

HOMOTOPY CLASSIFICATION OF NANOPHRASES WITH LESS THAN OR EQUAL TO FOUR LETTERS

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ABSTRACT

In this paper we give the classification of stable equivalence classes of ordered, pointed, oriented multi-component curves on surfaces with minimum crossing number less than or equal to 2 which any curve in its equivalent class has no simply closed curves in its components. To do this, we use the theory of words and phrases which was introduced by V. Turaev. Indeed we give the homotopy classification of nanophrases with less than or equal 4 letters. This is an extension of the classification of nanophrases of length 2 with less than or equal to 4 letters which was given by the author in the recent paper.

Keywords: Nanophrases, Homotopy, Multi-component curves, Stably equivalent

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1. Introduction.

The study of curves via words was introduced by C. F. Gauss [2]. Gauss encoded closed planar curves by words of certain type which are now called Gauss words. We can apply this method to encode multi-component curves on surfaces. For instance, in [6] and [7] V. Turaev studied stable equivalence classes of curves on surfaces by using generalized Gauss words (called nanowords).

More precisely a nanoword over an alphabet α endowed with an involution $\tau : \alpha \longrightarrow \alpha$ is a word in an alphabet \mathcal{A} endowed with a projection $\mathcal{A} \ni A \mapsto |A| \in \alpha$ such that every letter appears twice or not at all. In the case where the alphabet α consists of two elements permuted by τ , the notion of a nanoword over α is equivalent to the notion of an open virtual string introduced in [8].

Turaev introduced an equivalence relation of homotopy on the set of nanowords over α . The relation of homotopy is generated by three transformations or moves on nanowords. The first move consists in deleting two consecutive entries of the same letter. The second move has the form $xAByBAz \mapsto xyz$ where x, y, z are words and A, B are letters such that $|A| = \tau(|B|)$. The third move has the form $xAByACzBCt \mapsto xBAyCAzCBt$ where x, y, z, t are words and A, B, C are letters

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such that $|A| = |B| = |C|$. These moves are suggested by the three local deformations of curves on surfaces (See Fig. 1 and [6] for more details). In [6] Turaev showed that a stable equivalence class of an oriented pointed curve on a surface is identified with a homotopy class of nanoword in a 2-letter alphabet. Moreover Turaev extended this result to multi-component curves. In fact a stable equivalence class of an oriented, ordered, pointed multi-component curve on a surface is identified with a homotopy class of a nanophrase in a 2-letter alphabet. Roughly speaking, a nanophrase is a sequence of words where concatenation of those words is a nanoword (See also section 3.2 and section 4 for more details). So if we use Turaev's theory of words and phrases, we can treat curves on surfaces algebraically.

Homotopy classification of nanowords was given by Turaev in [5]. Turaev gave the classification of nanowords less than or equal to 6 letters. Moreover using the Turaev's classification result and new homotopy invariants of nanophrases, the author gave the homotopy classification of nanophrases of length 2 with less than or equal to 4 letters in [1].

Now the purpose of this paper is to classify the multi-component curves with minimum crossing number less than or equal to 2 which has no "untide" components up to stable equivalence. To do this we use Turaev's theory of words and phrases. Indeed we give the classification theorem of nanophrases over arbitrary alphabet with less than or equal 4 letters without the condition on length and some lemmas. Then we obtain the proof of the main theorem Theorem 2.2 as a corollary.

The constitution of this paper is as follows. In sections 2-4 we review the theory of multi-component curves and the homotopy theory of words and phrases. In section 5 we introduce known results about classification of nanowords and nanophrases up to homotopy and we extend these results to phrases of an arbitrary length. Finally in section 6 we give the proof of the main theorem in this paper.

2. Stable Equivalence of Multi-component Curves.

2.1. Multi-component curves.

In this paper a *curve* means the image of a generic immersion of an oriented circle into an oriented surface. The word "generic" means that the curve has only a finite set of self-intersections which are all double and transversal. A *k-component curve* is defined in the same way as a curve with the difference that they may be formed by k curves rather than only one curve. These curves are *components* of the k -component curve. A k -component curves are *pointed* if each component is endowed with a base point (the origin) distinct from the crossing points of the k -component curve. A k -component curve is *ordered* if its components are numerated. Two ordered, pointed curves are *stably homeomorphic* if there is an orientation preserving homeomorphism of their regular neighborhoods in the ambient surfaces mapping the first multi-component curve onto second one and preserving the order,

the origins, and the orientations of the components.

Now we define stable equivalence of ordered, pointed multi-component curves [4]: Two ordered, pointed multi-component curves are *stably equivalent* if they can be related by finite sequence of the following transformations: (i) a trace formation replacing a ordered, pointed multi-component curve with homeomorphic one; (ii) a deformation of a pointed curve in its ambient surface away from the origin (such a deformation may push a branch of the multi-component curves across another branch or a double point but not across the origin of the curves) as in Fig. 1.

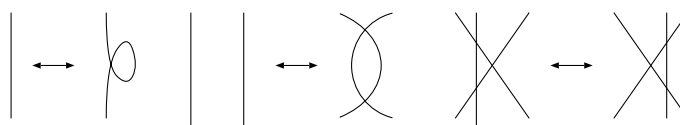


Fig. 1. Three local deformations of curves.

We denote the set of stable equivalence classes of ordered, pointed k -component curves by \mathcal{C}_k .

Remark 2.1. The theory of stable equivalence class of multi-component curves on surfaces is closely related to the theory of virtual strings. See [3] and [8] for more details.

The purpose of this paper is to show the following theorem.

Theorem 2.2. *Any ordered, pointed multi-component curve on surface with minimum crossing number less than or equal to 2 which any curve in its equivalent class has no simply closed curves in its components is stably equivalent to one of the ordered, pointed multi-component curves obtained from the following list:*

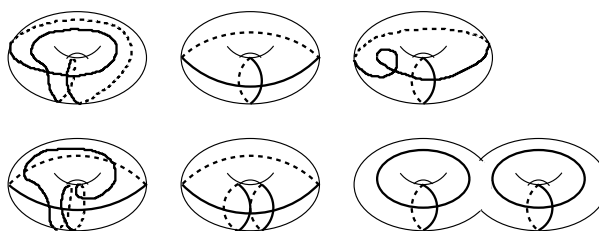


Fig. 2. The list of curves.

Remark 2.3. We list up the stable equivalence classes of ordered, pointed multi-component curves on surfaces with minimum crossing number less than or equal

to 2 which has no untide components. However there are too many curves to list up. So we make list of multi-component curves without order, orientation of the components and the base points. If we choose order, orientation and the base points, then we obtain a ordered, pointed multi-component curve. By the Theorem 5.15, these curves are stably equivalent if and only if nanophrases associated these curves are homotopic and we can obtain all of the stable equivalent classes of ordered, pointed multi-component curves on surfaces with minimum crossing number less than or equal to 2 which has no untide components are obtained by specifying order, orientation and the base points for following multi-component curves.

To proof the Theorem 2.2, we use Turaev's theory of words and phrases which is introduced by V. Turaev in [5] and [6].

3. Turaev's Theory of Words and Phrases.

In this section we review the theory of topology of words and phrases.

3.1. Nanowords and their homotopy.

An *alphabet* is a set and *letters* are its elements. A *word of length* $n \geq 1$ *on an alphabet* \mathcal{A} is a mapping $w : \hat{n} \rightarrow \mathcal{A}$ where $\hat{n} = \{1, 2, \dots, n\}$. A word is usually encoded by the sequence of letters $w(1)w(2) \cdots w(n)$. A word $w : \hat{n} \rightarrow \mathcal{A}$ is a *Gauss word* if each element of \mathcal{A} is the image of precisely two elements of \hat{n} .

For a set α , an α -alphabet is a set \mathcal{A} endowed with a mapping $\mathcal{A} \rightarrow \alpha$ called *projection*. The image of $A \in \mathcal{A}$ under this mapping is denoted $|A|$. An *étale word* over α is a pair (an α -alphabet, a word on \mathcal{A}). A *nanoword over* α is a pair (an α -alphabet, a Gauss word on α). An empty étale word in an empty α -alphabet is a nanoword called the *empty nanoword*. It is written \emptyset and has length 0.

A *morphism* of α -alphabets $\mathcal{A}_1, \mathcal{A}_2$ is a set-theoric mapping $f : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ such that $|A| = |f(A)|$ for all $A \in \mathcal{A}_1$. If f is bijective, then this morphism is an *isomorphism*. Two étale words (\mathcal{A}_1, w_1) and (\mathcal{A}_2, w_2) over α are *isomorphic* if there is an isomorphism $f : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ such that $w_2 = f \circ w_1$.

To define homotopy of nanowords we fix a finite set α with an involution $\tau : \alpha \rightarrow \alpha$ and a subset $S \subset \alpha \times \alpha \times \alpha$. We call the pair (α, S) *homotopy data*.

Definition 3.1. Let (α, S) be homotopy data. We define *homotopy moves* (1) - (3) as follows:

- (1) $(\mathcal{A}, xAAy) \rightarrow (\mathcal{A} \setminus \{A\}, xy)$
for all $A \in \mathcal{A}$ and x, y are words in $\mathcal{A} \setminus \{A\}$ such that xy is a Gauss word.
- (2) $(\mathcal{A}, xAByBAz) \rightarrow (\mathcal{A} \setminus \{A, B\}, xyz)$
if $A, B \in \mathcal{A}$ with $|B| = \tau(|A|)$. x, y, z are words in $\mathcal{A} \setminus \{A, B\}$ such that xyz is a Gauss word.
- (3) $(\mathcal{A}, xAByACzBCt) \rightarrow (\mathcal{A}, xBAyCAzCBt)$
if $A, B, C \in \mathcal{A}$ satisfy $(|A|, |B|, |C|) \in S$. x, y, z, t are words in \mathcal{A} such that

$xyzt$ is a Gauss word.

Definition 3.2. Let (α, S) be homotopy data. Then nanowords (\mathcal{A}_1, w_1) and (\mathcal{A}_2, w_2) over α are S -homotopic (denoted $(\mathcal{A}_1, w_1) \simeq_S (\mathcal{A}_2, w_2)$) if (\mathcal{A}_2, w_2) can be obtained from (\mathcal{A}_1, w_1) by a finite sequence of isomorphism, S -homotopy moves (1) - (3) and the inverse of moves (1) - (3).

The set of S -homotopy classes of nanowords over α is denoted $\mathcal{N}(\alpha, S)$.

To define S -homotopy of étale words we define *desingularization* of étale words (\mathcal{A}, w) over α as follows: $\mathcal{A}^d := \{A_{i,j} := (A, i, j) | A \in \mathcal{A}, 1 \leq i < j \leq m_w(A)\}$ with projection $|A_{i,j}| := |A| \in \alpha$ for all $A_{i,j}$ (where $m_w(A) := \text{Card}(w^{-1}(A))$). The word w^d is obtained from w by first deleting all $A \in \mathcal{A}$ with $m_w(A) = 1$. Then for each $A \in \mathcal{A}$ with $m_w(A) \geq 2$ and each $i = 1, 2, \dots, m_w(A)$, we replace the i -th entry of A in w by

$$A_{1,i}A_{2,i} \dots A_{i-1,i}A_{i,i+1}A_{i,i+2} \dots A_{i,m_w(A)}.$$

The resulting (\mathcal{A}^d, w^d) is a nanoword of length $\sum m_w(A)(m_w(A) - 1)$ and called a *desingularization* of (\mathcal{A}, w) . Then we define S -homotopy of étale words as follows:

Definition 3.3. Let w_1 and w_2 be étale words over α . Then w_1 and w_2 are S -homotopic if w_1^d and w_2^d are S -homotopic.

3.2. Nanophrases and their homotopy.

In [6], Turaev proceeded similar arguments for phrases (sequence of words).

Definition 3.4. A *nanophrase* $(\mathcal{A}, (w_1|w_2|\dots|w_k))$ of length $k \geq 0$ over a set α is a pair consisting of an α -alphabet \mathcal{A} and a sequence of k words w_1, \dots, w_k on \mathcal{A} such that $w_1w_2 \dots w_k$ is a Gauss word on \mathcal{A} . We denote it simply by $(w_1|w_2|\dots|w_k)$.

By definition, there is a unique *empty nanophrase* of length 0 (the corresponding α -alphabet \mathcal{A} is void).

Remark 3.5. Any nanoword w over α yields a nanophrase (w) of length 1.

A mapping $f : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is *isomorphism of two nanophrases* if f is an isomorphism of α -alphabets transforming the first nanophrase into the second one.

Given homotopy data (α, S) , we define homotopy moves on nanophrases as in section 3.1 with the only difference that the 2-letter subwords AA, AB, BA, AC and BC modified by these moves may occur in different words of phrase. Isomorphism and homotopy moves generate an equivalence relation \simeq_S of S -homotopy on the classes of nanophrases over α . We denote the set of S -homotopy classes of nanophrases of length k by $\mathcal{P}_k(\alpha, S)$.

4. Nanophrases versus Multi-component Curves

In [6], Turaev showed that the special case of the study of homotopy theory of nanophrases is equivalent to the study of \mathcal{C}_k . More precisely, Turaev showed following theorem.

Theorem 4.1. (Turaev [6]). *Let α_0 is the set $\{a, b\}$ with involution $\tau : \alpha_0 \rightarrow \alpha_0$ permuting a, b and S_0 is the diagonal of $\alpha_0 \times \alpha_0 \times \alpha_0$. Then there is a canonical bijection \mathcal{C}_k to $\mathcal{P}_k(\alpha_0, S_0)$.*

The method of making nanophrase $P(C)$ over α_0 from ordered, pointed k -component curve C is as follows. Let us label the double points of C by distinct letters A_1, \dots, A_n . Starting at the origin of first component of C and following along C in the positive direction, we write down the labels of double points which we passes until the return to the origin. Then we obtain a word w_1 . Similarly we obtain words w_2, \dots, w_k from second component, \dots , k -th component. On the alphabet $\mathcal{A} = \{A_1, \dots, A_n\}$. Let t_i^1 (respectively, t_i^2) be the tangent vector to C at the double point labeled A_i appearing at the first (respectively, second) passage through this point. Set $|A_i| = a$, if the pair (t_i^1, t_i^2) is positively oriented, and $|A_i| = b$ otherwise. Then we obtain a required nanophrase $P(C) := (\mathcal{A}, (w_1 | \dots | w_k))$.

By the above theorem if we classify the homotopy classes of nanophrases, then we obtain the classification of ordered, pointed multi-component curves under the stable equivalence as a corollary.

5. Classification of Nanophrases.

In this section, we give the homotopy classification of nanophrases with less than or equal to 4 letters under the assumption that a homotopy data S is diagonal. In the remaining part of the paper we always assume that homotopy data is diagonal. Note that this assumption is not obstruct the our purpose.

5.1. The case of nanophrases of length 1.

The case of nanophrases of length 1 (in other words the case of nanowords), Turaev gave the following classification theorem.

Theorem 5.1. (Turaev [5]). *Let w be a nanoword of length 4 over α . Then w is either homotopic to the empty nanoword or isomorphic to the nanoword $w_{a,b} := (\mathcal{A} = \{A, B\}, ABAB)$ where $|A| = a, |B| = b \in \alpha$ with $a \neq \tau(b)$. Moreover for $a \neq \tau(b)$, the nanoword $w_{a,b}$ is non-contractible and two nanowords $w_{a,b}$ and $w_{a',b'}$ are homotopic if and only if $a = a'$ and $b = b'$.*

Remark 5.2. In the paper [5], Turaev gave the classification of nanowords of length 6. But in this paper we do not use this result. Classification problem of nanowords of length more than or equal to 8 is still open(See [7]) .

5.2. The case of nanophrases of length 2.

First we prepare following notations: $P_a := (A|A)$, $P_{a,b}^{4,0} := (ABAB|\emptyset)$, $P_{a,b}^{3,1} := (ABA|B)$, $P_{a,b}^{2,2I} := (AB|AB)$, $P_{a,b}^{2,2II} := (AB|BA)$, $P_{a,b}^{1,3} := (A|BAB)$ and $P_{a,b}^{0,4} := (\emptyset|ABAB)$ with $|A| = a$, $|B| = b \in \alpha$. If $a = \tau(b)$, then $P_{a,b}^{4,0} \simeq P_{a,b}^{2,2I} \simeq P_{a,b}^{2,2II} \simeq P_{a,b}^{0,4} \simeq (\emptyset|\emptyset)$. So in this paper, if we write $P_{a,b}^{4,0}$, $P_{a,b}^{2,2I}$, $P_{a,b}^{2,2II}$, $P_{a,b}^{0,4}$ then we always assume that $a \neq \tau(b)$.

In [1], the author gave the classification of nanophrases of length 2 with less than or equal to 4 letters.

Theorem 5.3. *Let P be a nanophrase of length 2 with 2 letters. Then P is not homotopic to $(\emptyset|\emptyset)$ if and only if P is isomorphic to P_a . Moreover P_a and $P_{a'}$ are homotopic if and only if $a = a'$.*

Theorem 5.4. *Let P be a nanophrase of length 2 with 4 letters, then P is homotopic to $(\emptyset|\emptyset)$ or homotopic to nanophrases of length 2 with 2 letters or isomorphic to one of the following nanophrases: $P_{a,b}^{4,0}$, $P_{a,b}^{3,1}$, $P_{a,b}^{2,2I}$, $P_{a,b}^{2,2II}$, $P_{a,b}^{1,3}$, $P_{a,b}^{0,4}$. For $(i, j) \in \{(4, 0), (3, 1), (2, 2I), (2, 2II), (1, 3), (0, 4)\}$ and any $a, b \in \alpha$, the nanophrase $P_{a,b}^{i,j}$ is neither homotopic to $(\emptyset|\emptyset)$ nor homotopic to nanophrases of length 2 with 2 letters. The nanophrases $P_{a,b}^{i,j}$ and $P_{a',b'}^{i,j}$ are homotopic if and only if $a = a'$ and $b = b'$. For $(i, j) \neq (i', j')$, the nanophrases $P_{a,b}^{i,j}$ and $P_{a',b'}^{i',j'}$ are not homotopic for any $a, b, a', b' \in \alpha$.*

In this paper, we give the classification of nanophrases of length more than or equal to 3 with 4 letters.

5.3. Homotopy invariants of nanophrases.

In this subsection we introduce some invariants of nanophrases over α (Some of them are defined in [1]).

Let Π be the group which is defined as follows:

$$\Pi := (\{z_a\}_{a \in \alpha} | z_a z_{\tau(a)} = 1 \text{ for all } a \in \alpha).$$

Definition 5.5. (cf. [1]). Let $P = (\mathcal{A}, (w_1|w_2|\dots|w_k))$ be a nanophrase of length k over α and n_i the length of nanoword w_i . Set $n = \sum_{1 \leq i \leq k} n_i$. Then we define n elements $\gamma_1^i, \gamma_2^i, \dots$ and $\gamma_{n_i}^i$ ($i \in \{1, 2, \dots, k\}$) of Π by $\gamma_i^j := z_{|w_j(i)|}$ if $w_j(i) \neq w_l(m)$ for all $l < j$ and for all $m < i$ when $l = j$. Otherwise $\gamma_i^j := z_{\tau(|w_j(i)|)}$. Then we define $\gamma(P) \in \Pi \times \dots \times \Pi$ (k times) by

$$\gamma(P) := (\gamma_1^1 \gamma_2^1 \cdots \gamma_{n_1}^1, \gamma_1^2 \gamma_2^2 \cdots \gamma_{n_2}^2, \dots, \gamma_1^k \gamma_2^k \cdots \gamma_{n_k}^k).$$

Then we obtain following proposition.

Proposition 5.6. *γ is a homotopy invariant of nanophrases.*

We define a invariant of nanophrases T .

First we prepare some notations. Since the set α is a finite set, we obtain following orbit decomposition of the $\tau : \alpha = \{\widetilde{a_{i_1}}, \widetilde{a_{i_2}}, \dots, \widetilde{a_{i_l}}, \widetilde{a_{i_{l+1}}}, \dots, \widetilde{a_{i_{l+m}}}\}$, where $\widetilde{a_{i_j}} := \{a_{i_j}, \tau(a_{i_j})\}$ such that $Card(\widetilde{a_{i_j}}) = 2$ for all $j \in \{1, \dots, l\}$ and $Card(\widetilde{a_{i_j}}) = 1$ for all $j \in \{l+1, \dots, l+m\}$ (we fix a complete representative system $\{a_{i_1}, a_{i_2}, \dots, a_{i_l}, a_{i_{l+1}}, \dots, a_{i_{l+m}}\}$ which satisfy the above condition). Let \mathcal{A} be a α -alphabet. For $A \in \mathcal{A}$ we define $\varepsilon(A) \in \{\pm 1\}$ by

$$\varepsilon(A) := \begin{cases} 1 & (\text{if } |A| = a_{i_j} \text{ for some } j \in \{1, \dots, l+m\}), \\ -1 & (\text{if } |A| = \tau(a_{i_j}) \text{ for some } j \in \{1, \dots, l\}). \end{cases}$$

Let $P = (\mathcal{A}, (w_1 | \dots | w_k))$ be a nanophrase over α and $A, B \in \mathcal{A}$. Then we define $\sigma_P(A, B) \in \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{l\text{-times}} \times \underbrace{\mathbb{Z}/2\mathbb{Z} \times \dots \times \mathbb{Z}/2\mathbb{Z}}_{m\text{-times}}$ as follows: If A and B form $\dots A \dots B \dots A \dots B \dots$ in P and $|B| = a_{i_j}$ for some $j \in \{1, \dots, l+m\}$, or $\dots B \dots A \dots B \dots A \dots$ in P and $|B| = \tau(a_{i_j})$ for some $j \in \{1, \dots, l+m\}$, then $\sigma_P(A, B) := (0, \dots, 0, \overset{j}{1}, 0, \dots, 0)$. If $\dots A \dots B \dots A \dots B \dots$ in P and $|B| = \tau(a_{i_j})$, or $\dots B \dots A \dots B \dots A \dots$ in P and $|B| = a_{i_j}$, then $\sigma_P(A, B) := (0, \dots, 0, \overset{j}{-1}, 0, \dots, 0)$. Otherwise $\sigma_P(A, B) := (0, \dots, 0)$. Under the above preparation, we define the invariant T as follows.

Definition 5.7. Let $P = (\mathcal{A}, (w_1 | w_2 | \dots | w_k))$ be a nanophrase of length k over α . For $A \in \mathcal{A}$ such that there exist $i \in \{1, 2, \dots, k\}$ with $Card(w_i^{-1}(A)) = 2$, we define $T_P(A) \in \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{l\text{-times}} \times \underbrace{\mathbb{Z}/2\mathbb{Z} \times \dots \times \mathbb{Z}/2\mathbb{Z}}_{m\text{-times}}$ by

$$T_P(A) := \varepsilon(A) \sum_{B \in \mathcal{A}} \sigma_P(A, B),$$

and $T_P(w_i) \in \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{l\text{-times}} \times \underbrace{\mathbb{Z}/2\mathbb{Z} \times \dots \times \mathbb{Z}/2\mathbb{Z}}_{m\text{-times}}$ by

$$T_P(w_i) := \sum_{A \in \mathcal{A}, Card(w_i^{-1}(A))=2} T_P(A).$$

Then we define $T(P) \in \underbrace{(\mathbb{Z} \times \dots \times \mathbb{Z})}_{l\text{-times}} \times \underbrace{(\mathbb{Z}/2\mathbb{Z} \times \dots \times \mathbb{Z}/2\mathbb{Z})}_{m\text{-times}}^k$ by

$$T(P) := (T_P(w_1), T_P(w_2), \dots, T_P(w_k)).$$

Proposition 5.8. T is a invariant of nanophrases over α .

Remark 5.9. This invariant T is the generalization of invariants T of nanophrases over α_0 and a 1-element set defined in [1]. If we use the invariant T defined in this paper, then we can classify nanophrases of length 2 with 4 letters without the Lemma 4.2 in [1]. Proposition is proved similarly as in [1].

Next we define another new invariant. Let π be the group which is defined as follows:

$$\pi := (a \in \alpha | a\tau(a) = 1, ab = ba \text{ for all } a, b \in \alpha) \simeq \Pi / [\Pi, \Pi].$$

Let $P = (\mathcal{A}, (w_1|w_2|\cdots|w_k))$ be a nanophrase of length k over α . We define $(w_i, w_j)_P \in \pi$ for $i < j$ by

$$(w_i, w_j)_P := \prod_{A \in w_i(\mathcal{A}) \cap w_j(\mathcal{A})} |A|.$$

Proposition 5.10. *If nanophrases over α , P_1 and P_2 are homotopic, then $(w_i, w_j)_{P_1} = (w_i, w_j)_{P_2}$.*

So we obtain an invariant $((w_i, w_j)_P)_{i < j} \in \pi \times \cdots \times \pi$ ($(k-1)$ -times).

5.4. The case of nanophrases of length more than or equal to 3.

Now using the invariants prepared in the last section and some lemmas, we classify the nanophrases of length more than or equal to 3 with less than or equal to 4 letters. First recall the following lemmas from [1].

Lemma 5.11. *Let $P_1 = (w_1|w_2|\cdots|w_k)$ and $P_2 = (v_1|v_2|\cdots|v_k)$ be nanophrases of length k over α . If P_1 and P_2 are homotopic as nanophrases, then w_i and v_i are homotopic as étale words for all $i \in \{1, 2, \dots, k\}$.*

Lemma 5.12. *Let $P_1 = (w_1|\cdots|w_k)$ and $P_2 = (v_1|\cdots|v_k)$ be nanophrases of length k over α . If P_1 and P_2 are homotopic, then the length of w_i is equal to length of v_i modulo 2 for all $i \in \{1, 2, \dots, k\}$.*

A following lemma is checked easily by definition of homotopy of nanophrases.

Lemma 5.13. *Let $P_1 = (w_1|\cdots|w_k)$ and $P_2 = (v_1|\cdots|v_k)$ be nanophrases of length k over α . If P_1 and P_2 are homotopic, then $(w_1|\cdots|w_l w_{l+1}|\cdots|w_k)$ and $(v_1|\cdots|v_l v_{l+1}|\cdots|v_k)$ are homotopic as nanophrases of length $k-1$ over α for all $l \in \{1, \dots, k-1\}$.*

Now we give the classification theorem of nanophrases with 2 letters. Set $P_a^{1,1;l_1,l_2} := (\emptyset|\cdots|\emptyset| \overset{l_1}{A} |\emptyset|\cdots|\emptyset| \overset{l_2}{A} |\emptyset|\cdots|\emptyset)$ with $|A| = a$ for $1 \leq l_1 < l_2 \leq k$

Theorem 5.14. *Let P be a nanophrase of length k with 2 letters. Then P is either homotopic to $(\emptyset|\cdots|\emptyset)$ or isomorphic to $P_a^{1,1;l_1,l_2}$ for some $l_1, l_2 \in \{1, \dots, k\}$, $a \in \alpha$. Moreover $P_a^{1,1;l_1,l_2}$ and $P_{a'}^{1,1;l'_1,l'_2}$ are homotopic if and only if $l_1 = l'_1$, $l_2 = l'_2$ and $a = a'$.*

Proof. Compare $((w_i, w_j)_{P_a^{1,1;l_1,l_2}})_{i < j}$ with $((w_i, w_j)_{P_{a'}^{1,1;l'_1,l'_2}})_{i < j}$.

□

To describe the classification theorem of nanophrases with 4 letters, we prepare following notations.

$$P_{a,b}^{4;l} := (\emptyset | \cdots | \emptyset | \overset{l}{ABAB} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{3,1;l_1,l_2} := (\emptyset | \cdots | \emptyset | \overset{l_1}{ABA} | \emptyset | \cdots | \emptyset | \overset{l_2}{B} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{2,2I;l_1,l_2} := (\emptyset | \cdots | \emptyset | \overset{l_1}{AB} | \emptyset | \cdots | \emptyset | \overset{l_2}{AB} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{2,2II;l_1,l_2} := (\emptyset | \cdots | \emptyset | \overset{l_1}{AB} | \emptyset | \cdots | \emptyset | \overset{l_2}{BA} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{3,1;l_1,l_2} := (\emptyset | \cdots | \emptyset | \overset{l_1}{A} | \emptyset | \cdots | \emptyset | \overset{l_2}{BAB} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{2,1,1I;l_1,l_2,l_3} := (\emptyset | \cdots | \emptyset | \overset{l_1}{AB} | \emptyset | \cdots | \emptyset | \overset{l_2}{A} | \emptyset | \cdots | \emptyset | \overset{l_3}{B} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{2,1,1II;l_1,l_2,l_3} := (\emptyset | \cdots | \emptyset | \overset{l_1}{BA} | \emptyset | \cdots | \emptyset | \overset{l_2}{A} | \emptyset | \cdots | \emptyset | \overset{l_3}{B} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{1,2,1I;l_1,l_2,l_3} := (\emptyset | \cdots | \emptyset | \overset{l_1}{A} | \emptyset | \cdots | \emptyset | \overset{l_2}{AB} | \emptyset | \cdots | \emptyset | \overset{l_3}{B} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{1,2,1II;l_1,l_2,l_3} := (\emptyset | \cdots | \emptyset | \overset{l_1}{A} | \emptyset | \cdots | \emptyset | \overset{l_2}{BA} | \emptyset | \cdots | \emptyset | \overset{l_3}{B} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{1,1,2I;l_1,l_2,l_3} := (\emptyset | \cdots | \emptyset | \overset{l_1}{A} | \emptyset | \cdots | \emptyset | \overset{l_2}{B} | \emptyset | \cdots | \emptyset | \overset{l_3}{AB} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{1,1,2II;l_1,l_2,l_3} := (\emptyset | \cdots | \emptyset | \overset{l_1}{A} | \emptyset | \cdots | \emptyset | \overset{l_2}{B} | \emptyset | \cdots | \emptyset | \overset{l_3}{BA} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{1,1,1,1I;l_1,l_2,l_3,l_4} := (\emptyset | \cdots | \emptyset | \overset{l_1}{A} | \emptyset | \cdots | \emptyset | \overset{l_2}{A} | \emptyset | \cdots | \emptyset | \overset{l_3}{B} | \emptyset | \cdots | \emptyset | \overset{l_4}{B} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{1,1,1,1II;l_1,l_2,l_3,l_4} := (\emptyset | \cdots | \emptyset | \overset{l_1}{A} | \emptyset | \cdots | \emptyset | \overset{l_2}{B} | \emptyset | \cdots | \emptyset | \overset{l_3}{A} | \emptyset | \cdots | \emptyset | \overset{l_4}{B} | \emptyset | \cdots | \emptyset),$$

$$P_{a,b}^{1,1,1,1III;l_1,l_2,l_3,l_4} := (\emptyset | \cdots | \emptyset | \overset{l_1}{A} | \emptyset | \cdots | \emptyset | \overset{l_2}{B} | \emptyset | \cdots | \emptyset | \overset{l_3}{B} | \emptyset | \cdots | \emptyset | \overset{l_4}{A} | \emptyset | \cdots | \emptyset),$$

with $|A| = a$, $|B| = b$. If $a = \tau(b)$, then nanophrases $P_{a,b}^{4;l}$, $P_{a,b}^{2,2I;l_1,l_2}$ and $P_{a,b}^{2,2II;l_1,l_2}$ are homotopic to $(\emptyset | \cdots | \emptyset)$. So when we write $P_{a,b}^{4;l}$, $P_{a,b}^{2,2I;l_1,l_2}$, $P_{a,b}^{2,2II;l_1,l_2}$ we always assume that $a \neq \tau(b)$.

Under the above notations the classification of nanophrases with 4 letter is described as follows.

Theorem 5.15. *Let P be a nanophrase of length k with 4 letters. Then P is either homotopic to nanophrase with less than or equal to 2 letters or isomorphic to $P_{a,b}^{X;Y}$ for some $X \in \{4, (3, 1), \dots, (1, 1, 1, 1III)\}$, $Y \in \{1, \dots, k, (1, 2), \dots, (k-3, k-2, k-1, k)\}$. Moreover $P_{a,b}^{X;Y}$ and $P_{a',b'}^{X';Y'}$ are homotopic if and only if $X = X'$, $Y = Y'$, $a = a'$ and $b = b'$.*

Proof. Using the invariants, lemmas and classification theorems which introduced in the above sections, we can proof this theorem. So we omit the proof of this theorem and we only show some examples in the remaining part of this section. □

Example 5.16. The nanophrases $P_{a,b}^{1,1,1,1I;l_1,l_2,l_3,l_4}$ and $P_{a',b'}^{1,1,1,1II;l'_1,l'_2,l'_3,l'_4}$ are not homotopic for all l_i, l'_i ($i \in \{1, 2, 3, 4\}$) and $a, b, a', b' \in \alpha$. Indeed if we assume $P_{a,b}^{1,1,1,1I;l_1,l_2,l_3,l_4}$ and $P_{a',b'}^{1,1,1,1II;l'_1,l'_2,l'_3,l'_4}$ are homotopic, then $l_i = l'_i$ for all $i \in \{1, 2, 3, 4\}$ since $((w_i, w_j)_{P_{a,b}^{1,1,1,1I;l_1,l_2,l_3,l_4}})_{i < j} = ((w_i, w_j)_{P_{a',b'}^{1,1,1,1II;l'_1,l'_2,l'_3,l'_4}})_{i < j}$. So $(A|BAB)$ must be homotopic to $(A|ABB)$ ($\simeq (A|A)$) by Lemma 5.13. But this contradicts Theorem 5.4.

Example 5.17. The nanophrases $P_{a,b}^{2,1,1I;l_1,l_2,l_3}$ and $P_{a',b'}^{1,2,1I;l'_1,l'_2,l'_3}$ are not homotopic for all l_i, l'_i ($i \in \{1, 2, 3\}$) and $a, b, a', b' \in \alpha$. Indeed if we assume they are homotopic, then $l_2 = l'_1$ and $l_3 = l'_3$ by Lemma 5.12. So $l_2 < l'_2 < l_3$. This implies $(w_{l'_2}, w_{l'_3})_{P_{a,b}^{2,1,1I;l_1,l_2,l_3}} = 1 \in \pi$ and $(w_{l'_2}, w_{l'_3})_{P_{a',b'}^{1,2,1I;l'_1,l'_2,l'_3}} = b \in \pi$. This contradicts Proposition 5.10.

6. Proof of The Theorem 2.2.

To complete the proof of the Theorem 2.2, we prepare a following lemma.

Lemma 6.1. *The nanophrases over α , $(A|A)$, $(AB|AB)$ with $|A| \neq \tau(|B|)$, $(AB|BA)$ with $|A| \neq \tau(|B|)$, $(ABA|B)$, $(A|BAB)$, $(AB|A|B)$, $(BA|A|B)$, $(A|AB|B)$, $(A|BA|B)$, $(A|B|AB)$, $(A|B|BA)$, $(A|A|B|B)$, $(A|B|A|B)$ and $(A|B|B|A)$ are not homotopic to nanophrases over α which have the empty words in its components.*

Proof. This lemma follows from Proposition 5.10 and Lemma 5.12. \square

Now the Theorem 2.2 immediately follows from the Theorem 5.15 and Lemma 6.1. It is sufficient to apply the above theorems to the case $\alpha = \alpha_0$ with involution $\tau : \alpha_0 \rightarrow \alpha_0$ permuting a, b .

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