$$|\lambda I - \Omega| = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & 1 \\ 1 & -\kappa_1 & \lambda - \kappa_2 \end{vmatrix} = \lambda^3 - \kappa_2\lambda^2 - \kappa_1\lambda + 1 = 0, \quad (2)$$

whose discriminants of multiple root are:

$$\begin{cases} A := \kappa_2^2 + 3\kappa_1, \\ B := \kappa_1\kappa_2 - 9, \\ C := \kappa_1^2 + 3\kappa_2, \end{cases} \quad (3)$$

and principal discriminant is:

$$\Delta := B^2 - 4AC = 3(-4\kappa_1^3 - 4\kappa_2^3 - 18\kappa_1\kappa_2 - \kappa_1^2\kappa_2^2 + 27), \quad (4)$$

then using the eigenvalues to get the fundamental system of solution of this ODE.

4. Centroaffine space curve with constant centroaffine curvatures

Theorem 4.1 Any nondegenerate centroaffine space curve with constant centroaffine curvatures and signature -1 is centroaffinely equivalent to one of the following curves:

- (1) $\varphi(s) = \{se^{-s}, e^{-s}, s^2e^{-s} + e^{-s}\}$, if $A^2 + B^2 = 0$,
- (2) $\varphi(s) = \{e^{s_1s}, e^{s_2s}, se^{s_1s}\}$, if $A^2 + B^2 \neq 0$ and $\Delta = 0$,
- (3) $\varphi(s) = e^{\kappa_2s/3}\{e^{\rho_1s}\sin(\rho_2s), e^{\rho_1s}\cos(\rho_2s), e^{-2\rho_1s}\}$, if $A^2 + B^2 \neq 0$ and $\Delta > 0$,
- (4) $\varphi(s) = e^{\kappa_2s/3}\{e^{-2\sigma_1s}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s}\}$, if $A^2 + B^2 \neq 0$ and $\Delta < 0$,

where s_i, ρ_i and σ_i ($i = 1, 2$), A, B, Δ are constants with respect to centroaffine curvatures.

5. Groups of centroaffine space curves with constant curvatures

As it is known, the homogeneous curves which are precisely the orbits of certain one parameter subgroups G of $GL(3; \mathbb{R})$, it is easy to check the nondegenerate centroaffine spaces curve with constant centroaffine curvatures are centroaffine homogeneous curves in \mathbb{R}^3 , their groups are the following, respectively

$$G_1 := \left\{ \begin{pmatrix} e^{-s} & se^{-s} & 0 \\ 0 & e^{-s} & 0 \\ 2se^{-s} & s^2e^{-s} & e^{-s} \end{pmatrix} \mid s \in \mathbb{R} \right\}, \quad (5)$$

which acts on the point ${}^t(0, 1, 1)$;

$$G_2 := \left\{ \begin{pmatrix} e^{s_1s} & 0 & 0 \\ 0 & e^{s_2s} & 0 \\ se^{s_1s} & 0 & e^{s_1s} \end{pmatrix} \mid s \in \mathbb{R} \right\}, \quad (6)$$

which acts on the point ${}^t(1, 1, 0)$;

$$G_3 := \left\{ e^{\kappa_2s/3} \begin{pmatrix} e^{\rho_1s}\cos(\rho_2s) & e^{\rho_1s}\sin(\rho_2s) & 0 \\ -e^{\rho_1s}\sin(\rho_2s) & e^{\rho_1s}\cos(\rho_2s) & 0 \\ 0 & 0 & e^{-2\rho_1s} \end{pmatrix} \mid s \in \mathbb{R} \right\}, \quad (7)$$

which acts on the point ${}^t(0, 1, 1)$;

$$G_4 := \left\{ e^{\kappa_2s/3} \begin{pmatrix} e^{-2\sigma_1s} & 0 & 0 \\ 0 & e^{(\sigma_1+\sigma_2)s} & 0 \\ 0 & 0 & e^{(\sigma_1-\sigma_2)s} \end{pmatrix} \mid s \in \mathbb{R} \right\}, \quad (8)$$

which acts on the point ${}^t(1, 1, 1)$.

6. Centroaffine homogeneous surfaces on which space curves with constant curvatures lie

First, we review the following fact in [4],

Theorem 6.1 (H.L.Liu and C.P.Wang) Let $x : M \rightarrow \mathbb{R}^3$ be a nondegenerate centroaffinely homogeneous surface. Then x is centroaffinely equivalent to one of the following surfaces in \mathbb{R}^3 :

- (i) $x^2 + y^2 + z^2 = 1$;
 - (ii) $x^\alpha y^\beta z^\gamma = 1, \alpha \neq 0, \alpha + \beta + \gamma \neq 0$;
 - (iii) $\exp(-\alpha \arctan \frac{x}{y})(x^2 + y^2)^\beta z^\gamma = 1, \alpha^2 + \beta^2 \neq 0, 2\beta + \gamma \neq 0$;
 - (iv) $z = -x(\alpha \log x + \beta \log y), \alpha + \beta \neq 0$;
 - (v) $yz = \pm x^2 + y^\alpha, \alpha \neq 1, 2$;
- where α, β, γ are constants.

6.1 The curve of type (2) and the surface of type (iv)

The curve of type (2) lies on the surface of type (iv) in Theorem 6.1. One can choose the parameters

$$\begin{cases} \alpha = \beta = 2\frac{1}{3}, \text{ if } \kappa_2 = 0, \\ \alpha = -2/\kappa_2, \beta = -1/\kappa_2, \text{ if } \kappa_2 \neq 0, \end{cases} \quad (9)$$

so that $\alpha + \beta \neq 0$. The surface can be written as

$$\tilde{G}_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \subset \mathbb{R}^3,$$

where

$$\tilde{G}_2 := \left\{ \begin{cases} \begin{pmatrix} e^u & 0 & 0 \\ 0 & e^v & 0 \\ -2\frac{1}{3}e^u(u+v) & 0 & e^u \end{pmatrix} \mid u, v \in \mathbb{R}, \text{ if } \kappa_2 = 0, \\ \begin{pmatrix} e^u & 0 & 0 \\ 0 & e^v & 0 \\ \frac{1}{\kappa_2}e^u(2u+v) & 0 & e^u \end{pmatrix} \mid u, v \in \mathbb{R}, \text{ if } \kappa_2 \neq 0, \end{cases} \right. \quad (10)$$

from (6), we can easily get $G_2 \subset \tilde{G}_2$.

6.2 The curve of type (3) and the surface of type (iii)

The curve of type (3) lies on the surface $\exp(-\frac{\kappa_2}{\rho_2} \arctan \frac{x}{y})(x^2 + y^2)z = 1$, which is given by (iii) with

$$\alpha = \kappa_2/\rho_2, \beta = \gamma = 1, \alpha^2 + \beta^2 \neq 0, 2\beta + \gamma \neq 0, \quad (11)$$

in Theorem 6.1 and can be written as

$$\tilde{G}_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \subset \mathbb{R}^3,$$

where

$$\tilde{G}_3 := \left\{ \begin{pmatrix} e^u \cos(v) & e^u \sin(v) & 0 \\ -e^u \sin(v) & e^u \cos(v) & 0 \\ 0 & 0 & e^{-2u + \frac{\kappa_2}{\rho_2}v} \end{pmatrix} \mid u, v \in \mathbb{R} \right\}, \quad (12)$$

from (7), we can easily get $G_3 \subset \tilde{G}_3$.

6.3 The curve of type (4) and the surface of type (ii)

The curve of type (4) lies on the surface of type (ii) in Theorem 6.1. One can choose the parameters

$$\begin{cases} \alpha = \beta = \gamma = 1, \text{ if } \kappa_2 = 0, \\ \alpha = -2\sigma_1 - \frac{2A}{3\kappa_2}, \beta = \sigma_1 + \sigma_2 - \frac{2A}{3\kappa_2}, \gamma = \sigma_1 - \sigma_2 - \frac{2A}{3\kappa_2}, \text{ if } \kappa_2 \neq 0, \end{cases} \quad (13)$$

so that $\alpha \neq 0, \alpha + \beta + \gamma \neq 0$. The surface can be written as

$$\tilde{G}_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \subset \mathbb{R}^3,$$

where

$$\tilde{G}_4 := \left\{ \begin{cases} \begin{pmatrix} e^u & 0 & 0 \\ 0 & e^v & 0 \\ 0 & 0 & e^{-u-v} \end{pmatrix} \mid u, v \in \mathbb{R}, \text{ if } \kappa_2 = 0, \\ \begin{pmatrix} e^{\gamma u} & 0 & 0 \\ 0 & e^{\gamma v} & 0 \\ 0 & 0 & e^{-\alpha u - \beta v} \end{pmatrix} \mid u, v \in \mathbb{R}, \text{ if } \kappa_2 \neq 0, \end{cases} \right. \quad (14)$$

from (8), we can easily get $G_4 \subset \tilde{G}_4$.

Theorem 6.2 The nondegenerate centroaffine space curve with constant centroaffine curvatures and signature -1 are homogeneous curves written as follows:

- (1) $G_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \subset \mathbb{R}^3$,
- (2) $G_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \subset \tilde{G}_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \{z = -x(\alpha \log x + \beta \log y)\} \subset \mathbb{R}^3$,
- (3) $G_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \subset \tilde{G}_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \{\exp(-\alpha \arctan \frac{x}{y})(x^2 + y^2)^\beta z^\gamma = 1\} \subset \mathbb{R}^3$,
- (4) $G_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \subset \tilde{G}_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \{x^\alpha y^\beta z^\gamma = 1\} \subset \mathbb{R}^3$.

where $G_i (i = 1, 2, 3, 4)$ are one parameter subgroups of $GL(3; \mathbb{R})$ given by (5) ~ (8), $\tilde{G}_j (j = 2, 3, 4)$ are two parameter subgroups of $GL(3; \mathbb{R})$ given by (10), (12), (14), α, β, γ are constants with respect to the centroaffine curvatures given by (9), (11), (13), respectively.

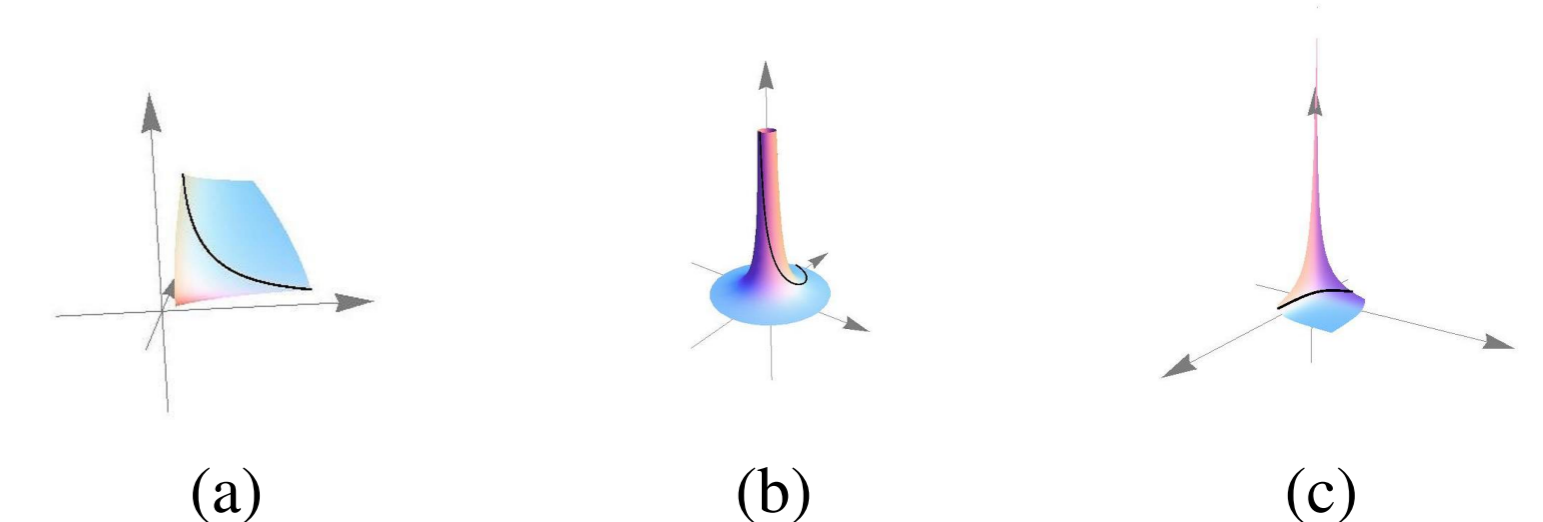


Figure 1: (a) the curve of type (2) ($\kappa_1 = \kappa_2 = 1$) and the surface $z = x(2 \log x + \log y)$, (b) the curve of type (3) ($\kappa_1 = \kappa_2 = 0$) and the surface $(x^2 + y^2)z = 1$, (c) the curve of type (4) ($\kappa_1 = 3, \kappa_2 = 0$) and the surface $xyz = 1$.

Theorem 6.3 Any nondegenerate centroaffine space curve with vanishing centroaffine second curvature is lying on a centroaffine minimal surface which is determined by the centroaffine first curvature of the curve.

References

- [1] S. J. Fan, A new extracting formula and a new distinguishing means on the one variable cubic equation, NATURAL SCIENCE JOURNAL OF HAINAN TEACHERS COLLEGE, (1989).
- [2] R. B. Gardner and G. R. Wilkens, The fundamental theorems of curves and hypersurfaces in centro-affine geometry, Bull. Belg. Math. Soc. Simon Stevin 4, 379(1997).
- [3] N.Hu, Centroaffine space curves with constant curvatures and homogeneous surfaces, preprint.
- [4] H.L.Liu and C. P. Wang, Centroaffinely homogeneous surfaces in \mathbb{R}^3 , Contrib. Algebra. Geom., 35(1994).
- [5] C.P.Wang, Centroaffine minimal hypersurfaces in \mathbb{R}^{n+1} , Geom. Dedicata 51 (1994), 63-74.