ON A RAMIFICATION BOUND OF TORSION
SEMI-STABLE REPRESENTATIONS OVER A LOCAL
FIELD

SHIN HATTORI

Abstract. Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors, $K$ be a finite totally ramified extension of Frac($W$) of degree $e$ and $r$ be a non-negative integer satisfying $r < p - 1$. In this paper, we prove the upper numbering ramification group $G^{(j)}_K$ for $j > u(K, r, n)$ acts trivially on the $p^n$-torsion semi-stable $G_K$-representations with Hodge-Tate weights in $\{0, \ldots, r\}$, where $u(K, 0, n) = 0$, $u(K, 1, n) = 1 + e(n + 1/(p - 1))$ and $u(K, r, n) = 1 - p^{-n} + e(n + r/(p - 1))$ for $1 < r < p - 1$.

1. Introduction

Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors and $K$ be a finite totally ramified extension of $K_0 = \text{Frac}(W)$ of degree $e = e(K)$. We normalize the valuation $v_K$ of $K$ as $v_K(p) = e$ and extend this to any algebraic closure of $K$. Let the maximal ideal of $K$ be denoted by $m_K$, an algebraic closure of $K$ by $\bar{K}$ and the absolute Galois group of $K$ by $G_K = \text{Gal}(\bar{K}/K)$. Let $G^{(j)}_K$ denote the $j$-th upper numbering ramification group in the sense of [10]. Namely, we put $G^{(j)}_K = G^{j-1}_K$, where the latter is the upper numbering ramification group defined in [18].

Consider a proper smooth scheme $X_K$ over $K$ and put $X_{\bar{K}} = X_K \times_K \bar{K}$. Let $\mathcal{L} \supseteq \mathcal{L}'$ be $G_K$-stable $\mathbb{Z}_p$-lattices in the $r$-th etale cohomology group $H^r_{\text{et}}(X_{\bar{K}}, \mathbb{Q}_p)$ such that the quotient $\mathcal{L}/\mathcal{L}'$ is killed by $p^n$. In [10], Fontaine conjectured the upper numbering ramification group $G^{(j)}_K$ acts trivially on the $G_K$-modules $\mathcal{L}/\mathcal{L}'$ and $H^r_{\text{et}}(X_K, \mathbb{Z}/p^n\mathbb{Z})$ for $j > e(n + r/(p - 1))$ if $X_K$ has good reduction. For $e = 1$ and $r < p - 1$, this conjecture was proved independently by himself ([11], for $n = 1$) and Abrashkin ([3], for any $n$), using the theory of Fontaine-Laffaille ([13]) and the comparison theorem of Fontaine-Messing ([14], see also [5] and [7]) between the etale cohomology groups of $X_K$ and the crystalline cohomology groups of the reduction of $X_K$. From these results, they also showed some rareness of a proper smooth

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scheme over $\mathbb{Q}$ with everywhere good reduction ([11, Théorème 1], [2, Section 7]). In fact, they proved this ramification bound for the torsion crystalline representations of $G_K$ with Hodge-Tate weights in $\{0, \ldots, r\}$ in the case where $K$ is absolutely unramified.

On the other hand, for a torsion semi-stable representation with Hodge-Tate weights in the same range, a similar ramification bound for $e = 1$ and $n = 1$ is obtained by Breuil (see [7, Proposition 9.2.2.2]). He showed, assuming the Griffiths transversality which in general does not hold, that if $e = 1$ and $r < p - 1$, then the ramification group $G_K^{(j)}$ acts trivially on the mod $p$ semi-stable representations for $j > 2 + 1/(p - 1)$.

In this paper, we prove a ramification bound for the torsion semi-stable representations of $G_K$ with Hodge-Tate weights in $\{0, \ldots, r\}$ with no assumption on $e$ but under the assumption $r < p - 1$. Let $\varpi$ be a uniformizer of $K$, $E(u) \in W[u]$ be the Eisenstein polynomial of $\varpi$ over $W$ and $S$ be the $p$-adic completion of the divided power envelope of $W[u]$ with respect to the ideal $(E(u))$. Consider a category $\text{Mod}^{r,\phi,N}_{/S_\infty}$ of filtered $(\phi, N)$-modules over the ring $S$ and a $G_K$-module

$$T_{\text{st},\varpi}(\mathcal{M}) = \text{Hom}_{S,\text{Fil}^{r,\phi,N}}(\mathcal{M}, \hat{A}_{\text{st},\infty})$$

for $\mathcal{M} \in \text{Mod}^{r,\phi,N}_{/S_\infty}$, where $\hat{A}_{\text{st},\infty}$ is a $p$-adic period ring ([6]). Then our main theorem is the following.

**Theorem 1.1.** Let $r$ be a non-negative integer such that $r < p - 1$. Let $\mathcal{M}$ be an object of the category $\text{Mod}^{r,\phi,N}_{/S_\infty}$ which is killed by $p^n$. Then the $j$-th upper numbering ramification group $G_K^{(j)}$ acts trivially on the $G_K$-module $T_{\text{st},\varpi}^*(\mathcal{M})$ for $j > u(K, r, n)$, where

$$u(K,r,n) = \begin{cases} 0 & (r = 0), \\
1 + e(n + \frac{1}{p-1}) & (r = 1), \\
1 - \frac{1}{p^n} + e(n + \frac{r}{p-1}) & (1 < r < p - 1). \end{cases}$$

We can check that this bound is sharp for $r \leq 1$ (Remark 5.15).

From this theorem and [10, Proposition 1.3], we have the following corollary.

**Corollary 1.2.** Let the notation be as in the theorem and $L$ be the finite extension of $K$ cut out by the $G_K$-module $T_{\text{st},\varpi}^*(\mathcal{M})$. Namely, the finite extension $L$ is defined by

$$G_L = \text{Ker}(G_K \to \text{Aut}(T_{\text{st},\varpi}^*(\mathcal{M}))).$$

Let $\mathfrak{D}_{L/K}$ denote the different of the extension $L/K$. Then we have the inequality

$$v_K(\mathfrak{D}_{L/K}) < u(K, r, n)$$

for $r > 0$ and $v_K(\mathfrak{D}_{L/K}) = 0$ for $r = 0$. 


Combining these results with a theorem of Liu ([17, Theorem 2.3.5]) or a theorem of Caruso ([8, Théorème 1.1]), we will show the corollary below.

**Corollary 1.3.** Let \( r \) be a non-negative integer such that \( r < p - 1 \). Then the same bounds as in Theorem 1.1 and Corollary 1.2 are also valid for the torsion \( G_K \)-modules of the following two cases:

1. the \( G_K \)-module \( L/L' \), where \( L \supseteq L' \) are \( G_K \)-stable \( \mathbb{Z}_p \)-lattices in a semi-stable \( p \)-adic representation \( V \) with Hodge-Tate weights in \( \{0, \ldots, r\} \) such that \( L/L' \) is killed by \( p^n \).
2. the \( G_K \)-module \( H^r_{et}(X_K, \mathbb{Z}/p^n\mathbb{Z}) \), where \( X_K \) is a proper smooth algebraic variety over \( K \) which has a proper semi-stable model over \( \mathcal{O}_K \) and \( r \) satisfies \( er < p - 1 \) for \( n = 1 \) and \( e(r + 1) < p - 1 \) for \( n > 1 \).

For the proof of Theorem 1.1, we basically follow a beautiful argument of Abrashkin ([3]). We may assume \( p \geq 3 \) and \( r \geq 1 \). Consider the finite Galois extension

\[
F_n = K(\pi^{1/p^n}, \zeta_{p^n+1})
\]

of \( K \) whose upper ramification is bounded by \( u(K, r, n) \). Let \( L_n \) be the finite Galois extension of \( F_n \) cut out by \( T_{st, \pi}(M)|_{G_{F_n}} \). Then we bound the ramification of \( L_n \) over \( K \). For this, we show that to study this \( G_{F_n} \)-module we can use a variant over a smaller coefficient ring \( \Sigma \) of filtered \((\mathbb{A}_r, \mathbb{N})\)-modules over \( S \). In precise, we set

\[
\Sigma = W[[u, E(u)^p/p]].
\]

This ring \( \Sigma \) is small enough for the method of Abrashkin, in which he uses filtered modules of Fontaine-Laffaille ([13]) whose coefficient ring is \( W \), to work also in the case where \( K \) is absolutely ramified.

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2. **Filtered \((\phi_r, N)\)-modules of Breuil**

In this section, we recall the theory of filtered \((\phi_r, N)\)-modules over \( S \) of Breuil, which is developed by himself and most recently by Caruso and Liu (see for example [6], [8], [17], [9]). In what follows, we always take the divided power envelope of a \( W \)-algebra with the compatibility condition with the natural divided power structure on \( pW \).

Let \( p \) be a rational prime and \( \sigma \) be the Frobenius endomorphism of \( W \). We fix once and for all a uniformizer \( \pi \) of \( K \) and a system \( \{\pi_n\}_{n \geq 0} \) of \( p \)-power roots of \( \pi \) in \( \bar{K} \) such that \( \pi_0 = \pi \) and \( \pi_n = \pi_{n+1}^{p^n} \) for any \( n \). Let \( E(u) \)
be the Eisenstein polynomial of \( \pi \) over \( W \) and set \( S = (W[u]^{\text{PD}})\wedge \), where PD means the divided power envelope and this is taken with respect to the ideal \((E(u))\), and \( \wedge \) means the \( p \)-adic completion. The ring \( S \) is endowed with the \( \sigma \)-semilinear endomorphism \( \phi : u \mapsto u^p \) and a natural filtration \( \text{Fil}^t S \) induced by the divided power structure such that \( \phi(\text{Fil}^t S) \subseteq \text{Fil}^{t+1} S \) for \( 0 \leq t \leq p - 1 \). We set \( \phi_t = p^{-t} \phi |_{\text{Fil}^t S} \) and \( c = \phi_1(E(u)) \in S^\infty \). Let \( N \) denote the \( W \)-linear derivation on \( S \) defined by the formula \( N(u) = -u \). We also define a filtration, \( \phi_t \) and \( N \) on \( S_n = S/p^n S \) similarly.

Let \( r \in \{0, \ldots, p - 2\} \) be an integer. Set \( \text{Mod}^{r, \phi, N}_{/S} \) to be the category consisting of the following data:

- an \( S \)-module \( \mathcal{M} \) and its \( S \)-submodule \( \text{Fil}^r \mathcal{M} \) containing \( \text{Fil}^r S \cdot \mathcal{M} \),
- a \( \phi \)-semilinear map \( \phi_r : \text{Fil}^r \mathcal{M} \to \mathcal{M} \) satisfying
  \[ \phi_r(s_r m) = \phi_r(s_r) \phi(m) \]
  for any \( s_r \in \text{Fil}^r S \) and \( m \in \mathcal{M} \), where we set \( \phi(m) = c^{-r} \phi_r(E(u)^r m) \),
- a \( W \)-linear map \( N : \mathcal{M} \to \mathcal{M} \) such that
  - \( N(sm) = N(s)m + sN(m) \) for any \( s \in S \) and \( m \in \mathcal{M} \),
  - \( E(u)N(\text{Fil}^r \mathcal{M}) \subseteq \text{Fil}^r \mathcal{M} \),
  - the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Fil}^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M} \\
E(u)N & \downarrow & \downarrow cN \\
\text{Fil}^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M},
\end{array}
\]

and the morphisms of \( \text{Mod}^{r, \phi, N}_{/S} \) are defined to be the \( S \)-linear maps preserving \( \text{Fil}^r \) and commuting with \( \phi_r \) and \( N \). The category defined in the same way but dropping the data \( N \) is denoted by \( \text{Mod}^{r, \phi}_{/S} \). These categories have obvious notions of exact sequences. Let \( \text{Mod}^{r, \phi, N}_{/S_1} \) denote the full subcategory of \( \text{Mod}^{r, \phi, N}_{/S} \) consisting of \( \mathcal{M} \) such that \( \mathcal{M} \) is free of finite rank over \( S_1 \) and generated as an \( S_1 \)-module by the image of \( \phi_r \). We write \( \text{Mod}^{r, \phi, N}_{/S_\infty} \) for the smallest full subcategory which contains \( \text{Mod}^{r, \phi, N}_{/S_1} \) and is stable under extensions. We let \( \text{Mod}^{r, \phi, N}_{/S} \) denote the full subcategory consisting of \( \mathcal{M} \) such that

- the \( S \)-module \( \mathcal{M} \) is free of finite rank and generated by the image of \( \phi_r \),
- the quotient \( \mathcal{M}/\text{Fil}^r \mathcal{M} \) is \( p \)-torsion free.

We define full subcategories \( \text{Mod}^{r, \phi}_{/S_1}, \text{Mod}^{r, \phi}_{/S_\infty} \) and \( \text{Mod}^{r, \phi}_{/S} \) of \( \text{Mod}^{r, \phi}_{/S} \) in a similar way. For \( \mathcal{M} \in \text{Mod}^{r, \phi, N}_{/S} \) (resp. \( \text{Mod}^{r, \phi}_{/S_\infty} \)), the quotient \( \mathcal{M}/p^n \mathcal{M} \) has a natural structure as an object of \( \text{Mod}^{r, \phi, N}_{/S_\infty} \) (resp. \( \text{Mod}^{r, \phi}_{/S_\infty} \)).
For \( p \)-torsion objects, we also have the following categories. Consider the \( k \)-algebra \( k[u]/(u^p) \cong S_1/\Fil^p S_1 \) and let this algebra be denoted by \( \tilde{S}_1 \). The algebra \( \tilde{S}_1 \) is equipped with the natural filtration, \( \phi \) and \( N \) induced by those of \( S_1 \). Namely, \( \Fil^i \tilde{S}_1 = u^i \tilde{S}_1 \), \( \phi(u) = u^p \) and \( N(u) = -u \). Let \( \Mod^{r,\phi,N}_{/\tilde{S}_1} \) denote the category consisting of the following data:

- an \( \tilde{S}_1 \)-module \( \tilde{M} \) and its \( \tilde{S}_1 \)-submodule \( \Fil^r \tilde{M} \) containing \( u^r \tilde{M} \),
- a \( \phi \)-semilinear map \( \tilde{\phi}_r : \Fil^r \tilde{M} \to \tilde{M} \),
- a \( k \)-linear map \( N : \tilde{M} \to \tilde{M} \) such that
  - \( N(sm) = N(s)m + sN(m) \) for any \( s \in \tilde{S}_1 \) and \( m \in \tilde{M} \),
  - \( u^r N(\Fil^r \tilde{M}) \subseteq \Fil^r \tilde{M} \),
- the following diagram is commutative:

\[
\begin{array}{ccc}
\Fil^r \tilde{M} & \xrightarrow{\tilde{\phi}_r} & \tilde{M} \\
\downarrow{u^r N} & & \downarrow{\epsilon N} \\
\Fil^r \tilde{M} & \xrightarrow{\tilde{\phi}_r} & \tilde{M},
\end{array}
\]

and whose morphisms are defined as before. Its full subcategory \( \Mod^{r,\phi,N}_{/\tilde{S}_1} \) is defined by the following condition:

- As an \( \tilde{S}_1 \)-module, \( \tilde{M} \) is free of finite rank and generated by the image of \( \tilde{\phi}_r \).

We define categories \( \Mod^{r,\phi}_{/\tilde{S}_1} \) and \( \Mod^{r,\phi}_{/\tilde{S}_1} \) similarly. Then we can show as in the proof of [4, Proposition 2.2.2.1] that the natural functor \( \mathcal{M} \to \mathcal{M}/\Fil^p S_1 \cdot \mathcal{M} \) induces equivalences of categories \( T : \Mod^{r,\phi,N}_{/\tilde{S}_1} \to \Mod^{r,\phi,N}_{/\tilde{S}_1} \) and \( T_0 : \Mod^{r,\phi}_{/\tilde{S}_1} \to \Mod^{r,\phi}_{/\tilde{S}_1} \).

For \( r = 0 \), let \( \Mod^{\phi}_{/W_\infty} \) be the category consisting of the following data:

- a finite torsion \( W \)-module \( \tilde{M} \),
- a \( \sigma \)-semilinear automorphism \( \phi : \tilde{M} \to \tilde{M} \).

Let \( \kappa \) be the kernel of the natural surjection \( S \to W \) defined by \( u \mapsto 0 \). Since \( \Tor^S_1(\mathcal{M}, S/\kappa S) = 0 \) for any \( \mathcal{M} \in \Mod^{0,\phi}_{/S_\infty} \), the proofs of [8, Lemme 2.2.7, Proposition 2.2.8] work also for the category \( \Mod^{0,\phi,N}_{/S_\infty} \) and we have a commutative diagram of categories

\[
\begin{array}{ccc}
\Mod^{0,\phi,N}_{/S_\infty} & \longrightarrow & \Mod^{0,\phi}_{/S_\infty} \\
\downarrow & & \downarrow \\
\Mod^{\phi}_{/W_\infty},
\end{array}
\]

where the downward arrows and horizontal arrow are defined by \( \mathcal{M} \to \mathcal{M}/\kappa \mathcal{M} \) and forgetting \( N \) respectively and these three arrows are equivalences of categories.
Let $A_{\text{crys}}$ and $\hat{A}_{\text{st}}$ be $p$-adic period rings. These are constructed as follows. Put $\hat{O}_K = \mathcal{O}_K/p\mathcal{O}_K$. Set $R$ to be the ring

$$R = \lim_{\longrightarrow}(\hat{O}_K \leftarrow \hat{O}_K \leftarrow \cdots),$$

where every arrow is the $p$-power map. For an element $x = (x_i)_{i \in \mathbb{Z}_{\geq 0}} \in R$ and an integer $n \geq 0$, we set

$$x^{(n)} = \lim_{m \to \infty} \hat{x}_i^{p^n} \in \mathcal{O}_C,$$

where $\hat{x}_i$ is a lift of $x_i$ in $\mathcal{O}_K$ and $\mathcal{O}_C$ is the $p$-adic completion of $\mathcal{O}_K$. Let $v_p$ denote the valuation of $\mathcal{O}_C$ normalized as $v_p(p) = 1$. Then the ring $R$ is a complete valuation ring whose valuation of an element $x \in R$ is given by $v_R(x) = v_p(x^{(0)})$. We define a natural ring homomorphism $\theta$ by

$$\theta : W(R) \to \mathcal{O}_C$$

$$(x_0, x_1, \ldots) \mapsto \sum_{n \geq 0} p^n x_n^{(n)}.$$

Then $A_{\text{crys}}$ is the $p$-adic completion of the divided power envelope of $W(R)$ with respect to the principal ideal $\text{Ker}(\theta)$ and $\hat{A}_{\text{st}}$ is the $p$-adic completion of the divided power polynomial ring $A_{\text{crys}}(X)$ over $A_{\text{crys}}$. We set $A_{\text{crys}, \infty} = A_{\text{crys}} \otimes_W K_0/W$ and $\hat{A}_{\text{st}, \infty} = \hat{A}_{\text{st}} \otimes_W K_0/W$. Put $\pi = (\pi_n)_{n \in \mathbb{Z}_{\geq 0}} \in R$, where we abusively let $\pi_n$ denote the image of $\pi_n \in \mathcal{O}_K$ in $\hat{O}_K$. These rings are considered as $S$-algebras by the ring homomorphisms $S \to \hat{A}_{\text{st}}$ and $S \to A_{\text{crys}}$ which are defined by $u \mapsto [\pi]/(1 + X)$ and $X \mapsto 0$, respectively. The ring $A_{\text{crys}}$ is endowed with a natural filtration induced by the divided power structure, a natural Frobenius endomorphism $\phi$ and the $\phi$-semilinear map $\phi_t = p^{-t}\phi|_{\text{Fil}^i A_{\text{crys}}}$. With these structures, $A_{\text{crys}}$ and $A_{\text{crys}, \infty}$ are considered as objects of $\text{Mod}[^i_{/S}]_{\phi}$. Moreover, the absolute Galois group $G_K$ acts naturally on these two rings. As for $\hat{A}_{\text{st}}$, its filtration is defined by

$$\text{Fil}^i \hat{A}_{\text{st}} = \left\{ \sum_{i \geq 0} a_i \frac{X^i}{i!} \mid a_i \in \text{Fil}^{i-i} A_{\text{crys}}, \lim_{i \to \infty} a_i = 0 \right\}$$

and the Frobenius structure of $A_{\text{crys}}$ extends to $\hat{A}_{\text{st}}$ by

$$\phi(X) = (1 + X)^p - 1,$$

$$\phi_t = p^{-t}\phi|_{\text{Fil}^i \hat{A}_{\text{st}}}.$$ 

We write $N$ also for the $A_{\text{crys}}$-linear derivation on $\hat{A}_{\text{st}}$ defined by $N(X) = 1 + X$. The rings $\hat{A}_{\text{st}}$ and $\hat{A}_{\text{st}, \infty}$ are objects of $\text{Mod}[^i_{/S}]_{\phi, N}$. The $G_K$-action on $A_{\text{crys}}$ naturally extends to an action on $\hat{A}_{\text{st}}$. Indeed, the action of $g \in G_K$ on $\hat{A}_{\text{st}}$ is defined by the formula

$$g(X) = [\varepsilon(g)](1 + X) - 1,$$
where \( g(\pi_n) = \varepsilon_n(g)\pi_n \) and \( \varepsilon(g) = (\varepsilon_n(g))_{n \in \mathbb{Z}_{\geq 0}} \in R \) with the abusive notation as above.

These rings have other descriptions, as follows. For an integer \( n \geq 1 \), put \( W_n = W/p^n W \) and let \( W_n(\hat{O}_K) \) be the ring of Witt vectors of length \( n \) associated to \( \hat{O}_K \). We define a \( W_n \)-algebra structure on \( W_n(\hat{O}_K) \) by twisting the natural \( W_n \)-algebra structure by \( \sigma^{-n} \). Then the natural ring homomorphism

\[
\theta_n : W_n(\hat{O}_K) \to \mathcal{O}_K/p^n \mathcal{O}_K
\]

\[
(a_0, \ldots, a_{n-1}) \mapsto \sum_{i=0}^{n-1} p^i \hat{a}_i^{p^{n-i}},
\]

where \( \hat{a}_i \) is a lift of \( a_i \) in \( \mathcal{O}_K \), is \( W_n \)-linear. Let us denote \( W_n^{PD}(\hat{O}_K) \) the divided power envelope of \( W_n(\hat{O}_K) \) with respect to the ideal \( \text{Ker}(\theta_n) \). This ring is considered as an \( S \)-algebra by \( u \mapsto [\pi_n] \). This ring also has a natural filtration defined by the divided power structure, and a natural \( G_K \)-module structure. The Frobenius endomorphism of the ring of Witt vectors induces on this ring a \( \phi \)-semilinear Frobenius endomorphism, which is denoted also by \( \phi \). Then, by the \( S \)-linear transition maps

\[
W_n^{PD}(\hat{O}_K) \to W_n^{PD}(\hat{O}_K)
\]

\[
(a_0, \ldots, a_n) \mapsto (a_0^p, \ldots, a_n^p),
\]

these \( S \)-algebras form a projective system compatible with all the structures. Using this transition map, a \( \phi \)-semilinear map

\[
\phi_r : \text{Fil}^r W_n^{PD}(\hat{O}_K) \to W_n^{PD}(\hat{O}_K)
\]

is defined by setting \( \phi_r(x) \) to be the image of \( p^{-r} \phi(\hat{x}) \), where \( \hat{x} \) is a lift of \( x \) in \( \text{Fil}^r W_n^{PD}(\hat{O}_K) \). By definition, the maps \( \phi_r \) are also compatible with the transition maps. The \( S \)-algebra \( W_n^{PD}(\hat{O}_K) \) is considered as an object of \( '\text{Mod}_{/S}^{\phi} \). Then we have a natural isomorphism in \( '\text{Mod}_{/S}^{\phi} \)

\[
A_{\text{crys}}/p^n A_{\text{crys}} \to W_n^{PD}(\hat{O}_K)
\]

\[
(x_0, \ldots, x_{n-1}) \mapsto (x_{0,n}, \ldots, x_{n-1,n}),
\]

where we set \( x_i = (x_{i,k})_{k \in \mathbb{Z}_{\geq 0}} \) for \( x_i \in R \).

Similarly, the divided power polynomial ring \( W_n^{PD}(\hat{O}_K)[X] \) over \( W_n^{PD}(\hat{O}_K) \) is considered as an \( S \)-algebra by \( u \mapsto [\pi_n]/(1 + X) \). This ring has a natural filtration coming from the divided power structure. We define a \( G_K \)-action on this ring by

\[
g(X) = [\varepsilon_n(g)](1 + X) - 1.
\]

We also define a \( \phi \)-semilinear Frobenius endomorphism, which we also write as \( \phi \), by \( \phi(X) = (1+X)^p - 1 \) and a \( W_n^{PD}(\hat{O}_K) \)-linear derivation \( N \) by \( N(X) = 1 + X \). These rings form a projective system of \( S \)-algebras compatible with
all the structures by the transition maps defined by the maps above and $X \hookrightarrow X$. We define $\phi$-semilinear maps
\[ \phi : \Fil^r W_n^{PD}(\tilde{O}_K)(X) \to W_n^{PD}(\tilde{O}_K)(X) \]
compatible with the transition maps as before. The $S$-algebra $W_n^{PD}(\tilde{O}_K)(X)$ is considered as an object of $\Mod_{/S}^r,\phi,N$ and there exists a natural isomorphism in $\Mod_{/S}^r,\phi,N$
\[ \hat{A}_{st}/p^n \hat{A}_{st} \to W_n^{PD}(\tilde{O}_K)(X) \]
\[ (x_0, \ldots, x_{n-1}) \mapsto (x_{0,n}, \ldots, x_{n-1,n}) \]
$X \hookrightarrow X$
which is $G_{K_n}$-linear.

Put $K_n = K(\pi_n)$ and $K_\infty = \bigcup_n K_n$. For $M \in \Mod_{/S_{\infty}}^r,\phi,N$, we define a $G_K$-module $T_{st,\phi}^{\#}(M)$ to be
\[ T_{st,\phi}^{\#}(M) = \Hom_{S,\Fil^r,\phi,N}(M, \hat{A}_{st,\infty}). \]
When $M$ is killed by $p^n$, we have a natural identification of $G_K$-modules
\[ T_{st,\phi}^{\#}(M) = \Hom_{S,\Fil^r,\phi,N}(M, W_n^{PD}(\tilde{O}_K)(X)). \]
Note that the $G_{K_n}$-module on the right-hand side is independent of the choice of $\pi_k$ for $k > n$. Since the natural map
\[ W_n^{PD}(\tilde{O}_K)(X) \to W_n^{PD}(\tilde{O}_K) \]
\[ X \hookrightarrow 0 \]
is $G_{K_n}$-linear, we also have a $G_{K_n}$-linear isomorphism ([6, Lemme 2.3.1.1])
\[ T_{st,\phi}^{\#}(M)|_{G_{K_n}} \to \Hom_{S,\Fil^r,\phi,N}(M, W_n^{PD}(\tilde{O}_K)). \]
On the other hand, for $r = 0$, the proof of [8, Proposition 2.3.13] shows that the $G_K$-module $T_{st,\phi}^{\#}(M)$ is unramified for any $M \in \Mod_{/S_{\infty}}^{0,\phi,N}$.

A variant of filtered $(\phi, N)$-modules over $S$ is also introduced by Breuil and Kisin, and developed also by Caruso and Liu (see for example [15], [16], [17], [9]). Put $\mathcal{G} = W[[u]]$ and let $\phi : \mathcal{G} \to \mathcal{G}$ be the $\sigma$-semilinear Frobenius endomorphism defined by $\phi(u) = u^p$. Let $\Mod_{/\mathcal{G}}^{r,\phi}$ denote the category consisting of the following data:
- an $\mathcal{G}$-module $\mathcal{M}$,
- a $\phi$-semilinear map $\mathcal{M} \to \mathcal{M}$, which is denoted also by $\phi$, such that the cokernel of the map $1 \otimes \phi : \phi^* \mathcal{M} \to \mathcal{M}$, where we set $\phi^* \mathcal{M} = \mathcal{G} \otimes_{\phi,\mathcal{G}} \mathcal{M}$, is killed by $E(u)^r$,
and whose morphisms are defined as before. The full subcategory of $\Mod_{/\mathcal{G}}^{r,\phi}$ consisting of $\mathcal{M}$ such that $\mathcal{M}$ is free of finite rank over $\mathcal{G}/p\mathcal{G}$ (resp. over $\mathcal{G}$) is denoted by $\Mod_{/\mathcal{G}_1}^{r,\phi}$ (resp. $\Mod_{/\mathcal{G}}^{r,\phi}$). We let $\Mod_{/\mathcal{G}_\infty}^{r,\phi}$ denote the smallest full subcategory which contains $\Mod_{/\mathcal{G}_1}^{r,\phi}$ and is stable under extensions, as
before. Then we have an exact functor ([9, Proposition 2.1.2], see also [15, Proposition 1.1.11])

\[ M_{\infty} : \text{Mod}^{r,\phi}_{/ S_{\infty}} \rightarrow \text{Mod}^{r,\phi}_{/ S_{\infty}}. \]

For \( M \in \text{Mod}^{r,\phi}_{/ S_{\infty}} \), the filtered \( \phi_r \)-module \( M = M_{\infty}(M) \) over \( S \) is defined as follows:

- \( M = S \otimes_{\phi_r, S} M \),
- \( \text{Fil}^r M = \text{Ker}(M \rightarrow S \otimes_{\phi_r} (S/\text{Fil}^r S) \otimes_{\phi_r} S) \),
- \( \phi_r : \text{Fil}^r M \rightarrow S \otimes_{\phi_r} M \rightarrow S \otimes_{\phi_r} S = M \).

We write \( M_{\infty} \) for the functor \( \text{Mod}^{r,\phi}_{/ S_{\infty}} \rightarrow \text{Mod}^{r,\phi}_{/ S_{\infty}} \) defined similarly.

3. Filtered \( \phi_r \)-modules over \( \Sigma \)

In this section, we define another variant \( \text{Mod}^{r,\phi}_{/ \Sigma_{\infty}} \) of the category \( \text{Mod}^{r,\phi}_{/ S_{\infty}} \) over a subring \( \Sigma \) of the ring \( S \), and prove that they are categorically equivalent.

Let \( p \) be a rational prime and \( r \) be an integer such that \( 0 \leq r < p - 1 \). Consider the \( W \)-algebra \( \Sigma = W[[u, Y]]/(E(u)^p - pY) \) as in [6, Subsection 3.2]. We regard \( \Sigma \) as a subring of \( S \) by the map sending \( Y \) to \( E(u)^p/p \). Then the element \( c = \phi_1(E(u)) \in S^{\times} \) is contained in \( \Sigma^{\times} \). We define on \( \Sigma \) a \( \sigma \)-semilinear Frobenius endomorphism \( \phi \) by \( \phi(u) = u^p \) and \( \phi(Y) = p^{p-1}c^p \).

Put \( \text{Fil}^r \Sigma = (E(u)^t, Y) \) for \( 0 \leq t \leq p - 1 \) and \( \text{Fil}^p \Sigma = (Y) \). Then we have \( \phi(\text{Fil}^t \Sigma) \subseteq p^t \Sigma \) for \( 0 \leq t \leq p - 1 \). We put \( \phi_t = p^{-t}\phi|_{\text{Fil}^t \Sigma} \). We also set \( \Sigma_{p} = \Sigma/p \Sigma \) and put on this ring the natural structures induced by those of \( \Sigma \).

We define a category \( \text{Mod}^{r,\phi}_{/ \Sigma} \) of filtered \( \phi_r \)-modules over \( \Sigma \) to be the category consisting of the following data:

- a \( \Sigma \)-module \( M \) and its \( \Sigma \)-submodule \( \text{Fil}^r M \) containing \( \text{Fil}^p \Sigma \cdot M \),
- a \( \phi \)-semilinear map \( \phi_r : \text{Fil}^r M \rightarrow M \) satisfying \( \phi_r(s \cdot m) = \phi_r(s) \cdot \phi(m) \) for any \( s \in \text{Fil}^r \Sigma \) and \( m \in M \), where we set \( \phi(m) = c^{-r} \phi_r(E(u)^r m) \),

and whose morphisms are defined in the same manner as \( \text{Mod}^{r,\phi}_{/ S} \). This category has a natural notion of exact sequences. We define its full subcategory \( \text{Mod}^{r,\phi}_{/ \Sigma_1} \) to be the category consisting of \( M \) which is free of finite rank and generated by the image of \( \phi_r \) as a \( \Sigma_1 \)-module. We also let \( \text{Mod}^{r,\phi}_{/ \Sigma_{\infty}} \) denote the smallest full subcategory of \( \text{Mod}^{r,\phi}_{/ \Sigma_1} \) which contains \( \text{Mod}^{r,\phi}_{/ \Sigma_1} \) and is stable under extensions. Moreover, we define a full subcategory \( \text{Mod}^{r,\phi}_{/ \Sigma} \) of \( \text{Mod}^{r,\phi}_{/ \Sigma_1} \) to be the category consisting of \( M \) such that

- the \( \Sigma \)-module \( M \) is free of finite rank and generated by the image of \( \phi_r \),
- the quotient \( M/\text{Fil}^r M \) is \( p \)-torsion free.
Then we see that for \( \hat{M} \in \text{Mod}^{r,\phi}_{\Sigma_1} \), the quotient \( \hat{M}/p^n\hat{M} \) is naturally considered as an object of \( \text{Mod}^{r,\phi}_{\Sigma_\infty} \).

The natural ring isomorphism \( \Sigma_1/Fil^p\Sigma_1 \cong \hat{S}_1 \) defines a functor \( T_{0,\Sigma} : \text{Mod}^{r,\phi}_{\Sigma_1} \to \text{Mod}^{r,\phi}_{\hat{S}_1} \) by \( M \mapsto M/Fil^p\Sigma_1 \cdot M \). Then just as in the case of the functor \( T_0 : \text{Mod}^{r,\phi}_{\hat{S}_1} \to \text{Mod}^{r,\phi}_{\hat{S}_1} \) ([4, Proposition 2.2.2.1]), we can show the following lemma.

**Lemma 3.1.** The functor \( T_{0,\Sigma} : \text{Mod}^{r,\phi}_{\Sigma_1} \to \text{Mod}^{r,\phi}_{\hat{S}_1} \) is an equivalence of categories.

On the other hand, [6, Proposition 2.2.1.3] and Nakayama’s lemma show the following.

**Lemma 3.2.** Let \( M \) be an object of \( \text{Mod}^{r,\phi}_{\Sigma_1} \) of rank \( d \) over \( \Sigma_1 \). Then there exists a basis \( \{e_1, \ldots, e_d\} \) of \( M \) such that \( Fil^rM = \Sigma_1u^{r_1}e_1 + \cdots + \Sigma_1u^{r_d}e_d + Fil^p\Sigma_1 \cdot M \) for some integers \( r_1, \ldots, r_d \) with \( 0 \leq r_i \leq er \) for any \( i \).

Then we can show the following lemma just as in the proof of [6, Lemme 2.3.1.3].

**Lemma 3.3.** The functor
\[
M \mapsto \text{Hom}_{\Sigma_1,Fil^r,\phi_r}(M, A_{cryst,\infty})
\]
from \( \text{Mod}^{r,\phi}_{\Sigma_1} \) to the category of \( G_{K_\infty} \)-modules is exact.

For \( M \in \text{Mod}^{r,\phi}_{\Sigma_1} \), we can show as in the case of the category \( \text{Mod}^{r,\phi}_{\hat{S}_1} \) that there is an isomorphism of \( G_{K_1} \)-modules
\[
\text{Hom}_{\Sigma_1,Fil^r,\phi_r}(M, (\hat{O}_R)^{PD}) \to \text{Hom}_{\hat{S}_1,Fil^r,\phi_r}(T_{0,\Sigma}(M), \hat{O}_R),
\]
where \( \hat{O}_R \) is considered as an object of \( \text{'Mod}^{r,\phi}_{\hat{S}_1} \) by the natural isomorphism
\[
(\hat{O}_R)^{PD}/Fil^p(\hat{O}_R)^{PD} \to \hat{O}_R.
\]
Thus [6, Lemme 2.3.1.2] implies the following.

**Lemma 3.4.** For \( M \in \text{Mod}^{r,\phi}_{\Sigma_1} \), we have
\[
\#\text{Hom}_{\Sigma_1,Fil^r,\phi_r}(M, (\hat{O}_R)^{PD}) = p^d,
\]
where \( d = \dim_{\Sigma_1} M \).

For the category \( \text{Mod}^{r,\phi}_{\Sigma_\infty} \), we have the following lemma.

**Lemma 3.5.** Let \( M \) be in \( \text{Mod}^{r,\phi}_{\Sigma_\infty} \). Then there exists \( \alpha_1, \ldots, \alpha_d \in Fil^rM \) such that \( Fil^rM = \Sigma\alpha_1 + \cdots + \Sigma\alpha_d + Fil^p\Sigma \cdot M \) and the elements \( e_1 = \phi_r(\alpha_1), \ldots, e_d = \phi_r(\alpha_d) \) form a system of generators of \( M \).
By induction and Lemma 3.2, we may assume that there exists an exact sequence of the category $\text{Mod}^{r,\phi}_{/\Sigma}$

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

such that the lemma holds for $M'$ and $M''$. Let $\alpha_1', \ldots, \alpha_{t'}'$ (resp. $\alpha_1'', \ldots, \alpha_{t''}'$) be elements of $\text{Fil}^r M'$ (resp. $\text{Fil}^r M''$) as in the lemma. Let $\alpha_i \in \text{Fil}^r M$ be a lift of $\alpha_i''$. Then the elements $\alpha_1', \ldots, \alpha_{t'}', \alpha_1, \ldots, \alpha_t$ satisfy the condition in the lemma for $M$.

**Corollary 3.6.** Let $M$ be an object of $\text{Mod}^{r,\phi}_{/\Sigma}$ and $C \in M_d(\Sigma)$ be a matrix satisfying

$$(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d)C$$

with the notation of the previous lemma. Let $A$ be an object of $'\text{Mod}^{r,\phi}_{/\Sigma}$. Then a $\Sigma$-linear homomorphism $f : M \rightarrow A$ preserving $\text{Fil}^r$ also commutes with $\phi_r$ if and only if

$$\phi_r(f(e_1, \ldots, e_d)C) = (f(e_1), \ldots, f(e_d)).$$

**Proof.** Suppose that the latter condition holds. Then we have $\phi_r(f(\alpha_i)) = f(\phi_r(\alpha_i))$ for any $i$. We only have to check the equality $\phi_r \circ f = f \circ \phi_r$ on $\text{Fil}^r M$. Suppose that this equality holds on the submodule $p^{l+1} \text{Fil}^r M$. For $m \in M$, we can take $m' \in \text{Fil}^r M$ such that $E(u)^r m = \sum_i s_i \alpha_i + m'$. Let $s$ be in $\text{Fil}^r M$. Then we have

$$f(\phi_r(p^i s m)) = p^i \phi_r(s) c^{-t} \sum_i \phi(s_i) f(\phi_r(\alpha_i)) + p^i \phi_r(s) c^{-t} f(\phi_r(m')).$$

Since $\phi_r(\text{Fil}^r M) \subseteq p \Sigma$, this equals to $\phi_r(f(p^i s m))$ by assumption. Thus the lemma follows by induction.

**Corollary 3.7.** Let $M$ and $A$ be as above and $J \subseteq \text{Fil}^r A$ be a $\Sigma$-submodule of $A$ such that $\phi_r(J) \subseteq J$. We can consider the $\Sigma$-module $A/J$ naturally as an object of $'\text{Mod}^{r,\phi}_{/\Sigma}$. Suppose that for any $x \in J$, there exists $t \in \mathbb{Z}_{>0}$ such that $\phi_r(x) = 0$. Then the natural homomorphism of abelian groups

$$\text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M, A) \rightarrow \text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M, A/J)$$

is an isomorphism.

**Proof.** The proof is similar to [3, Subsection 2.2]. We consider the $\Sigma$-submodule $J$ as an object of the category $'\text{Mod}^{r,\phi}_{/\Sigma}$ by putting $\text{Fil}^r J = J$.

By devissage, it is enough to show that, for any $M \in \text{Mod}^{r,\phi}_{/\Sigma}$, we have $\text{Ext}^{1}_{'\text{Mod}^{r,\phi}_{/\Sigma}}(M, J) = 0$ and the map in the corollary is an isomorphism. For the first assertion, let

$$
\begin{align*}
0 & \rightarrow J \rightarrow E \rightarrow M \rightarrow 0 
\end{align*}
$$
be an extension in the category \(\text{’Mod}^{r,\phi}_{/\Sigma}\). Let \(e_i, \alpha_i\) and \(C\) be as in Corollary 3.6 such that \(e_1, \ldots, e_d\) form a basis of \(M\). Let \(\hat{e}_i \in \mathcal{E}\) be a lift of \(e_i \in M\). Then we have \((\hat{e}_1, \ldots, \hat{e}_d)C \in (\text{Fil}^r \mathcal{E})^{\oplus d}\) and
\[
\phi_r((\hat{e}_1, \ldots, \hat{e}_d)C) = (\hat{e}_1 + \delta_1, \ldots, \hat{e}_d + \delta_d)
\]
for some \(\delta_1, \ldots, \delta_d \in J\). On the other hand, there exists a unique \(d\)-tuple \((x_1, \ldots, x_d) \in J^{\oplus d}\) satisfying the equation
\[
\phi_r((\hat{e}_1 + x_1, \ldots, \hat{e}_d + x_d)C) = (\hat{e}_1 + 1, \ldots, \hat{e}_d + x_d).
\]
Indeed, the \(d\)-tuple
\[
\sum_{i=0}^{t} (\phi^i_r(\delta_1), \ldots, \phi^i_r(\delta_d)) \phi(C) \cdots \phi^{i-1}(C) \phi^i(C)
\]
is stable for sufficiently large \(t\) by assumption and this limit gives a unique solution of the equation. Then we have
\[
(p(\hat{e}_1 + x_1), \ldots, p(\hat{e}_d + x_d)) = \phi_r(p(\hat{e}_1 + x_1), \ldots, p(\hat{e}_d + x_d))(\phi(C)).
\]
Since the \(d\)-tuple on the left-hand side is contained in \(J^{\oplus d}\), we see that this \(d\)-tuple is zero and \(e_i \mapsto \hat{e}_i + x_i\) defines a section \(M \to \mathcal{E}\). We can prove the second assertion similarly.

Next we show that the two categories \(\text{Mod}^{r,\phi}_{/\Sigma}\) and \(\text{Mod}^{r,\phi}_{/\Sigma_{\infty}}\) are in fact equivalent. For \(M \in \text{Mod}^{r,\phi}_{/\Sigma_{\infty}}\), we associate to it an \(S\)-module \(\mathcal{M}\) by setting \(\mathcal{M} = S \otimes_{\Sigma} M\). We also define its \(S\)-submodule \(\text{Fil}^r \mathcal{M}\) by
\[
\text{Fil}^r \mathcal{M} = \text{Ker}(\mathcal{M} = S \otimes_{\Sigma} M \to S/\text{Fil}^r S \otimes_{\Sigma} M/\text{Fil}^r M \simeq M/\text{Fil}^r M),
\]
where the last isomorphism is induced by the natural isomorphisms of \(W\)-algebras
\[
W[u]/(E(u)^r) \to \Sigma/\text{Fil}^r \Sigma \to S/\text{Fil}^r S.
\]
These associations induce two functors from \(\text{Mod}^{r,\phi}_{/\Sigma_{\infty}}\) to the category of \(S\)-modules, \(M \mapsto \mathcal{M}\) and \(M \mapsto \text{Fil}^r \mathcal{M}\). Since the rings \(S\) and \(W[u]/(E(u)^r)\) are \(p\)-torsion free, we have \(\text{Tor}^{\Sigma}_1(\Sigma_1, S) = \text{Tor}^{\Sigma}_1(\Sigma_1, \Sigma/\text{Fil}^r \Sigma) = 0\) and thus \(\text{Tor}^{\Sigma}_1(M, S) = \text{Tor}^{\Sigma}_1(M, \Sigma/\text{Fil}^r \Sigma) = 0\) for any \(M \in \text{Mod}^{r,\phi}_{/\Sigma_{\infty}}\). Hence we see that these two functors are exact.

We define \(\phi_r : \text{Fil}^r \mathcal{M} \to \mathcal{M}\) as follows. Note that \(\text{Fil}^r S \otimes_{\Sigma} M \subseteq \mathcal{M}\) and \(\text{Fil}^r \mathcal{M}\) is equal to \(\text{Fil}^r S \otimes_{\Sigma} M + \text{Im}(S \otimes \text{Fil}^r M \to \mathcal{M})\). Set \(\phi'_r : \text{Fil}^r S \otimes_{\Sigma} M \to \mathcal{M}\) to be \(\phi'_r = \phi_r \otimes \phi\).

**Lemma 3.8.** The map \(\phi \otimes \phi_r : S \otimes_{\Sigma} \text{Fil}^r M \to \mathcal{M}\) induces a \(\phi\)-semilinear map \(\phi''_r : \text{Im}(S \otimes \text{Fil}^r M \to \mathcal{M}) \to \mathcal{M}\).

**Proof.** Let \(z = \sum_i s_i \otimes m_i\) be in \(S \otimes \text{Fil}^r M\) with \(s_i \in S\) and \(m_i \in \text{Fil}^r M\). Let \(\bar{z}\) be its image in \(\mathcal{M}\) and suppose that \(\bar{z} = 0\). Write \(s_i = s'_i + s''_i\) with \(s'_i \in \Sigma\) and \(s''_i \in \text{Fil}^r S\). Since we have an isomorphism \(\mathcal{M}/\text{Fil}^r S \cdot \mathcal{M} \simeq M/\text{Fil}^r \Sigma \cdot M\),
we can find elements \( s^{(j)} \in \text{Fil} \Sigma \) and \( m^{(j)} \in M \) such that the equality 
\[
\sum_i s^i m_i = \sum_j s^{(j)} m^{(j)}
\]
holds in \( M \). Then we have
\[
0 = \bar{z} = \sum_i 1 \otimes s^i m_i + \sum_i s''_i \otimes m_i = \sum_j s^{(j)} \otimes m^{(j)} + \sum_i s''_i \otimes m_i
\]
in \( \mathcal{M} \). On the other hand, the element \( (\phi \otimes \phi_r)(z) \in \mathcal{M} \) is equal to
\[
\sum_j 1 \otimes \phi_r(s^{(j)} m^{(j)}) + \sum_i \phi(s''_i) \otimes \phi_r(m_i).
\]
Since \( \phi = p^* \phi_r \), this equals \( \phi'_i(\sum_j s^{(j)} \otimes m^{(j)} + \sum_i s''_i \otimes m_i) = 0 \).

**Lemma 3.9.** The maps \( \phi'_i \) and \( \phi''_i \) patch together and define a \( \phi \)-semilinear map \( \phi_r : \text{Fil} \Sigma M \to \mathcal{M} \).

**Proof.** Since \( \phi'_i \) and \( \phi''_i \) coincide on \( \text{Im}(\text{Fil}^r S \otimes \Sigma \text{Fil}^r M \to \mathcal{M}) \), it is enough to show that \( 1 \otimes \phi_r(m) = \phi'_i(\sum_i s_i \otimes m_i) \) for any \( m \in \text{Fil}^r M \), \( s_i \in \text{Fil}^r S \) and \( m_i \in M \) satisfying \( 1 \otimes m = \sum_i s_i \otimes m_i \) in \( \mathcal{M} \). As in the proof of Lemma 3.8, the element \( m \) can be written as \( m = \sum_j s^{(j)} m^{(j)} \) for some \( s^{(j)} \in \text{Fil} \Sigma \) and \( m^{(j)} \in M \). By assumption, we have \( \sum_i s_i \otimes m_i = \sum_j s^{(j)} \otimes m^{(j)} \) in \( \text{Fil}^r S \otimes \Sigma M \). Hence the lemma follows.

Then we see that this construction defines a functor \( \mathcal{M}_{\Sigma \infty} : \text{Mod}^r_{/\Sigma_{\infty}} \to \text{Mod}^r_{/S_{\infty}} \).

**Lemma 3.10.** The functor \( \mathcal{M}_{\Sigma \infty} \) induces an equivalence of categories \( \text{Mod}^r_{/\Sigma_{\infty}} \to \text{Mod}^r_{/S_{\infty}} \).

**Proof.** Consider the diagram of functors
\[
\text{Mod}^r_{/\Sigma_{\infty}} \xrightarrow{\mathcal{M}_{\Sigma \infty}} \text{Mod}^r_{/S_{\infty}} \xrightarrow{T_0} \text{Mod}^r_{/S_{\infty}}.
\]
From the definition, we see that this diagram is commutative. By Lemma 3.1, the downward arrows are equivalences of categories. Thus the lemma follows.

Then a devissage argument as in [15, Proposition 1.1.11] shows the following corollary.

**Corollary 3.11.** The functor \( \mathcal{M}_{\Sigma \infty} : \text{Mod}^r_{/\Sigma_{\infty}} \to \text{Mod}^r_{/S_{\infty}} \) is fully faithful.

To show the essential surjectivity of the functor \( \mathcal{M}_{\Sigma \infty} \), we define another functor \( \mathcal{M}_{\Sigma_{\infty}} : \text{Mod}^r_{/\Sigma_{\infty}} \to \text{Mod}^r_{/S_{\infty}} \) which is defined in a similar way to the functor \( \mathcal{M}_{\Sigma_{\infty}} : \text{Mod}^r_{/\Sigma_{\infty}} \to \text{Mod}^r_{/S_{\infty}} \). For an \( \Sigma \)-module \( \mathfrak{M} \) in \( \text{Mod}^r_{/\Sigma_{\infty}} \), we associate to it a \( \Sigma \)-module \( M \in \text{Mod}^r_{/\Sigma} \) as follows:
• $M = \Sigma \otimes_{\phi, E} \mathcal{M}$,
• $\text{Fil}^r M = \text{Ker}(M \xrightarrow{1 \otimes \phi} \Sigma \otimes_{E} \mathcal{M} \to (\Sigma/\text{Fil}^r \Sigma) \otimes_{E} \mathcal{M})$,
• $\phi_r : \text{Fil}^r M \xrightarrow{1 \otimes \phi} \text{Fil}^r \Sigma \otimes_{E} \mathcal{M} \xrightarrow{\phi_r \otimes 1} \Sigma \otimes_{\phi, E} \mathcal{M} = M$.

We can check that this defines an exact functor $\text{Mod}_{/S_\infty}^{r, \phi} \to \text{Mod}_{/S_\infty}^{r, \phi}$ as in the proof of [15, Proposition 1.1.11]. We let this functor be denoted by $M_{S_\infty}$.

**Lemma 3.12.** The diagram of functors

\[
\begin{array}{ccc}
\text{Mod}_{/S_\infty}^{r, \phi} & \xrightarrow{M_{S_\infty}} & \text{Mod}_{/\Sigma_\infty}^{r, \phi} \\
\downarrow & & \downarrow \\
\text{Mod}_{/S_\infty}^{r, \phi} & \xrightarrow{M_{S_\infty}} & \text{Mod}_{/\Sigma_\infty}^{r, \phi}
\end{array}
\]

is commutative.

**Proof.** For $\mathcal{M} \in \text{Mod}_{/S_\infty}^{r, \phi}$, put $M = M_{S_\infty}(\mathcal{M})$ and $\mathcal{M} = M_{S_\infty}(\mathcal{M})$. Then $\mathcal{M} = S \otimes_{\Sigma} M$ as an $S$-module. Let $\text{Fil}^r \mathcal{M}$ and $\phi_r : \text{Fil}^r \mathcal{M} \to \mathcal{M}$ denote the filtration and Frobenius structure defined by the functor $M_{S_\infty}$. We also let $\hat{\text{Fil}}^r \mathcal{M}$ and $\hat{\phi}_r : \hat{\text{Fil}}^r \mathcal{M} \to \mathcal{M}$ denote those defined by $M_{\Sigma_\infty}$.

The $S$-module $\text{Fil}^r \mathcal{M}$ contains $\hat{\text{Fil}}^r \mathcal{M}$. Conversely, let $z$ be an element of $\text{Fil}^r \mathcal{M}$. Note that $\text{Fil}^p S \cdot M \subseteq \hat{\text{Fil}}^r \mathcal{M}$. Thus, to show $z \in \text{Fil}^r \mathcal{M}$, we may assume that $z \in \text{Im}(M \to \mathcal{M})$. Then the commutative diagram whose right vertical arrow is an isomorphism

\[
\begin{array}{ccc}
M = \Sigma \otimes_{\phi, E} \mathcal{M} & \xrightarrow{1 \otimes \phi} & \Sigma \otimes_{E} \mathcal{M} \xrightarrow{\Sigma/\text{Fil}^r \Sigma \otimes_{E} \mathcal{M}} \\
\downarrow & & \downarrow \\
\mathcal{M} = S \otimes_{\phi, E} \mathcal{M} & \xrightarrow{1 \otimes \phi} & S \otimes_{E} \mathcal{M} \xrightarrow{S/\text{Fil}^r S \otimes_{E} \mathcal{M}}
\end{array}
\]

implies that $z \in \text{Im}(\text{Fil}^r M \to \text{Fil}^r \mathcal{M}) \subseteq \hat{\text{Fil}}^r \mathcal{M}$ and hence $\text{Fil}^r \mathcal{M} = \hat{\text{Fil}}^r \mathcal{M}$.

From the definition, we also can show $\phi_r = \hat{\phi}_r$. This implies the lemma. □

**Proposition 3.13.** The functor $M_{\Sigma_\infty} : \text{Mod}_{/\Sigma_\infty}^{r, \phi} \to \text{Mod}_{/S_\infty}^{r, \phi}$ is an equivalence of categories.

**Proof.** Since the functor $M_{S_\infty}$ is an equivalence of categories for $p \geq 3$ ([9, Theorem 2.3.1]), Corollary 3.11 and Lemma 3.12 imply the proposition in this case. For $r = 0$, put $\kappa_{\Sigma} = \kappa \cap \Sigma$, where $\kappa = \text{Ker}(S \to W)$. Then, by using a natural isomorphism $\Sigma \simeq W[[u, u^{np}/p]]$, we can show that the functor $M \mapsto M/\kappa_{\Sigma} M$ defines an equivalence of categories $\text{Mod}_{/\Sigma_\infty}^{0, \phi} \to \text{Mod}_{/W_\infty}^{\phi}$, as
in the case of the category \( \text{Mod}^{0,\phi}_{/S_{\infty}} \). Since the diagram

\[
\begin{array}{ccc}
\text{Mod}^{0,\phi}_{/S_{\infty}} & \stackrel{M_{\Sigma_{\infty}}}{\longrightarrow} & \text{Mod}^{0,\phi}_{/S_{\infty}} \\
\downarrow & & \downarrow \\
\text{Mod}^{\phi}_{/W_{\infty}}
\end{array}
\]

is commutative and the downward arrows are equivalences of categories, the proposition follows also for \( p = 2 \).

\[\square\]

**Remark 3.14.** We can also define a fully faithful functor \( M_{\Sigma} : \text{Mod}^{r,\phi}_{/\Sigma} \to \text{Mod}^{r,\phi}_{/S} \) in a similar way to \( M_{\Sigma_{1}} \) and prove that this is an equivalence of categories. Indeed, the claim for \( p \geq 3 \) follows from [9, Theorem 2.2.1]. Let \( M \) be in \( \text{Mod}^{r,\phi}_{/S} \) and \( e_{1}, \ldots, e_{d} \) be a basis of \( M \) over \( S \). Let \( C \in GL_{d}(S) \) be the matrix such that

\[
\phi(e_{1}, \ldots, e_{d}) = (e_{1}, \ldots, e_{d})C.
\]

Then the elements \( \phi(e_{1}), \ldots, \phi(e_{d}) \) also form a basis of \( M \) and

\[
\phi(\phi(e_{1}), \ldots, \phi(e_{d})) = (\phi(e_{1}), \ldots, \phi(e_{d}))\phi(C).
\]

Since \( \phi(S) \subseteq \Sigma \), the \( \Sigma \)-module \( M \) defined by \( M = \Sigma\phi(e_{1}) \oplus \cdots \oplus \Sigma\phi(e_{d}) \) is stable under \( \phi \). Hence we see that \( M \in \text{Mod}^{0,\phi}_{/\Sigma} \) and \( M = M_{\Sigma}(M) \).

**Proposition 3.15.** Let \( M \) be an object of \( \text{Mod}^{r,\phi}_{/\Sigma_{\infty}} \) and set \( M = M_{\Sigma_{\infty}}(M) \). Then there exists a natural isomorphism of \( G_{K_{\infty}} \)-modules

\[
\text{Hom}_{\Sigma,\text{Fil}^{r,\phi}_{/\Sigma_{\infty}}}(M, A_{\text{crys},\infty}) \to \text{Hom}_{S,\text{Fil}^{r,\phi}_{/S}}(M, A_{\text{crys},\infty}).
\]

Moreover, this induces for any \( n \) an isomorphism of \( G_{K_{n}} \)-modules

\[
\text{Hom}_{\Sigma,\text{Fil}^{r,\phi}_{/\Sigma_{\infty}}}(M, W^{PD}_{n}(\tilde{\mathcal{O}}_{K})) \to \text{Hom}_{S,\text{Fil}^{r,\phi}_{/S}}(M, W^{PD}_{n}(\tilde{\mathcal{O}}_{K})).
\]

**Proof.** By definition, \( M = S \otimes_{\Sigma} M \) and we have a natural isomorphism

\[
\text{Hom}_{\Sigma}(M, A_{\text{crys},\infty}) \to \text{Hom}_{S}(M, A_{\text{crys},\infty}).
\]

From the definition, we can check that this isomorphism induces the map in the proposition, which is injective. To prove the bijectivity, by descent we may assume that \( pM = 0 \). Then both sides of this injection have the same cardinality by Lemma 3.4 and the first assertion follows. Since the sequence

\[
0 \longrightarrow W^{PD}_{n}(\tilde{\mathcal{O}}_{K}) \longrightarrow A_{\text{crys},\infty} \longrightarrow A_{\text{crys},\infty}^{p} \longrightarrow 0
\]

of the category \( \text{Mod}^{r,\phi}_{/\Sigma} \) is exact, the first assertion implies the second one.

\[\square\]
4. A METHOD OF ABRASHKIN

In this section, we study the $G_{K_n}$-module $\text{Hom}_{\Sigma, \text{Fil}^r}(M, W^\text{PD}_n(\hat{O}_K))$ following Abrashkin ([3]).

Let $p$ and $0 \leq r < p - 1$ be as before. We fix a system of $p$-power roots of unity $\{\zeta_p^n\}_{n \in \mathbb{Z}_{\geq 0}}$ in $\hat{K}$ such that $\zeta_p \neq 1$ and $\zeta_p^n = \zeta_{p^{n+1}}$ for any $n$, and set an element $\hat{e}$ of $\hat{R}$ to be $(\zeta_p^n)_{n \in \mathbb{Z}_{\geq 0}}$. Then the elements $\lfloor \hat{e} \rfloor - 1$ and $\lfloor \hat{e}^{1/p} \rfloor - 1$ are topologically nilpotent in $W(R)$. The element of $W(R)$

$$t = (\lfloor \hat{e} \rfloor - 1)/(\lfloor \hat{e}^{1/p} \rfloor - 1) = 1 + \lfloor \hat{e}^{1/p} \rfloor + \lfloor \hat{e}^{1/p} \rfloor^2 + \cdots + \lfloor \hat{e}^{1/p} \rfloor^{p-1}$$

is a generator of the principal ideal $\text{Ker}(\theta)$. We define an element $a \in W(R)$ to be

$$a = \left\{ \begin{array}{ll}
\sum_{k=1}^{p-2} p^{-1}((-1)^{p-1-k}C_k - 1)\lfloor \hat{e}^{1/p} \rfloor^k & (p \geq 3) \\
-p^{-1}C_{p-1} & (p = 2),
\end{array} \right.$$ 

where $p^{-1}C_k = (p - 1)!/(k!(p - 1 - k)!)$. The coefficient of $\lfloor \hat{e}^{1/p} \rfloor^k$ in each term is an integer. The element $a$ is invertible in the ring $W(R)$, since $\theta(a) = (\zeta_p - 1)^{p-1}/p \in O_{\hat{C}}^\times$ and the ideal $\text{Ker}(\theta)$ is topologically nilpotent in $W(R)$.

The element $Z = (\lfloor \hat{e} \rfloor - 1)/(\lfloor \hat{e}^{1/p} \rfloor - 1)/p$ of $A_{\text{crys}}$ is topologically nilpotent and we have $\phi(t) = p(Z - \phi(a))$. Consider the formal power series ring $W(R)[[u']]$ with the $(t, u')$-adic topology and the continuous ring homomorphism $W(R)[[u']] \to A_{\text{crys}}$ which sends $u'$ to $Z$. Let $\hat{A}$ denote the image of this homomorphism. Then we see that the ring $\hat{A}$ is $(t, Z)$-adically complete. Since we have $Z = at^{p-1} + t^p/p$, the element $t^p/p$ of $A_{\text{crys}}$ is contained in the subring $\hat{A}$ and topologically nilpotent in this subring. Hence we can consider the ring $\hat{A}$ as a $\Sigma$-algebra by $u \mapsto \hat{u}$. Put $\text{Fil}^i\hat{A} = (t^i, Z)$ for $0 \leq i \leq p - 1$. The Frobenius endomorphism $\phi$ of $A_{\text{crys}}$ preserves $\hat{A}$ and satisfies $\phi(\text{Fil}^i\hat{A}) \subseteq p^i\hat{A}$ for $0 \leq i \leq p - 1$. Set $\phi_r = p^{-r}\phi|_{\text{Fil}^r\hat{A}}$. Then we can consider the ring $\hat{A}$ also as an object of the category $\text{Mod}^{\Sigma, \phi}_{\bar{K}}$. Put $\hat{A}_n = \hat{A}/p^n\hat{A}$ and $\hat{A}_\infty = \hat{A}_{\bar{K}}W K_0/W$.

We include here a proof of the following lemma stated in [3, Subsection 3.2].

**Lemma 4.1.** The natural inclusion $W(R) \to \hat{A}$ induces isomorphisms of $W(R)$-algebras $W(R)/((\lfloor \hat{e} \rfloor - 1)^{p-1}) \to \hat{A}/(Z)$ and $W_n(R)/((\lfloor \hat{e} \rfloor - 1)^{p-1}) \to \hat{A}_n/(Z)$.

**Proof.** For a subring $B$ of $A_{\text{crys}}$,

$$I^{[s]}B = \{ x \in B \mid \phi^i(x) \in \text{Fil}^s A_{\text{crys}} \text{ for any } i \}$$

as in [12, Subsection 5.3]. Then we have $I^{[s]}W(R) = (\lfloor \hat{e} \rfloor - 1)^s W(R)$ and the natural ring homomorphism

$$W(R)/I^{[s]}W(R) \to A_{\text{crys}}/I^{[s]}A_{\text{crys}}.$$
is an injection ([12, Proposition 5.1.3, Proposition 5.3.5]). Since the element $Z$ is contained in the ideal $I^{[p-1]}A_{\text{crys}}$, this injection factors as

$$W(R)/I^{[p-1]}W(R) \to \hat{A}/(Z) \to A_{\text{crys}}/I^{[p-1]}A_{\text{crys}}.$$  

Hence the former arrow is an isomorphism and the lemma follows.

Therefore $\hat{A}/\text{Fil}^r \hat{A}$ is $p$-torsion free and $p^n \text{Fil}^r \hat{A} = \text{Fil}^r \hat{A} \cap p^n \hat{A}$. Thus we can also consider $\hat{A}_n$ and $\hat{A}_\infty$ as objects of the category $\text{Mod}^{\phi}_{\Sigma}$. The absolute Galois group $G_{K_\infty}$ acts naturally on these $\Sigma$-modules.

**Lemma 4.2.** We have a natural decomposition as an $R$-module

$$\hat{A}_1 = R/((t^p) \oplus (Z)).$$

**Proof.** Consider the natural inclusion $W(R) \to \hat{A}$. We claim that this induces an injection $R/((t^p) \to \hat{A}_1$. Let $x$ be in the ring $R$. If the element $[x] \in W(R)$ is contained in $p \hat{A}$, then its image in $A_{\text{crys}}/p A_{\text{crys}}$ is zero. We have an isomorphism of $R$-algebras

$$R[Y_1, Y_2, \ldots]/((t^p, Y_1^p, Y_2^p, \ldots) \to A_{\text{crys}}/pA_{\text{crys}}$$

which sends $Y_i$ to the image of $t^p/p^i$. Thus the element $x$ is contained in the ideal $(t^p)$. Conversely, if $v_R(x) \geq p$, then we have

$$[x] = w([x] - 1)^{p-1} + pw'$$

for some $w, w' \in W(R)$ and this implies $[x] \in p \hat{A}$. Now we have the commutative diagram of $R$-algebras

$$\begin{array}{ccc}
R/((t^p) & \to & \hat{A}_1 \\
\downarrow & & \downarrow \\
\hat{A}_1/(Z) & \to & \hat{A}_{1}/(Z)
\end{array}$$

and the map $f : R/((t^p) \to \hat{A}_1/(Z)$ is an isomorphism by Lemma 4.1. Hence the lemma follows.

Since $r < p - 1$, from this lemma we can show the following lemma as in the proof of [6, Lemme 2.3.1.3].

**Lemma 4.3.** The functor

$$M \mapsto \text{Hom}_{\Sigma, \text{Fil}^r, \phi}(M, \hat{A}_\infty)$$

from $\text{Mod}^{\phi, \Sigma}$ to the category of $G_{K_\infty}$-modules is exact.

**Corollary 4.4.** For any $M \in \text{Mod}^{\phi, \Sigma}$, the natural map

$$\text{Hom}_{\Sigma, \text{Fil}^r, \phi}(M, \hat{A}_\infty) \to \text{Hom}_{\Sigma, \text{Fil}^r, \phi}(M, A_{\text{crys, } \infty})$$
is an isomorphism of $G_{K_{1\infty}}$-modules. Moreover, for any $n$, we have an isomorphism of $G_{K_{1\infty}}$-modules

$$\text{Hom}_{\Sigma, \text{Fil}^r_{\phi_r}}(M, \hat{A}_n) \rightarrow \text{Hom}_{\Sigma, \text{Fil}^r_{\phi_r}}(M, A_{\text{crys}}/p^nA_{\text{crys}}).$$

**Proof.** Let us prove the first assertion. By Lemma 3.3 and Lemma 4.3, we may assume $pM = 0$. Consider the commutative diagram of rings

$$\begin{array}{ccc}
\hat{A}_1 & \rightarrow & A_{\text{crys}}/pA_{\text{crys}} \\
\downarrow & & \downarrow \\
R/(t^{p-1}) & & \\
\end{array}$$

whose downward arrows are defined by modulo $\text{Fil}^{p-1}$ of the rings $\hat{A}_1$ and $A_{\text{crys}}/pA_{\text{crys}}$, respectively. Since $r < p - 1$, we have $\phi_r(\text{Fil}^{p-1}\hat{A}_1) = 0$ and similarly for the ring $A_{\text{crys}}/pA_{\text{crys}}$. Thus these two surjections induce on the ring $R/(t^{p-1})$ the same structure of a filtered $\phi_r$-module over $\Sigma$. By Corollary 3.7, we have a commutative diagram

$$\begin{array}{ccc}
\text{Hom}_{\Sigma, \text{Fil}^r_{\phi_r}}(M, \hat{A}_1) & \rightarrow & \text{Hom}_{\Sigma, \text{Fil}^r_{\phi_r}}(M, A_{\text{crys}}/pA_{\text{crys}}) \\
& \downarrow & \downarrow \\
& \text{Hom}_{\Sigma, \text{Fil}^r_{\phi_r}}(M, R/(t^{p-1})) & \\
\end{array}$$

whose downward arrows are isomorphisms. This concludes the proof of the first assertion. Since we have an exact sequence

$$0 \rightarrow \hat{A}_n \rightarrow \hat{A}_{\infty} \rightarrow \hat{A}_0 \rightarrow 0$$

in the category $'\text{Mod}^r_{\Sigma}$, the second assertion follows. $\Box$

Since the ideal $(Z)$ of $\hat{A}_n$ satisfies the condition of Corollary 3.7, the $\Sigma$-algebra $\hat{A}_n/(Z)$ is naturally considered as an object of $'\text{Mod}^r_{\Sigma}$. We also give the ring $W_n(R)/((Z-1)^{p-1})$ the structures of a $\Sigma$-algebra and a filtered $\phi_r$-module over $\Sigma$ induced from those of $\hat{A}_n/(Z)$ by the isomorphism in Lemma 4.1. The map

$$\Sigma \rightarrow W_n(R)/((Z-1)^{p-1})$$

sends the element $u \in \Sigma$ to the image of $[\pi]$ in the ring on the right-hand side. Put $v = \bar{t}/E([\pi]) \in W(R)^\times$. As for the element $Y \in \Sigma$, the equality

$$Y = -av^{-1}E([\pi])^{p-1} + v^{-p}Z$$

holds in $\hat{A}$. Hence the above homomorphism sends the element $Y$ to the image of $-av^{-1}E([\pi])^{p-1}$.

Consider the surjective ring homomorphism

$$R \rightarrow \hat{O}_K$$

$$x = (x_0, x_1, \ldots) \mapsto x_n$$
and the induced surjection $\beta_n : W_n(R) \to W_n(\bar{O}_K)$. Let
$$J = \{(x_0, \ldots, x_{n-1}) \in W_n(R) \mid v_R(x_i) \geq p^n \text{ for any } i\}$$
be the kernel of the latter surjection.

**Lemma 4.5.** The ideal $J$ is contained in the ideal $((\mathbb{F}_p - 1)^{p-1})$ of the ring $W_n(R)$.

**Proof.** Write the element $((\mathbb{F}_p - 1)^{p-1})$ also as $x = (x_0, \ldots, x_{n-1}) \in W_n(R)$ with $v_R(x_0) = p$. Take an element $z = (z_0, \ldots, z_{n-1})$ of the ideal $J$. We construct $y \in W_n(R)$ such that $xy = z$. By induction, it is enough to show that if $z_0 = \cdots = z_{i-1} = 0$ for some $0 \leq i \leq n-1$ and $(x_0, \ldots, x_i)(0, \ldots, 0, y_i) = (0, \ldots, 0, z_i)$ in $W_{i+1}(R)$, then $x(0, \ldots, 0, y_i, 0, \ldots, 0) \in J$. Let us write this element as $(0, \ldots, 0, w_i, \ldots, w_{n-1})$ with $w_i = z_i$. We have $v_R(y_i) \geq p^n - p^{i+1}$. In the ring of Witt vectors $W_n(F_p[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}])$, the $k$-th entry of the vector
$$(X_0, \ldots, X_{n-1})(0, \ldots, 0, Y_i, 0, \ldots, 0)$$
is $X_{p^k-1}Y_i^{k-1}$ for any $k \geq i$. Thus we have $v_R(w_k) \geq p^n$. \hfill $\square$

Note that the elements $[\mathbb{F}_p] - 1$ and $[\mathbb{F}_p^{p+1}] - 1$ are nilpotent in $W_n(\bar{O}_K)$. By the above lemma, we have an isomorphism of rings
$$W_n(R)/((\mathbb{F}_p - 1)^{p-1}) \to W_n(\bar{O}_K)/(((\mathbb{F}_p^{p+1}) - 1)^{p-1}).$$
We let $\tilde{A}_{n,p-1}$ denote the ring on the right-hand side and give the ring $\tilde{A}_{n,p-1}$ the structure of a filtered $\phi$-module over $\Sigma$ induced by this isomorphism.

For an algebraic extension $F$ of $K$, we put
$$b_F = \{x \in O_F \mid v_K(x) > cr/(p - 1)\}.$$Note that the ring $O_F/b_F$ is killed by $p$. We consider the ring of Witt vectors $W_n(O_F/b_F)$ as a $W_n(O_F)$-algebra by the natural ring surjection $W_n(O_F) \to W_n(O_F/b_F)$ and as a $W_n$-algebra by twisting the natural action by $\sigma^{-n}$, as before. For a ring $B$ and its ideal $I$, we define an ideal $W_n(I)$ of the ring $W_n(B)$ to be
$$W_n(I) = \{(x_0, \ldots, x_{n-1}) \in W_n(B) \mid x_i \in I \text{ for any } i\}.$$Put $F_n = K_n(\zeta_{p^{n+1}})$. For an algebraic extension $F$ of $F_n$ in $\bar{K}$, the elements $[\mathbb{F}_p] - 1$ and $[\mathbb{F}_p^{p+1}] - 1$ of $W_n(m_F)$ are topologically nilpotent non-zero divisors in $W_n(O_F)$. Let the ring
$$W_n(O_F/b_F)/(((\mathbb{F}_p^{p+1}) - 1)^rW_n(m_F/b_F)$$be denoted by $\tilde{A}_{n,F,r+}$. We also put $\tilde{A}_{n,r+} = \tilde{A}_{n,K,r+}$.

**Lemma 4.6.** The ideal $((\mathbb{F}_p^{p+1}) - 1)^rW_n(m_F)$ of $W_n(O_F)$ contains the ideal $W_n(b_F)$ for any $r \in \{0, \ldots, p-2\}$. We also have $((\mathbb{F}_p^{p+1}) - 1)^{p-1} \supseteq W_n(pO_F)$. 


The proof is similar to the proof of Lemma 4.5. Let us show the first assertion. Since this is trivial for \( r = 0 \), we may assume \( r \geq 1 \). Put \( x = (x_0, \ldots, x_{n-1}) = ([\zeta_{p^n}] - 1)^r \in W_n(\mathcal{O}_F) \). Then we have \( v_p(x_0) = r/(p^{n-1}(p-1)) \). By induction, it is enough to show that for \( 0 \leq i \leq n-1 \), if \( (x_0, \ldots, x_i)(0, \ldots, 0, y_i) \in W_{i+1}(b_F) \), then \( y_i \in m_F \) and \( x(0, \ldots, 0, y_i, 0, \ldots, 0) \in W_n(b_F) \). By assumption, we have

\[
v_p(y_i) > \frac{r}{p-1} \left( 1 - \frac{1}{p^{n-i-1}} \right) \geq 0.
\]

Put \( (0, \ldots, 0, w_1, \ldots, w_{n-1}) = x(0, \ldots, 0, y_i, 0, \ldots, 0) \). We show \( w_l \in b_F \) for any \( l \) by induction. Indeed, let us suppose that \( w_l \in b_F \) for any \( i \leq l \leq k-1 \) with some \( i + 1 \leq k \leq n-1 \). We have the equality

\[
p^i y_l^{p^i-1}(x_0^i + px_1^{p^i-1} + \cdots + p^k x_k) = (p^i w_l^{p^i-1} + p^{i+1} w_{i+1}^{p^{i+1}-1} + \cdots + p^k w_k).
\]

Since \( r \geq 1 \), we have \( (p^{k-l} - 1)r/(p-1) \geq k - l \) for \( 0 \leq l \leq k-1 \). This implies \( v_p(p^i w_l^{p^i-1}) > k + r/(p-1) \) for \( 0 \leq l \leq k-1 \). The valuation of the left-hand side of the above equality also satisfies this inequality. Thus we have \( v_p(w_k) > r/(p-1) \) and the assertion follows. We can show the second assertion similarly. \( \square \)

By this lemma, the natural surjections of rings

\[
W_n(\mathcal{O}_F)/([\zeta_{p^n}] - 1)^r W_n(m_F) \to W_n(\mathcal{O}_F/p\mathcal{O}_F)/([\zeta_{p^n}] - 1)^r W_n(m_F/p\mathcal{O}_F) \to \hat{A}_{n,F,r+}
\]

are isomorphisms. Then we see that the natural injection \( F \to \bar{K} \) induces an injection of rings \( \hat{A}_{n,F,r+} \to \hat{A}_{n,r+} \).

Write \( Z_n \) for the image of the element \( Z \) of \( A_{\text{crys}} \) in \( W_n^{\text{PD}}(\hat{\mathcal{O}}_K) \). Then we have a commutative diagram of \( \Sigma \)-algebras

\[
\begin{array}{ccc}
\hat{A}_n & \longrightarrow & A_{\text{crys}}/p^n A_{\text{crys}} \\
& & \downarrow \downarrow \\
W_n^{\text{PD}}(\hat{\mathcal{O}}_K) & \longrightarrow & W_n^{\text{PD}}(\hat{\mathcal{O}}_R)/(Z_n),
\end{array}
\]

where all the vertical arrows are surjections satisfying the condition of Corollary 3.7. Hence this is also a commutative diagram in \( '\text{Mod}^r_{\Sigma} \). Note
that these rings and homomorphisms are independent of the choice of a system \(\{\zeta_{p^n}\}_{n \in \mathbb{Z}_{\geq 0}}\). We also note that \(\text{Fil}^r \check{A}_{n,r+} = E(\{\pi_n\})^r \check{A}_{n,r+}\) and \(\phi_r(E(\{\pi_n\})^r y) = c^r \phi(y)\) for any \(y \in \check{A}_{n,r+}\), where \(\phi\) denotes the Frobenius endomorphism of \(\check{A}_{n,r+}\) induced from that of the ring \(W_n(\mathcal{O}_K/\mathfrak{b}_K)\). Moreover, let \(M\) be an object of \(\text{Mod}_{\Sigma}^r/_{\Sigma_{\infty}}\). Then, by Corollary 3.7 and Corollary 4.4, we have a natural isomorphism of abelian groups

\[
\text{Hom}_{\Sigma, \text{Fil}^r}^r(\check{M}, W_n^{PD}(\check{\mathcal{O}}_K)) \rightarrow \text{Hom}_{\Sigma, \text{Fil}^r}^r(\check{M}, \check{A}_{n,r+}).
\]

Next we investigate the module on the right-hand side of this isomorphism, and prove this is in fact an isomorphism of \(G_{F_n}\)-modules. Consider the element \(E(\{\pi_n\}) \in W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})\) and let us fix its lift \(\hat{\gamma} \in W_n(\mathcal{O}_{F_n})\) by the natural surjection \(W_n(\mathcal{O}_{F_n}) \rightarrow W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})\). Let \(a \in W(R)^{\times}\) and \(v = t/E(\{\pi\}) \in W(R)^{\times}\) as before. We let \(a_n, t_n\) and \(v_n\) denote the images of \(a, t\) and \(v\) by the surjection \(W(R) \rightarrow W_n(\check{\mathcal{O}}_K)\) induced by \(\beta_n\), respectively. The elements \(a_n\) and \(t_n\) of the ring \(W_n(\check{\mathcal{O}}_K)\) are contained in the subring \(W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})\). We abusively let them also denote their images by the natural surjections \(W_n(\check{\mathcal{O}}_K) \rightarrow W_n(\mathcal{O}_K/\mathfrak{b}_K) \rightarrow \check{A}_{n,r+}\).

**Lemma 4.7.** The element

\[
i_n = 1 + [\zeta_{p^{n+1}}] + [\zeta_{p^{n+1}}]^2 + \cdots + [\zeta_{p^{n+1}}]^{p-1} = \frac{[\zeta_{p^n}] - 1}{[\zeta_{p^{n+1}}] - 1}
\]

is divisible by \(\hat{\gamma}\) in the ring \(W_n(\mathcal{O}_{F_n})\). In particular, \(\hat{\gamma}\) is a non-zero divisor of the ring \(W_n(\check{\mathcal{O}}_K)\).

**Proof.** It is enough to show the divisibility in the ring \(W_n(\mathcal{O}_K)\). Note that the element \(t_n\) is also the image of \(t_n\) by the natural map \(W_n(\mathcal{O}_{F_n}) \rightarrow W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})\). Let \(\tilde{v}_n\) be a lift of \(v_n\) by the natural surjection \(W_n(\mathcal{O}_K) \rightarrow W_n(\check{\mathcal{O}}_K)\). Then we have \(t_n - \hat{\gamma} \tilde{v}_n \in W_n(p\mathcal{O}_K)\). By Lemma 4.6, there exists \(\tilde{y} \in W_n(m_{\mathbb{C}})\) such that \(t_n - \hat{\gamma} \tilde{v}_n = t_n \tilde{y}\). Hence we have \(t_n(1 - \tilde{y}) = \hat{\gamma} \tilde{v}_n\). Since \(\tilde{y}\) is topologically nilpotent in the ring \(W_n(\mathcal{O}_K)\), the element \(1 - \tilde{y}\) is invertible and the lemma follows. \(\square\)

**Lemma 4.8.** The image of \(Y \in \Sigma\) in the ring \(\check{A}_{n,r+}\) (resp. \(\check{A}_{n,p-1}\)) is contained in its subring \(\check{A}_{n,F_n,r+}\) (resp. \(W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})/(\{[\zeta_{p^n}] - 1\}^{p-1})\)).

**Proof.** We have the equality

\[
E(\{\pi_n\})v_n = t_n = 1 + [\zeta_{p^{n+1}}] + [\zeta_{p^{n+1}}]^2 + \cdots + [\zeta_{p^{n+1}}]^{p-1}
\]

in the ring \(W_n(\check{\mathcal{O}}_K)\). Note that any element \(v'_n \in W_n(\check{\mathcal{O}}_K)\) satisfying the same equality is invertible and thus the elements \(v'_n^{1-E(\{\pi_n\})} = 1\) to each other. Since \(Y = -a_n^{-1}E(\{\pi_n\})^{p-1}\) in the rings \(\check{A}_{n,r+}\) and \(\check{A}_{n,p-1}\), it suffices to construct an element \(v'_n\) of the ring \(W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})\) such that the equality \(E(\{\pi_n\})v'_n = t_n\) holds. This follows from Lemma 4.7. \(\square\)

From this lemma, we see that the natural \(G_{F_n}\)-actions on the rings \(\check{A}_{n,p-1}\) and \(\check{A}_{n,r+}\) are compatible with the filtered \(\phi_r\)-module structures over \(\Sigma\). In
the big commutative diagram above, the lowest horizontal arrow and lower right vertical arrow are $G_K$-linear by definition. Hence we have shown the following proposition.

**Proposition 4.9.** Let $M$ be an object of $\text{Mod}^{\text{Fil}, \phi}_{\Sigma_{\infty}}$. Then the map

$$\text{Hom}_{\Sigma_{\infty}, \text{Fil}, \phi}(M, W_n^p (\hat{O}_K)) \rightarrow \text{Hom}_{\Sigma_{\infty}, \text{Fil}, \phi}(M, \bar{A}_{n,r+})$$

is an isomorphism of $G_{F_n}$-modules.

Let $M$ be as in the proposition. Let $e_1, \ldots, e_d$ be a system of generators of $M$ as in Lemma 3.5 and $C = (c_{i,j}) \in M_d(\Sigma)$ be a matrix representing $\phi_r$ as in Corollary 3.6. Consider the surjection $\Sigma^{\geq d} \rightarrow M$ defined by $(s_1, \ldots, s_d) \mapsto s_1e_1 + \cdots + s_de_d$ and let $(s_{i,1}, \ldots, s_{i,d}), \ldots, (s_{q,1}, \ldots, s_{q,d})$ be a system of generators of its kernel. Then the underlying $G_{F_n}$-set of the $G_{F_n}$-module

$$\text{Hom}_{\Sigma_{\infty}, \text{Fil}, \phi}(M, \bar{A}_{n,r+})$$

is identified with the set of $d$-tuples $(\bar{x}_1, \ldots, \bar{x}_d)$ in $\bar{A}_{n,r+}$ such that the following three conditions hold:

- $s_{i,1}\bar{x}_1 + \cdots + s_{i,d}\bar{x}_d = 0$ for any $l$,
- $c_{i,1}\bar{x}_1 + \cdots + c_{i,d}\bar{x}_d \in \text{Fil}^r A_{n,r+}$ for any $i$,
- the following equality holds:

$$\begin{cases} 
\phi_r(c_{i,1}\bar{x}_1 + \cdots + c_{i,d}\bar{x}_d) = \bar{x}_1 \\
\vdots \\
\phi_r(c_{1,1}\bar{x}_1 + \cdots + c_{d,1}\bar{x}_d) = \bar{x}_d.
\end{cases}$$

We choose lifts $\bar{c}, \bar{c}_{i,j}$ and $\bar{s}_{i,j}$ in $W_n(\hat{O}_{F_n})$ of the images of $c, c_{i,j}$ and $s_{i,j}$ in $\bar{A}_{n,r+}$ by the natural ring homomorphism

$$W_n(\hat{O}_K) \rightarrow W_n(\hat{O}_K) \rightarrow W_n(\hat{O}_K/\mathfrak{b}_K) \rightarrow \bar{A}_{n,r+},$$

respectively. Recall that we have already chosen a lift $\hat{\gamma} \in W_n(\hat{O}_{F_n})$ of $E([\pi_n]) \in W_n(\hat{O}_{F_n}/p\hat{O}_{F_n})$.

Fix a polynomial $\Phi_i \in \mathbb{Z}[X_0, \ldots, X_{n-1}]$ such that $\Phi_i \equiv X_i^p \mod p$. This induces for any commutative ring $B$ a map $\Phi = (\Phi_0, \ldots, \Phi_{n-1}) : W_n(B) \rightarrow W_n(B)$ which is a lift of the Frobenius endomorphism on $W_n(B/pB)$. In particular, set $B$ to be the polynomial ring $\mathbb{Z}[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]$. Put $X = (X_0, \ldots, X_{n-1})$ and $Y = (Y_0, \ldots, Y_{n-1})$ in the ring $W_n(B)$. Then we see that there exists elements $U_0, \ldots, U_{n-1}$ and $U'_0, \ldots, U'_{n-1}$ of the polynomial ring $B$ such that

$$\Phi(X + Y) = \Phi(X) + \Phi(Y) + (pU_0, \ldots, pU_{n-1}),$$
$$\Phi(XY) = \Phi(X)\Phi(Y) + (pU'_0, \ldots, pU'_{n-1})$$

in the ring $W_n(B)$.

**Proposition 4.10.** Every $d$-tuple $(\bar{x}_1, \ldots, \bar{x}_d)$ in $\bar{A}_{n,r+}$ satisfying the above three conditions uniquely lifts to a $d$-tuple $(\hat{x}_1, \ldots, \hat{x}_d)$ in $W_n(\hat{O}_K)$ such that

- $\hat{s}_{l,1}\hat{x}_1 + \cdots + \hat{s}_{l,d}\hat{x}_d \in ([\zeta_p^l] - 1)^r W_n(m_K)$ for any $l$,,
\[ \hat{c}_1, \hat{x}_1 + \cdots + \hat{c}_d, \hat{x}_d \in \hat{\gamma}^r W_n(\mathcal{O}_K) \text{ for any } i, \]

the following equality holds:

\[
\begin{align*}
\hat{c}^r \Phi((\hat{c}_1, \hat{x}_1 + \cdots + \hat{c}_d, \hat{x}_d)/\hat{\gamma}^r) &= \hat{x}_1 \\
\vdots \\
\hat{c}^r \Phi((\hat{c}_1, \hat{x}_1 + \cdots + \hat{c}_d, \hat{x}_d)/\hat{\gamma}^r) &= \hat{x}_d.
\end{align*}
\]

Proof. Fix a lift \( \hat{x}_i \) of \( \bar{x}_i \) in \( W_n(\mathcal{O}_K) \). Recall that the kernel of the surjection \( W_n(\mathcal{O}_K) \to \bar{A}_{n,r+} \) is equal to the ideal \( (\zeta_p^n - 1)^r W_n(m_K) \). The first condition in the proposition holds automatically for \( (\hat{x}_1, \ldots, \hat{x}_d) \). By Lemma 4.7, the element \( \hat{c}_1, \hat{x}_1 + \cdots + \hat{c}_d, \hat{x}_d \) is contained in \( \hat{\gamma}^r W_n(\mathcal{O}_K) \) for any \( i \). Since the map \( \phi_r : \text{Fil}^r \bar{A}_{n,r+} \to A_{n,r+} \) satisfies \( \phi_r(E([\pi])^r \bar{x}) = c^r \phi(\bar{x}) \) for any \( \bar{x} \in \bar{A}_{n,r+} \), we have \( \hat{c}^r \Phi((\hat{c}_1, \hat{x}_1 + \cdots + \hat{c}_d, \hat{x}_d)/\hat{\gamma}^r) = \hat{x}_1 + ([\zeta_p^n] - 1)^r \hat{\delta}_1 + 
\vdots + \hat{x}_d + ([\zeta_p^n] - 1)^r \hat{\delta}_d \]

for some \( \hat{\delta}_1, \ldots, \hat{\delta}_d \in W_n(m_K) \). It suffices to show that there exists a unique \( d \)-tuple \( (\hat{y}_1, \ldots, \hat{y}_d) \) in \( W_n(m_K) \) such that

\[ \hat{c}^r \Phi((\hat{c}_1, \hat{x}_1 + ([\zeta_p^n] - 1)^r \hat{y}_1) + \cdots + \hat{c}_d, \hat{x}_d + ([\zeta_p^n] - 1)^r \hat{y}_d)/\hat{\gamma}^r) \]

\[ = \hat{x}_1 + ([\zeta_p^n] - 1)^r \hat{y}_i \]

for any \( i \). For this, we need the following lemma.

**Lemma 4.11.** Let \( N \) be a complete discrete valuation field and \( m_N \) be the maximal ideal of \( N \). Let \( \epsilon_1, \ldots, \epsilon_d \) be in \( m_N \). Let \( P_1, \ldots, P_d \) and \( P'_1, \ldots, P'_d \) be elements of \( \mathcal{O}_N[[Y_1, \ldots, Y_d]] \) such that \( P_i \in (Y_1, \ldots, Y_d)^2 \). Then the equation

\[
\begin{align*}
Y_1 - P_1(Y_1, \ldots, Y_d) - \epsilon_1 P'_1(Y_1, \ldots, Y_d) &= 0 \\
\vdots \\
Y_d - P_d(Y_1, \ldots, Y_d) - \epsilon_d P'_d(Y_1, \ldots, Y_d) &= 0
\end{align*}
\]

has a unique solution in \( m_N \).

Proof. By assumption, we see that for any integer \( l \geq 1 \), a \( d \)-tuple \( (y_1, \ldots, y_d) \) in \( m_N/m_N^l \) satisfying the above equation lifts uniquely to a \( d \)-tuple in \( m_N/m_N^{l+1} \) satisfying the same equation. Thus the lemma follows.

Let us write as \( \hat{y}_i = (\hat{y}_{i,0}, \ldots, \hat{y}_{i,n-1}) \). Since the image of \( \Phi((\zeta_p^{n+1} - 1)^r) \) in \( \bar{A}_{n,r+} \) is equal to \( ([\zeta_p^n] - 1)^r \), we can find \( \hat{b} \in W_n(\mathcal{O}_K) \) such that

\[ \Phi(([\zeta_p^n] - 1)^r/\hat{\gamma}^r) = ([\zeta_p^n] - 1)^r \hat{b}. \]
Then there exists polynomials $U_{i,m}$ over $\mathcal{O}_K$ of the indeterminates $y = (Y_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1}$ such that the equation we have to solve is

\[
\hat{x}_i + (\hat{\zeta}_p^n - 1)^r \hat{y}_i = \hat{x}_i + (\hat{\zeta}_p^n - 1)^r \hat{\delta}_i + (\hat{\zeta}_p^n - 1)^r b^{q^r}(\Phi(\hat{c}_{1,i})\Phi(\hat{y}_1) + \cdots + \Phi(\hat{c}_{d,i})\Phi(\hat{y}_d)) + (pU_{i,0}(\hat{y}), \ldots, pU_{i,n-1}(\hat{y}))
\]

for any $i$, where we put $\hat{y} = (\hat{y}_{i,m})_{1 \leq i \leq d, 0 \leq m \leq n-1}$. As in the proof of Lemma 4.6, we see that, for any elements $P_0, \ldots, P_{n-1}$ of the polynomial ring $\mathcal{O}_K[y]$, we can uniquely find elements $Q_0, \ldots, Q_{n-1}$ of this ring such that the coefficients of these polynomials are in the maximal ideal $m_K$ and the equality

\[
(pP_0, \ldots, pP_{n-1}) = ([\zeta_p^n] - 1)^r(Q_0, \ldots, Q_{n-1})
\]

holds in the ring of Witt vectors $W_n(\mathcal{O}_K[y])$. Therefore, this equation is equivalent to the equation

\[
y_i = \hat{\delta}_i + b^{q^r}(\Phi(\hat{c}_{1,i})\Phi(\hat{y}_1) + \cdots + \Phi(\hat{c}_{d,i})\Phi(\hat{y}_d)) + (V_{i,0}(\hat{y}), \ldots, V_{i,n-1}(\hat{y}))
\]

where $V_{i,m}$ is a polynomial of $y$ over $\mathcal{O}_K$ whose coefficients are in the maximal ideal $m_K$. From the definition of $\Phi$, we see that $\hat{y} = (\hat{y}_{i,m})_{i,m}$ is a solution of a system of equations

\[
Y_{i,m} - P_{i,m}(y) - e_{i,m}P'_{i,m}(y) = 0
\]

satisfying the condition of Lemma 4.11 for a sufficiently large finite extension $N$ of $K$. Then, by this lemma, we can solve the equation uniquely in $m_K$. \qed

Let $F$ be an algebraic extension of $F_n$ in $K$ and consider the ring $\tilde{A}_{n,F,r+}$. By Lemma 4.8, we can consider this ring as a $\Sigma$-subalgebra of $\tilde{A}_{n,r+}$. Put $\text{Fil}^r \tilde{A}_{n,F,r+} = E([\pi_n])^r \tilde{A}_{n,F,r+}$. Then Lemma 4.7 implies that

\[
\tilde{A}_{n,F,r+} \cap \text{Fil}^r \tilde{A}_{n,r+} = \text{Fil}^r \tilde{A}_{n,F,r+}.
\]

Moreover, the Frobenius endomorphism $\phi$ of the ring $\tilde{A}_{n,r+}$ preserves the subalgebra $\tilde{A}_{n,F,r+}$ and thus $\phi_r : \text{Fil}^r \tilde{A}_{n,r+} \to \tilde{A}_{n,r+}$ induces a $\phi$-semilinear map $\phi_r : \text{Fil}^r \tilde{A}_{n,F,r+} \to \tilde{A}_{n,F,r+}$. Hence $\tilde{A}_{n,F,r+}$ is a subobject of $\tilde{A}_{n,r+}$ in the category $\text{Mod}_\Sigma^{r,\phi}$. For $M \in \text{Mod}_\Sigma^{r,\phi}$, let us set

\[
T_{crys,\pi_n,F}(M) = \text{Hom}_{\Sigma,\text{Fil}^r,\phi_r}(M, \tilde{A}_{n,F,r+}).
\]

We see that

\[
\tilde{A}_{n,r+} = \tilde{A}_{n,K,r+} = \bigcup_{F/F_n} \tilde{A}_{n,F,r+}
\]

in $\text{Mod}_\Sigma^{r,\phi}$ and thus we have a natural identification of abelian groups

\[
T_{crys,\pi_n,K}(M) = \bigcup_{F/F_n} T_{crys,\pi_n,F}(M).
\]

The absolute Galois group $G_{F_n}$ acts on the abelian group on the left-hand side.
Lemma 4.12. Let $F$ be an algebraic extension of $F_n$ in $\bar{K}$. Then the $G_F$-fixed part $T^*_{\text{crys}, \pi_n, \bar{K}}(M)^{G_F}$ is equal to $T^*_{\text{crys}, \pi_n, F}(M)$.

Proof. From Proposition 4.10, we see that the elements of $T^*_{\text{crys}, \pi_n, \bar{K}}(M)$ correspond bijectively to the $d$-tuples in $W_n(\mathcal{O}_K)$ satisfying the three conditions in this proposition. The uniqueness assertion of the proposition shows that $g \in G_F$ fixes such a $d$-tuple in $W_n(\mathcal{O}_K)$ if and only if $g$ fixes its image in $A_{n,r+}$. Hence an element of $T^*_{\text{crys}, \pi_n, \bar{K}}(M)$ is fixed by $G_F$ if and only if it is contained in the image of $W_n(\mathcal{O}_F)$. Thus the lemma follows. $\square$

Corollary 4.13. Let $L_n$ be the finite Galois extension of $F_n$ corresponding to the kernel of the map

$$G_{F_n} \to \text{Aut}(T^*_{\text{crys}, \pi_n, \bar{K}}(M)).$$

Then an algebraic extension $F$ of $F_n$ in $\bar{K}$ contains $L_n$ if and only if

$$\#T^*_{\text{crys}, \pi_n, F}(M) = \#T^*_{\text{crys}, \pi_n, \bar{K}}(M).$$

Proof. An algebraic extension $F$ of $F_n$ contains $L_n$ if and only if the action of $G_F$ on $T^*_{\text{crys}, \pi_n, \bar{K}}(M)$ is trivial. By Lemma 4.12, this is equivalent to $T^*_{\text{crys}, \pi_n, F}(M) = T^*_{\text{crys}, \pi_n, \bar{K}}(M)$. $\square$

5. Ramification bound

In this section, we prove Theorem 1.1. Let $\mathcal{M}$ be an object of $\text{Mod}^{r,\phi,N}_{/S_{\infty}}$ which is killed by $p^n$ and let $\bar{L}$ be the finite Galois extension of $K$ corresponding to the kernel of the map

$$G_K \to \text{Aut}(T^*_{\text{st}, \pi}(\mathcal{M})).$$

Then the theorem is equivalent to the inequality $u_{L/K} \leq u(K, r, n)$, where $u_{L/K}$ denotes the greatest upper ramification break of the Galois extension $L/K$ ([10]). For $r = 0$, the $G_K$-module $T^*_{\text{st}, \pi}(\mathcal{M})$ is unramified and the assertion is trivial. Thus we may assume $p \geq 3$ and $r \geq 1$.

Let $L_n$ be the finite Galois extension of $F_n$ corresponding to the kernel of the map

$$G_{F_n} \to \text{Aut}(T^*_{\text{st}, \pi}(\mathcal{M})).$$

Since $F_n$ is Galois over $K$, the extension $L_n = LF_n$ is also a Galois extension of $K$. Let $M \in \text{Mod}^{\phi, \Sigma}_{/S_{\infty}}$ be the filtered $\phi$-module over $\Sigma$ which corresponds to $\mathcal{M}$ by the equivalence $\mathcal{M}_{S_{\infty}}$ of Proposition 3.13. Then Proposition 3.15 and Proposition 4.9 show that $L_n$ is also the finite extension of $F_n$ cut out by the $G_{F_n}$-module $T^*_{\text{crys}, \pi_n, K}(M)$. It is enough to prove the inequality $u_{L_n/K} \leq u(K, r, n)$.

Before proving this, we state some general lemmas to calculate the ramification bound. Let $N$ be a complete discrete valuation field of positive residue characteristic, $v_N$ be its valuation normalized as $v_N(N^\times) = \mathbb{Z}$ and $N^{\text{sep}}$ be its separable closure. We extend $v_N$ to any algebraic closure of $N$. 

- **Lemma** Let $L$ be an algebraic extension of $F_n$ in $\bar{K}$. Then the $G_F$-fixed part $T^*_{\text{crys}, \pi_n, \bar{K}}(M)^{G_F}$ is equal to $T^*_{\text{crys}, \pi_n, F}(M)$. 

- **Proof.** From Proposition 4.10, we see that the elements of $T^*_{\text{crys}, \pi_n, \bar{K}}(M)$ correspond bijectively to the $d$-tuples in $W_n(\mathcal{O}_K)$ satisfying the three conditions in this proposition. The uniqueness assertion of the proposition shows that $g \in G_F$ fixes such a $d$-tuple in $W_n(\mathcal{O}_K)$ if and only if $g$ fixes its image in $A_{n,r+}$. Hence an element of $T^*_{\text{crys}, \pi_n, \bar{K}}(M)$ is fixed by $G_F$ if and only if it is contained in the image of $W_n(\mathcal{O}_F)$. Thus the lemma follows. $\square$

- **Corollary** Let $L_n$ be the finite Galois extension of $F_n$ corresponding to the kernel of the map

$$G_{F_n} \to \text{Aut}(T^*_{\text{crys}, \pi_n, \bar{K}}(M)).$$

Then an algebraic extension $F$ of $F_n$ in $\bar{K}$ contains $L_n$ if and only if

$$\#T^*_{\text{crys}, \pi_n, F}(M) = \#T^*_{\text{crys}, \pi_n, \bar{K}}(M).$$

- **Proof.** An algebraic extension $F$ of $F_n$ contains $L_n$ if and only if the action of $G_F$ on $T^*_{\text{crys}, \pi_n, \bar{K}}(M)$ is trivial. By Lemma 4.12, this is equivalent to $T^*_{\text{crys}, \pi_n, F}(M) = T^*_{\text{crys}, \pi_n, \bar{K}}(M)$. $\square$
Lemma 5.1. Let \( f(T) \in \mathcal{O}_N[T] \) be a separable monic polynomial and \( z_1, \ldots, z_d \) be the zeros of \( f \) in \( \mathcal{O}_N^{\text{sep}} \). Suppose that the set \( \{ v_N(z_k - z_i) | k = 1, \ldots, d, k \neq i \} \) is independent of \( i \). Put
\[
s_f = \sum_{k=1, \ldots, d \atop k \neq i} v_N(z_k - z_i) \text{ and } \alpha_f = \sup_{k=1, \ldots, d \atop k \neq i} v_N(z_k - z_i),
\]
which are independent of \( i \) by assumption. If \( j > s_f + \alpha_f \), then we have the decomposition
\[
\{ x \in \mathcal{O}_N^{\text{sep}} | v_N(f(x)) \geq j \} = \prod_{i=1, \ldots, d} \{ x \in \mathcal{O}_N^{\text{sep}} | v_N(x - z_i) \geq j - s_f \}.
\]
Otherwise, the set on the left-hand side contains
\[
\{ x \in \mathcal{O}_N^{\text{sep}} | v_N(x - z_i) \geq \alpha_f \},
\]
which contains at least two zeros of \( f \).

Proof. A verbatim argument in the proof of [1, Lemma 6.6] shows the claim.

Corollary 5.2. Let \( f(T) \) be as above and put \( B = \mathcal{O}_N[T]/(f(T)) \). Let us write the \( N \)-algebra \( N' = B \otimes_{\mathcal{O}_N} N \) as the product \( N_1 \times \cdots \times N_t \) of finite separable extensions \( N_1, \ldots, N_t \) of \( N \). If \( j > s_f + \alpha_f \), then the \( j \)-th upper numbering ramification group \((N)\), which we let be denoted by \( G^{(j)}_N \), is contained in \( G_N^{(j)} \) for any \( i \). Moreover, if \( N' \) is a field and \( B \) coincides with \( \mathcal{O}_N' \), then \( j > s_f + \alpha_f \) if and only if \( G_N^{(j)} \subseteq G_N' \).

Proof. Note that the algebra \( B \) is finite flat and of relative complete intersection over \( \mathcal{O}_N \). By the previous lemma, the conductor \( c(B) \) of the \( \mathcal{O}_N \)-algebra \( B \) ([1, Proposition 6.4]) is equal to \( s_f + \alpha_f \). Thus we have the inequality
\[
c(\mathcal{O}_N' \times \cdots \times \mathcal{O}_N') \leq c(B) = s_f + \alpha_f
\]
by the definition of the conductor and a functoriality of the functor \( \mathcal{F}j \) defined in [1]. This implies the corollary.

Corollary 5.3. We have the inequality
\[
u_K(\zeta_{p^{n+1}})/K \leq 1 - \frac{1}{e(K(\zeta_p)/K)} + e(n + \frac{1}{p - 1}),
\]
where \( e(K(\zeta_p)/K) \) denotes the relative ramification index of \( K(\zeta_p) \) over \( K \).

Proof. Since the Herbrand function is transitive and the finite extension \( K(\zeta_p) \) is tamely ramified over \( K \), it is enough to show the inequality
\[
u_K(\zeta_{p^{n+1}})/K(\zeta_p) \leq e(K(\zeta_p))(n + \frac{1}{p - 1}).
\]

Put \( N = K(\zeta_p) \) and \( f(T) = T^{p^n} - \zeta_p \). These satisfy the assumptions of Corollary 5.2. We have \( s_f = ne(K(\zeta_p)) \) and \( \alpha_f = e(K(\zeta_p))/(p - 1) \) in this case. Hence the corollary follows.
Corollary 5.4. Consider the finite Galois extension $F_n = K_n(\zeta_{p^n+1})$ of $K$. Then we have the equality
\[ u_{F_n/K} = 1 + e(n + \frac{1}{p-1}) \]

Proof. Applying Corollary 5.2 to the Eisenstein polynomial $f(T) = T^{p^n} - \pi$ and $N = K$ shows that $j > 1 + e(n + 1/(p-1))$ if and only if $G^{(j)}_N \subseteq G^{(j)}_K$. From Corollary 5.3, we see that if $j > 1 + e(n + 1/(p-1))$, then $G^{(j)}_N \subseteq G^{(j)}_K$. Since $G^{(j)}_{F_n} = G^{(j)}_N \cap G^{(j)}_K(\zeta_{p^n+1})$, we conclude that $j > 1 + e(n + 1/(p-1))$ if and only if $G^{(j)}_N \subseteq G^{(j)}_N$.

Remark 5.5. Note that this argument also shows the equality
\[ u_{K_n(\zeta_{p^n})/K} = 1 + e(n + \frac{1}{p-1}) \]

Next we assume that the residue field of $N$ is perfect. For an algebraic extension $F$ of $N$, we put
\[ \mathfrak{a}^j_F = \{ x \in \mathcal{O}_F \mid v_N(x) \geq j \} \]

Let $Q$ be a finite Galois extension of $N$ and consider the property
\[ (P_j) \begin{cases} 
\text{for any algebraic extension } F \text{ of } N, \text{ if there exists an } \mathcal{O}_N\text{-algebra homomorphism } \mathcal{O}_Q \rightarrow \mathcal{O}_F/\mathfrak{a}_F^j, \\
\text{then there exists an } N\text{-algebra injection } Q \rightarrow F \end{cases} \]

for $j \in \mathbb{R}_{\geq 0}$, as in [10, Proposition 1.5]. Then we have the following proposition, which is due to Yoshida. Here we reproduce his proof for the convenience of the reader.

Proposition 5.6 ([19]).
\[ u_{Q/N} = \inf \{ j \in \mathbb{R}_{\geq 0} \mid \text{the property } (P_j) \text{ holds} \} \]

Proof. By [10, Proposition 1.5 (i)], it is enough to show that the property $(P_j)$ does not hold for $j = u_{Q/N} - (e')^{-1}$ with an arbitrarily large $e' > 0$. As in the proof of [10, Proposition 1.5 (ii)], we may assume that $Q$ is totally and wildly ramified over $N$. Take an arbitrarily large integer $e'' > 0$ with $(e'', pe(Q/N)) = 1$. We may also assume that $N$ contains a primitive $e''$-th root of unity. Set $N' = N(\pi^{1/e''}_N)$ and $Q' = QN'$. Note that we have $u_{Q'/N} = u_{Q/N}$ by assumption. From this proposition in [10], we see that for some algebraic extension $F$ of $N$, there exists an $\mathcal{O}_N$-algebra homomorphism $\mathcal{O}_{Q'} \rightarrow \mathcal{O}_F/\mathfrak{a}_F^j$ for $j = u_{Q/N} - e(Q'/N)^{-1}$ but no $N$-algebra injection $Q' \rightarrow F$. Since $Q'/N$ is wildly ramified, we see that $e(Q'/N)u_{Q/N} - 1 > e(Q/N)$. Hence we have $u_{Q/N} - e(Q'/N)^{-1} > 1 \geq u_{Q'/N}$ and there exists an $N$-algebra injection $N' \rightarrow F$ also by this proposition. Thus there exists no $N$-algebra injection $Q \rightarrow F$ and the property $(P_j)$ for $Q/N$ does not hold. Since $e(Q'/N) = e'e(Q/N)$, the proposition follows.
We see from Proposition 5.6 that to bound the greatest upper ramification break $u_{L_n/K}$, it is enough to show the following proposition.

**Proposition 5.7.** Let $F$ be an algebraic extension of $K$. If $j > u(K, r, n)$ and there exists an $O_K$-algebra homomorphism

$$\eta: O_{L_n} \to O_F/a_F^j/K,$$

then there exists a $K$-algebra injection $L_n \to F$.

**Proof.** We may assume that $F$ is contained in $K$. By assumption, we have $j > er/(p-1)$ and we see that the ideal $b_F = \{ x \in O_F \mid v_K(x) > er/(p-1) \}$ contains $a_F^j$. Thus $\eta$ induces an $O_K$-algebra homomorphism

$$O_{L_n} \to O_F/b_F.$$

Since $\eta$ also induces an $O_K$-algebra homomorphism $O_{F_n} \to O_F/a_F^j/K$ and $r \geq 1$, from Corollary 5.4 and [10, Proposition 1.5] we get a $K$-linear injection $F_n \to F$. Thus we see that $F$ contains $\pi_n$ and $\zeta_{p^n+1}$. More precisely, we have the following two lemmas.

**Lemma 5.8.** There exists $i \in \mathbb{Z}$ such that $\eta(\pi_n) \equiv \pi_n\zeta_{p^n}^i \mod b_F$.

**Proof.** Since the map $\eta$ is $O_K$-linear, the equality $\eta(\pi_n)^{p^n} = \pi$ holds in $O_F/a_F^j/K$. Set $\hat{x}$ to be a lift of $\eta(\pi_n)$ in $O_F$. Then we have

$$v_K(\hat{x}^{p^n} - \pi) = \sum_{i=0}^{p^n-1} v_K(\hat{x} - \pi_n\zeta_{p^n}^i) \geq j.$$

Let us apply Lemma 5.1 to $f(T) = T^{p^n} - \pi \in O_K[T]$. Then, with the notation of the lemma, we have

$$s_f = 1 - \frac{1}{p^n} + ne \quad \text{and} \quad \alpha_f = 1 - \frac{e}{p-1}.$$

Since $j - s_f > er/(p-1)$ by assumption, we have

$$\hat{x} \equiv \pi_n\zeta_{p^n}^i \mod b_F$$

for some $i$. \hfill \square

**Lemma 5.9.** There exists $g' \in G_K$ such that $\eta(\zeta_{p^n+1}) \equiv g'(\zeta_{p^n+1}) \mod b_F$.

**Proof.** Set $N$ to be the maximal unramified subextension of $K(\zeta_{p^n+1})/K$. Since the map $O_K \to O_N$ is etale, there exists a $K$-algebra injection $g_0 : N \to F$ such that $\eta(x) \equiv g_0(x) \mod a_F^j/K$ for any $x \in O_N$. Let $\varpi$ be a uniformizer of $K(\zeta_{p^n+1})$ and $f(T) \in O_N[T]$ be the Eisenstein polynomial of $\varpi$ over $O_N$. We let $f^{g_0}(T) \in O_N[T]$ denote the conjugate of $f$ by $g_0$. Then $f^{g_0}$ satisfies the conditions of Lemma 5.1. By definition we have $s_{f^{g_0}} = s_f$ and $\alpha_{f^{g_0}} = \alpha_f$. Since the roots of $f^{g_0}(T)$ are conjugates of $\varpi$ over $K$, Lemma 5.1 implies as in the previous lemma that there exists $g' \in G_K$ such...
that \( g' | N = g_0 \) and \( \eta(\varpi) \equiv g'(\varpi) \bmod a_F^{1-s_f} \). Since \( \mathcal{O}_K(\zeta_{p^{n+1}}) \) is generated by \( \varpi \) over \( \mathcal{O}_N \), we see that \( \eta(\zeta_{p^{n+1}}) \equiv g'(\zeta_{p^{n+1}}) \bmod a_F^{1-s_f} \).

Thus it is enough to check the inequality \( j - s_f > er/(p-1) \). Note that \( s_f \) is equal to the valuation \( v_K(\mathcal{D}_K(\zeta_{p^{n+1}})/N) \) of the different of the totally ramified Galois extension \( K(\zeta_{p^{n+1}})/N \). To bound this, put \( G = \text{Gal}(K(\zeta_{p^{n+1}})/N(\zeta_p)) \) and \( e' = e(N(\zeta_p)/N) \). We have

\[
v_K(\tau(\varpi) - \varpi) \leq v_K(\tau(\zeta_{p^{n+1}}) - \zeta_{p^{n+1}})
\]

for any \( \tau \in G \) and thus

\[
v_K(\mathcal{D}_K(\zeta_{p^{n+1}})/N(\zeta_p)) \leq \sum_{\tau \neq 1 \in G} v_K(\tau(\zeta_{p^{n+1}}) - \zeta_{p^{n+1}}) \leq ne.
\]

We also have the equality \( v_K(\mathcal{D}_N(\zeta_p)/N) = 1 - 1/e' \) and hence we get

\[s_f = v_K(\mathcal{D}_K(\zeta_{p^{n+1}})/N) \leq 1 - 1/e' + ne.\]

Since \( e' \leq p-1 \), the inequality \( j - s_f > er/(p-1) \) holds. \( \square \)

**Corollary 5.10.** There exists \( g \in G_K \) such that \( \eta(\pi_n) \equiv g(\pi_n) \bmod b_F \) and \( \eta(\zeta_{p^{n+1}}) \equiv g(\zeta_{p^{n+1}}) \) modulo \( b_F \).

**Proof.** Let \( i \in \mathbb{Z} \) and \( g' \in G_K \) be as in Lemma 5.8 and Lemma 5.9, respectively. Since \( K_n \cap K(\zeta_{p^{n+1}}) = K \) (see for example [17, Lemma 5.1.2]), we can find an element \( g \in G_K \) such that \( g(\pi_n) = \pi_n \zeta_{p^n} \) and \( g(\zeta_{p^{n+1}}) = g'(\zeta_{p^{n+1}}) \). \( \square \)

**Lemma 5.11.** For \( m \in \mathbb{Z}_{\geq 0} \), set an ideal \( b_{L_n}^{(m)} \) of \( \mathcal{O}_{L_n} \) to be

\[b_{L_n}^{(m)} = \{ x \in \mathcal{O}_{L_n} \mid v_K(x) > \frac{er}{p^m(p-1)} \}\]

and similarly for \( F \). Then the \( \mathcal{O}_K \)-algebra homomorphism \( \eta \) induces an \( \mathcal{O}_K \)-algebra injection

\[\eta^{(m)} : \mathcal{O}_{L_n}/b_{L_n}^{(m)} \to \mathcal{O}_F/b_F^{(m)}\]

for any \( m \).

**Proof.** We may assume that \( L_n \) is totally ramified over \( K \). We write the Eisenstein polynomial of a uniformizer \( \pi_{L_n} \) of \( L_n \) over \( \mathcal{O}_K \) as

\[P(T) = T^d + c_1T^{d-1} + \cdots + c_{d-1}T + c_d,\]

where \( c'_e = e(L_n/K) \). Then \( z = \eta(\pi_{L_n}) \) satisfies \( P(z) = 0 \) in \( \mathcal{O}_F/a_F^{1}/K \). Let \( \hat{z} \) be a lift of \( z \) in \( \mathcal{O}_F \). Since \( j > 1 \), we have \( v_K(\hat{z}) = 1/e' \). The condition \( i > e(L_n)r/(p^m(p-1)) \) is equivalent to the condition

\[v_K(\hat{z}^i) > \frac{e(L_n)r}{p^m(p-1)} \cdot \frac{1}{e'} = \frac{er}{p^m(p-1)}.\]

Thus the claim follows. \( \square \)
Since $L_n$ contains $F_n$, we can consider the ring
\[ \bar{A}_{n,L,n} = W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n})/([\zeta_p^n] - 1)^t W_n(m_{L_n}/\mathfrak{b}_{L_n}) \]
and similarly $\bar{A}_{n,F,r,+}$ for $F$. We give these rings structures of $\Sigma$-algebras as follows. The ring $\bar{A}_{n,L,n} +$ is considered as a $\Sigma$-algebra by using the system \{ $\tilde{\pi}_n$ \}$_{n \in \mathbb{Z}_{\geq 0}}$ we chose of $p$-power roots of $\pi$, as in the previous section. On the other hand, using $g \in G_K$ in Corollary 5.10, put $\tilde{\pi}_n = g(\pi_n)$ and $\tilde{\zeta}_p^{n+1} = g(\zeta_p^{n+1})$. Then we consider the ring $\bar{A}_{n,F,r,+}$ as a $\Sigma$-algebra by using a system of $p$-power roots of $\pi$ containing $\tilde{\pi}_n$. We define $\text{Fil}^r$ and $\phi_r$ of these rings in the same way as before.

**Lemma 5.12.** The induced ring homomorphism
\[ \bar{\eta} : \bar{A}_{n,L,n} \to \bar{A}_{n,F,r,+} \]
is a morphism of the category $\text{Mod}^{\phi_r}_{/\Sigma}$.

**Proof.** Firstly, we check that $\bar{\eta}$ is $\Sigma$-linear. By definition, this homomorphism commutes with the action of the element $u \in \Sigma$. To show the compatibility with the element $Y \in \Sigma$, let us consider the commutative diagram
\[
\begin{array}{ccc}
W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n}) & \xrightarrow{\eta_n} & W_n(\mathcal{O}_F/\mathfrak{b}_F) \\
\downarrow & & \downarrow \\
\bar{A}_{n,L,n} & \xrightarrow{\bar{\eta}} & \bar{A}_{n,F,r,+}
\end{array}
\]
where the horizontal arrows are induced by $\eta$. Note that we have $\eta_n([\pi_n]) = [\tilde{\pi}_n]$ and $\eta_n([\zeta_p^n]) = [\tilde{\zeta}_p^{n+1}]$. Let $a \in W(R)^\times$ and $v = t/E([\pi]) \in W(R)^\times$ be as in the previous section. Let $a_n$ and $v_n$ denote the images of $a$ and $v$ in $W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n})$, respectively. Then the element $v_n$ is a solution of the equation
\[ E([\pi_n])v_n = 1 + [\zeta_p^n] + \cdots + [\zeta_p^{n+1}]^{p-1}. \]
Similarly, we define elements $\tilde{\eta}_n$ and $\tilde{v}_n$ of $W_n(\mathcal{O}_F/\mathfrak{b}_F)$ using $\tilde{\pi}_n$ and $\tilde{\zeta}_p^{n+1}$. By definition, the element $\tilde{v}_n$ is a solution of the equation
\[ E([\tilde{\pi}_n])\tilde{v}_n = 1 + [\tilde{\zeta}_p^{n+1}] + \cdots + [\tilde{\zeta}_p^{n+1}]^{p-1}. \]
Now what we have to show is the equality
\[ \bar{\eta}(a_nv_n^{-1}E([\pi_n])^{p-1}) = \tilde{\eta}_n\tilde{v}_n^{-1}E([\tilde{\pi}_n])^{p-1} \]
in the ring $\bar{A}_{n,F,r,+}$. Since the element $a_n$ of $W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n})$ is a linear combination of the elements $1, [\zeta_p^n], \ldots, [\zeta_p^{n+1}]^{p-1}$ over $\mathbb{Z}$, we have $\bar{\eta}(a_n) = \tilde{\eta}_n$ in $\bar{A}_{n,F,r,+}$. The elements $\tilde{v}_n$ and $\tilde{\eta}(v_n)$ satisfy the same equation in $\bar{A}_{n,F,r,+}$. Since these two elements are invertible, we get $\tilde{\eta}(v_n)^{-1}E([\tilde{\pi}_n]) = \tilde{v}_n^{-1}E([\tilde{\pi}_n])$ and the equality holds. Since the diagram above is compatible with the Frobenius endomorphisms, we see from the definition that $\bar{\eta}$ also preserves $\text{Fil}^r$ and commutes with $\phi_r$ of both sides. \qed
Thus we get a homomorphism of abelian groups

\[ T^*_{\text{crys}, L_n, \pi_n}(M) \to T^*_{\text{crys}, F, \tilde{\pi}_n}(M). \]

Then the following lemma, whose proof is omitted in [3, Subsection 3.13], implies that this homomorphism is an injection. We insert here a proof of this lemma for the convenience of the reader.

**Lemma 5.13.** The ring homomorphism \( \tilde{\eta} : \tilde{A}_{n, L_n, r^+} \to \tilde{A}_{n, F, r^+} \) is an injection.

**Proof.** Let \( x = (x_0, \ldots, x_{n-1}) \) be an element of \( W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n}) \) such that

\[
(\eta^{(0)}(x_0), \ldots, \eta^{(0)}(x_{n-1})) \in ((\zeta_{\rho^n}) - 1)^r W_n(\mathfrak{m}_F/\mathfrak{b}_F),
\]

where \( \eta^{(0)} \) is as in Lemma 5.11. Suppose that \( x_0 = \cdots = x_{m-1} = 0 \) for some \( 0 \leq m \leq n - 1 \). Let \( \tilde{z}_i \in \mathcal{O}_F \) be a lift of \( \eta^{(0)}(x_i) \). By Lemma 4.6, we have

\[
(0, \ldots, 0, \tilde{z}_m, \ldots, \tilde{z}_{n-1}) = ((\zeta_{\rho^n}) - 1)^r (\tilde{y}_0, \ldots, \tilde{y}_{n-1})
\]

for some \( \tilde{y}_0, \ldots, \tilde{y}_{n-1} \in m_F ). Thus we get \( \tilde{y}_0 = \cdots = \tilde{y}_{m-1} = 0 \) and \( v_K(\tilde{z}_m) > er/(p^{n-1-m}(p - 1)) \). Then Lemma 5.11 implies that \( x_m \) is contained in the ideal \( \mathfrak{b}_{L_n}^{(n-1-m)}/\mathfrak{b}_{L_n} \) and

\[
x = ((\zeta_{\rho^n}) - 1)^r (0, \ldots, 0, y, 0, \ldots, 0) + (0, \ldots, 0, x'_{m+1}, \ldots, x'_{n-1})
\]

for some \( y \in m_{L_n}/\mathfrak{b}_{L_n} \) and \( x'_{m+1}, \ldots, x'_{n-1} \in \mathcal{O}_{L_n}/\mathfrak{b}_{L_n} \). Repeating this, we see that \( x \) is zero in \( \tilde{A}_{n, L_n, r^+} \) and the lemma follows. \( \square \)

Now Corollary 4.13 shows that the abelian group \( T^*_{\text{crys}, L_n, \pi_n}(M) \) has the same cardinality as \( T^*_{\text{crys}, K, \pi_n}(M) \). This implies that the abelian group \( T^*_{\text{crys}, F, \tilde{\pi}_n}(M) \) has cardinality no less than \( \# T^*_{\text{crys}, K, \pi_n}(M) \). Let \( g \in G_K \) be as in Corollary 5.10. Then we have the following lemma.

**Lemma 5.14.** The \( G_F \)-module \( T^*_{\text{crys}, K, \tilde{\pi}_n}(M) \) is isomorphic to the conjugate of the \( G_F \)-module \( T^*_{\text{crys}, K, \pi_n}(M) \) by the element \( g \).

**Proof.** Let us consider the composite

\[
\Sigma \to \tilde{A}_{n, r^+} \xrightarrow{g} \tilde{A}_{n, r^+}
\]

of the ring homomorphism defined by \( u \mapsto [\pi_n] \) and the map induced by \( g \). We can check that this is the natural ring homomorphism defined by \( u \mapsto [\tilde{\pi}_n] \) as in the proof of Lemma 5.12. Thus we have an isomorphism of abelian groups

\[
\text{Hom}_\Sigma(M, \tilde{A}_{n, r^+}) \to \text{Hom}_\Sigma(M, \tilde{A}_{n, r^+})
\]

\[
f \mapsto g \circ f,
\]

where we consider on the ring \( \tilde{A}_{n, r^+} \) on the right-hand side the filtered \( \phi_r \)-module structure over \( \Sigma \) defined by \( \tilde{\pi}_n \). As in the proof of Lemma 5.12, we can check that this isomorphism induces an injection

\[
\text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, \tilde{A}_{n, r^+}) \to \text{Hom}_{\Sigma, \text{Fil}^r, \phi_r}(M, \tilde{A}_{n, r^+}).
\]
This is also an isomorphism, for the map \( f \mapsto g^{-1} \circ f \) defines its inverse.

Thus we have \( \#T^*_{\text{crys}, K, \tilde{\tau}_n}(M) = \#T^*_{\text{crys}, K, \tilde{\tau}_n}(M) \). Since \( L_n \) is Galois over \( K \), this lemma also shows that the finite Galois extension of \( F_n \) cut out by the action on \( T^*_{\text{crys}, K, \tilde{\tau}_n}(M) \) is \( L_n \). Hence we see from Corollary 4.13 that \( F \) contains \( L_n \) and Proposition 5.7 follows. This concludes the proof of Theorem 1.1.

**Proof of Corollary 1.3.** The second assertion follows immediately from Theorem 1.1 and [8, Théorème 1.1]. As for the first assertion, note that if \( r = 0 \) then \( V \) is unramified and the assertion is trivial. Thus we may assume \( p \geq 3 \).

Since we have the natural surjection \( L/p^nL \to L/L' \), we may also assume \( L' = p^nL \). For \( \mathcal{M} \in \text{Mod}^{r, \phi, N}_{/S} \), let us consider the \( G_K \)-module

\[
T^*_{\text{st}, \mathcal{M}} = \text{Hom}_{S, \text{Fil} r, \phi, N}(\mathcal{M}, \hat{A}_\text{st}).
\]

By [17, Theorem 2.3.5], there exists \( \hat{\mathcal{M}} \in \text{Mod}^{r, \phi, N}_{/S} \) such that the \( G_K \)-module \( \mathcal{L} \) is isomorphic to \( T^*_\text{st}\mathcal{M} \). Then we see that the \( G_K \)-module \( L/p^nL \) is isomorphic to \( T^*_\text{st}\mathcal{M}/p^n\mathcal{M} \) and the assertion follows from Theorem 1.1.

**Remark 5.15.** The ramification bound in Theorem 1.1 is sharp for \( r \leq 1 \). Indeed, the greatest upper ramification break \( 1 + e(n + 1/(p - 1)) \) for \( r = 1 \) is obtained by the \( p^n \)-torsion of the Tate curve \( \bar{K}^\times/\pi^Z \) (see Remark 5.5).

The author does not know whether these bounds are sharp also for \( r \geq 2 \).

**References**


E-mail address: shin-h@math.kyushu-u.ac.jp

Faculty of Mathematics, Kyushu University