A uniqueness theorem for holomorphic functions of exponential type

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§ 1. Introduction.

In this paper we treat a uniqueness theorem for holomorphic functions of exponential type on a half plane from the point of view of the theory of analytic functionals with non-compact carrier.

Avanissian and Gay [1] proved among others the following Theorem 1 using the theory of analytic functionals of Martineau [4].

**Theorem 1.** Let \( F(\zeta) \) be an entire function of type \(<\pi\). If we have \( F(-n)=0 \) for every \( n=1, 2, 3, \ldots \), then the entire function \( F(\zeta) \) vanishes identically.

Theorem 1 is a corollary to Carlson's theorem (see Boas [2] p. 153).

**Theorem 2.** (Carlson) Let \( F(\zeta) \) be a holomorphic function on the half plane \( \{\zeta=\xi+i\eta; \xi=\text{Re} \zeta<0\} \). Suppose that there exist real numbers \( a, k \) with \( 0\leq k<\pi \) and \( C\geq 0 \) such that

\[
|F(\zeta)| \leq C \exp(a\xi+k|\eta|) \quad \text{for Re} \zeta<0.
\]

If we have \( F(-n)=0 \) for \( n=1, 2, 3, \ldots \), then the function \( F(\zeta) \) vanishes identically.

We will prove Carlson's theorem by means of the theory of analytic functionals with non-compact carrier, which was introduced by the first named author [5] in connection with the theory of ultra-distributions of exponential growth of Sebastiañê-Silva [6], namely Fourier ultra-hyperfunction.

Following the sections we outline the results. In § 2 we define the fundamental space \( \mathcal{Q}(L; k') \), the element of which is a holomorphic function in a tubular neighborhood \( L \) of the closed half strip \( L=[a, \infty)+i[k, k_2] \). A continuous linear functional on the space \( \mathcal{Q}(L; k') \) is, by definition, an analytic functional with carrier in \( L \) and of (exponential) type \( \leq k' \). The image of the Laplace transformation of \( \mathcal{Q}'(L; k') \) is characterized in Theorem 3. As in Avanissian-Gay [1] we define in § 3 the transformation \( G_\mu \).
of an analytic functional $\mu \in \mathscr{Q}'(L; k')$ by the formula: $G_\mu(\zeta) = \langle \mu_z, (1 - \zeta e^z)^{-1} \rangle$ and call it the Avanissian-Gay transformation. The Avanissian-Gay transformation is defined for $0 \leq k' < 1$ and $G_\mu$ is a holomorphic function on the complement of the set $\exp(-L)$, vanishes at the infinity and satisfies a certain growth condition at the origin. We show in § 4 the Avanissian-Gay transformation is injective if the width of the half strip $L$ is less than $2\pi$, proving the inversion formula (Theorem 4). As a corollary, we have the above mentioned Carlson theorem. In the last section, we determine the image of the Avanissian-Gay transformation (Theorem 6).

§ 2. Analytic functionals with half strip carrier and their Laplace transformation.

In this section we recall the definition of analytic functionals with non-compact carrier and characterize their Laplace transformation.

We begin with some notations. In the sequel, $L$ denotes the closed half strip in the complex number plane $C$: $L = A + iK$, $A = [a, \infty)$, $K = [k_1, k_2]$ and $i = \sqrt{-1}$, namely, $L = \{z = x + iy \in C; x \geq a, k_1 \leq y \leq k_2\}$. By $L$, we denote the $\varepsilon$-neighborhood of $L$: $L_\varepsilon = L + [-\varepsilon, \varepsilon] + i[-\varepsilon, \varepsilon]$.

For $\varepsilon > 0$, $\varepsilon' > 0$ and $0 \leq k' < \infty$, we define the function space $\mathscr{Q}_b(L_\varepsilon; k' + \varepsilon')$ as follows:

$$\mathscr{Q}_b(L_\varepsilon; k' + \varepsilon') = \{f \in \mathscr{Q}(\text{int } L_\varepsilon) \cap \mathscr{C}(L_\varepsilon); \sup_{z \in L_\varepsilon} |f(z)| \exp((k' + \varepsilon')x) < \infty\},$$

where $\mathscr{Q}(\text{int } L_\varepsilon)$ denotes the space of holomorphic functions on the interior int $L_\varepsilon$ of $L_\varepsilon$ and $\mathscr{C}(L_\varepsilon)$ denotes the space of continuous functions on $L_\varepsilon$. Endowed with the norm

$$\sup_{z \in L_\varepsilon} |f(z)| \exp((k' + \varepsilon')x),$$

the space $\mathscr{Q}_b(L_\varepsilon; k' + \varepsilon')$ becomes a Banach space. If $\varepsilon_1 < \varepsilon$ and $\varepsilon_1' < \varepsilon'$, the restriction mapping

$$\mathscr{Q}_b(L_\varepsilon; k' + \varepsilon') \rightarrow \mathscr{Q}_b(L_{\varepsilon_1}; k' + \varepsilon_1')$$

is defined and a continuous linear injection. Following the mappings (2.1), we from the locally convex inductive limit:

$$\mathscr{Q}(L; k') = \lim_{\varepsilon_1 \rightarrow 0, \varepsilon_1' \rightarrow 0} \mathscr{Q}_b(L_{\varepsilon_1}; k' + \varepsilon_1').$$
If we put $X_n = \mathcal{O}(L_1/n ; k' + 1/n)$, then with mappings (2.1) we have a sequence of Banach spaces with compact injective mappings $X_j \rightarrow X_{j+1}$:

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \cdots$$

As we have clearly $\mathcal{O}(L ; k') = \lim \text{ind} X_j$, the locally convex space $\mathcal{O}(L ; k')$ is a DFS space (namely the dual space of a Fréchet-Schwartz space). We denote the dual space of $\mathcal{O}(L ; k')$ by $\mathcal{O}'(L ; k')$, an element of which is, by definition, an analytic functional with carrier in $L$ and of type $\leq k'$.

We denote by $h_L(\zeta)$ the supporting function of the half strip $L$:

$$h_L(\zeta) = \sup_{z \in L} \Re \zeta z = \begin{cases} a\xi - k_1 \eta & \text{if } \xi \leq 0 \text{ and } \eta > 0 \\ a\xi - k_2 \eta & \text{if } \xi \leq 0 \text{ and } \eta \leq 0 \end{cases}$$

Remark $h_L(\zeta) = \infty$ if $\Re \zeta > 0$. For $k' \geq 0$ we denote by $\text{Exp}((\infty, -k') + i\mathbb{R} ; L)$ the space of all holomorphic functions $\varphi$ on the open half plane $(-\infty, -k') + i\mathbb{R}$ for which

$$\sup_{\Re \zeta \leq -k' - \epsilon'} |\varphi(\zeta)| \exp(-h_L(\zeta) - \epsilon |\zeta|) < \infty \quad (2.2)$$

for every $\epsilon > 0$ and $\epsilon' > 0$. An element of the space $\text{Exp}((\infty, -k') + i\mathbb{R} ; L)$ is said to be a holomorphic function of exponential type in $L$. Endowed with the norms (2.2), the space $\text{Exp}((\infty, -k') + i\mathbb{R} ; L)$ is an FS space (namely a Fréchet-Schwartz space). (As for the DFS spaces and FS spaces, we refer the reader to Komatsu [3].)

We define the Laplace transformation of an analytic functional $\mu$ with carrier in $L$ and of type $\leq k'$ as follows:

$$\tilde{\mu}(\zeta) = \langle \mu_z, \exp(z \zeta) \rangle. \quad (2.3)$$

Remark that $\tilde{\mu}(\zeta)$ is defined for $\zeta$ of the half plane $\{ \zeta ; \Re \zeta < -k' \}$. The next Paley-Wiener type theorem characterizes the Laplace transformation of the analytic functionals with carrier in $L$ and of type $\leq k'$.

**Theorem 3.** (Morimoto [5]) The Laplace transformation (2.3) is a linear topological isomorphism of the space $\mathcal{O}'(L ; k')$ onto the space $\text{Exp}((\infty, -k') + i\mathbb{R} ; L)$.

We have the following density theorem.

**Proposition 1.** For $h \in \mathcal{O}(L ; k')$, we have

$$\lim_{\delta \downarrow 0} h(z) \exp(-\delta z^2) = h(z)$$

in the topology of $\mathcal{O}(L ; k')$. 

PROOF. By the definition of the space $\mathscr{B}(L; k')$, there exist $\varepsilon > 0$ and $\varepsilon' > 0$ such that $h \in \mathscr{B}_{b}(L_{2*}; k' + 2\varepsilon')$.
In particular, we have
\[
\sup_{z \in L_{2*}} |h(z)| \exp((k' + 2\varepsilon') x) = M > \infty.
\]
Then we have
\[
\sup_{z \in L_{2*}} |h(z)| \left| 1 - \exp(-\delta z^{2}) \right| \exp((k' + \varepsilon') x) \leq M \sup_{z \in L_{2*}} \left| 1 - \exp(-\delta z^{2}) \right| \exp(-\varepsilon' x).
\]
As the righthand side tends to 0 as $\delta \downarrow 0$, $h(z) \exp(-\delta z^{2})$ tends to $h(z)$ in the topology of $\mathscr{B}_{b}(L_{2*}; k' + \varepsilon')$ as $\delta \downarrow 0$.

q.e.d.

COROLLARY. If $k_{1}' > k'$, then the space $\mathscr{B}(L; k_{1}')$ is a dense subspace of the space $\mathscr{B}(L; k')$. The dual space $\mathscr{B}'(L; k')$ can be considered as a subspace of $\mathscr{B}'(L; k_{1}')$.

PROOF. If $\delta > 0$ and $h \in \mathscr{B}(L; k')$, then $h(z) \exp(-\delta z^{2})$ belongs to $\mathscr{B}(L; k_{1}')$. The second assertion results from the Hahn-Banach theorem.

q.e.d.

§ 3. The Avanissian-Gay transformation.

If $0 \leq k' < 1$ and $\zeta \notin \exp(-L)$, then the function of $z$, $(1 - \zeta e^{z})^{-1}$ belongs to the space $\mathscr{B}(L; k')$. Following Avanissian-Gay [1] we define the transformation $G_{\mu}$ of an analytic functional $\mu \in \mathscr{B}'(L; k')$ as follows:
\[
G_{\mu}(\zeta) = \langle \mu_{z}, (1 - \zeta e^{z})^{-1} \rangle.
\]

$G_{\mu}(\zeta)$ is a function of $\zeta \notin \exp(-L)$ and has the following properties.

PROPOSITION 2. Suppose $\mu \in \mathscr{B}'(L; k')$, $0 \leq k' < 1$.

(i) $G_{\mu}(\zeta)$ is a holomorphic function on the complement of $\exp(-L)$.

(ii) The following Laurent expansion is valid:
\[
G_{\mu}(\zeta) = -\sum_{n=1}^{\infty} \zeta^{-n} \mu(-n)
\]
for $|\zeta| > e^{-a}$.

(iii) $\lim_{|\zeta| \to \infty} |G_{\mu}(\zeta)| = 0$.

PROOF. (i) can be derived from Morera’s theorem.

(ii) We have the following expansion:
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\[(1 - \zeta e^z)^{-1} = -\sum_{n=1}^{\infty} \zeta^{-n} \exp(-nz) .\]

By elementary calculations, we can show that this series converges uniformly with respect to \(\zeta\) with \(|\zeta| \geq e^{-a+}\), \(\epsilon > 0\), in the topology of \(\mathcal{O}(L; k')\). Hence we have

\[G_\mu(\zeta) = -\sum_{n=1}^{\infty} \zeta^{-n} \langle \mu, \exp(-nz) \rangle = -\sum_{n=1}^{\infty} \zeta^{-n} \tilde{\mu}(-n) .\]

(iii) is a trivial consequence of (ii).

If the half strip \(L\) has the width \(k_2 - k_1 < 2\pi\), the complement of the set \(\exp(-L)\) contains the open angular domain

\[A(-k_1, -k_2 + 2\pi) = \{\zeta \in \mathbb{C} \setminus (0) ; \ -k_1 < \arg \zeta < -k_2 + 2\pi \} .\]

We shall investigate, for further purposes, the behavior of the function \(G_\mu(\zeta)\) in this angular domain.

**Proposition 3.** Suppose the half strip \(L = [a, \infty) + i [k_1, k_2]\) has the width \(k_2 - k_1 < 2\pi\) and \(0 \leq k' < 1\). If \(\mu \in \mathcal{Q}'(L; k')\), then, for any \(\epsilon\) with \(0 < 2\epsilon < 2\pi + k_1 - k_2\) and any \(\epsilon'\) with \(0 < \epsilon' < 1 - k'\), there exists a constant \(C \geq 0\) such that

\[|G_\mu(\zeta)| \leq C|\zeta|^{-k' - \epsilon'}\]

in the closed angular domain

\[\bar{A}(-k_1 + \epsilon, -k_2 + 2\pi - \epsilon) = \{\zeta \in \mathbb{C} \setminus (0) ; \ -k_1 + \epsilon \leq \arg \zeta \leq -k_2 + 2\pi - \epsilon \} .\]

**Proof.** By the continuity of \(\mu \in \mathcal{Q}'(L; k')\), there exists a constant \(C' \geq 0\) such that, for \(\zeta \in \exp(-L_{\epsilon/2})\), we have
\[ \left| G_\mu(\zeta) \right| = \langle \mu_z, (1-\zeta e^z)^{-1} \rangle \]
\[ \leq C' \sup_{z \in L_{1/2}} \left|1 - \zeta e^z\right|^{-1} \exp \left((k' + \epsilon') x\right) \]
\[ = C' \sup_{z \in L_{1/2}} \left|e^{-z} - \zeta\right|^{-1} \exp \left((k' + \epsilon' - 1) x\right) \]
\[ \leq C' \sup_{z \in L_{1/2}} \left|e^{-z} - \zeta\right|^{-k' - \epsilon'} \sup_{z \in L_{1/2}} \left|e^{-z} - \zeta\right|^{k' + \epsilon' - 1} \exp \left((k' + \epsilon' - 1) x\right) \].

Therefore with another constant \( C'' \geq 0 \), we have
\[ \left| G_\mu(\zeta) \right| \leq C \]
\[ \text{dist} \left(\zeta, \exp(-L_{1/2})\right)^{-k' - \epsilon'} \sup_{z \in L_{1/2}} \left|1 - \zeta e^z\right|^{k' + \epsilon' - 1} \]
for \( \zeta \in \exp(-L_{1/2}) \). On the other hand, as the set \( \exp(-L_{1/2}) \) is contained in the closed angular domain \( \bar{A}(-k_2 - \epsilon/2, -k_1 + \epsilon/2) \), we have
\[ \text{dist} \left(\zeta, \exp(-L_{1/2})\right) \geq |\zeta| \sin(\epsilon/2) \]
for \( \zeta \in \bar{A}(-k_1 + \epsilon, -k_2 + 2\pi - \epsilon) \).

If \( z \in L_{1/2} \) and \( \zeta \in \bar{A}(-k_1 + \epsilon, -k_2 + 2\pi - \epsilon) \), then
\[ |\arg \zeta e^z| \geq \epsilon/2 \mod 2\pi \]
Therefore we have
\[ \inf_{z \in L_{1/2}} |1 - \zeta e^z| \geq \sin(\epsilon/2) \]
for \( \zeta \in \bar{A}(-k_1 + \epsilon, -k_2 + 2\pi - \epsilon) \).

As \( -k' - \epsilon < 0 \) and \( k' + \epsilon' - 1 < 0 \) by the choice of \( \epsilon' \), putting \( C = C'(\sin(\epsilon/2))^{-1} \)
we obtain the desired estimate of \( G_\mu(\zeta) \). q. e. d.

Suppose always \( L = [a, \infty) + i[k_1, k_2] \) has the width \( k_2 - k_1 < 2\pi \) and \( 0 \leq k' < 1 \). We denote by \( \mathcal{O}_0(\mathbb{C} \setminus \exp(-L); k') \) the space of all holomorphic functions \( \phi \) on the domain \( \mathbb{C} \setminus \exp(-L) \) which satisfy following two conditions:

(1) \[ |\phi(\zeta)| \to 0 \]
(2) \[ \sup \{ |\phi(\zeta)|^{k' + \epsilon'}; \zeta \in \bar{A}(-k_1 + \epsilon, -k_2 + 2\pi - \epsilon) \} < \infty \]
for any \( \epsilon \) with \( 0 < 2\epsilon < 2\pi - k_1 - k_2 \) and any \( \epsilon' \) with \( 0 < \epsilon' < 1 - k' \). The space \( \mathcal{O}_0(\mathbb{C} \setminus \exp(-L); k') \) equipped with the seminorms \( \sup \{ |\phi(\zeta)|; |\zeta| \geq e^{-a+}\epsilon \} \) and
\[ \sup \{ |\phi(\zeta)|^{k' + \epsilon'}; \zeta \in \bar{A}(-k_1 + \epsilon, -k_2 + 2\pi - \epsilon) \}, \]
is clearly a Fréchet (Schwartz) space. As a corollary to Propositions 2 and 3, we have the following proposition.

Proposition 4. Suppose the width of \( L \) is less than \( 2\pi \) and \( 0 \leq k \leq 1 \).
Then the Avanissian-Gay transformation \( G \) is a continuous linear mapping of \( \mathcal{O}'(L; k') \) into \( \mathcal{O}_0(\mathbb{C} \setminus \exp(-L); k') \).
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PROOF. The continuity results from the boundedness of the set \((1-\zeta e^{z})^{-1}; |\zeta| \geq e^{-a+\epsilon}\) and the set \(\zeta^{e'+\epsilon'}(1-\zeta e^{z})^{-1}; \zeta \in \tilde{\mathbb{A}}(-k_{1}+\epsilon, -k_{2}+2\pi-\epsilon)\) in the space \(\mathscr{O}(L; l')\).

§ 4. Inversion formula for \(G_{\rho}(\zeta)\).

In the sequel we suppose the half strip \(L\) has the form

\[ L = [a, \infty) + i[k_{1}, k_{2}], \quad k_{2} - k_{1} < 2\pi \]

and \(0 \leq k' < 1\).

**Lemma 1.** (An integral formula) Let \(h \in \mathscr{O}(L; k')\). Choose positive numbers \(\epsilon\) and \(\epsilon'\) so small that \(0 < 2\epsilon < 2\pi + k_{1} - k_{2}\), \(0 < \epsilon' < 1 - k'\) and that \(h \in \mathscr{O}_{\rho}(L_{\epsilon}; k' + \epsilon')\).

(i) For any \(R > 0\), the function of \(z\)

\[ H_{R}(z) = \int_{\partial L_{\epsilon}, R} h(w) \left(1 - \exp(z - w)\right)^{-1} dw \]

belongs to the space \(\mathscr{O}(L; 1)\), consequently to the space \(\mathscr{O}(L; \rho)\), where we denote \(\partial L_{\epsilon}, R = \partial L_{\epsilon}(w; \Re w \leq R)\).

(ii) We have

\[ 2\pi i h(z) = \int_{\partial L_{\epsilon}} h(w) \left(1 - \exp(z - w)\right)^{-1} dw \quad \text{for } z \in \text{int } L_{\epsilon}. \]

(iii) In the topology of \(\mathscr{O}(L; k')\), we have

\[ \lim_{R \to \infty} \int_{\partial L_{\epsilon}, R} h(w) \left(1 - \exp(z - w)\right)^{-1} dw = 2\pi i h(z). \]

**Proof.** (i) It is clear the function \(H_{R}(z)\) is holomorphic in \(\text{int } L_{\epsilon}\). On the other hand we have

\[ \sup_{z \in L_{\epsilon}} \left|H_{R}(z) e^{z}\right| \]

\[ = \sup_{z \in L_{\epsilon}} \left|\int_{\partial L_{\epsilon}, R} h(w) \left(e^{-z} - e^{-w}\right)^{-1} dw\right| \]

\[ \leq \int_{\partial L_{\epsilon}, R} \left|h(w)\right| \text{dist} \left(e^{-w}, \exp(-L_{\epsilon, 2})\right)^{-1} |dw| < \infty, \]

because the integrand is continuous and \(\partial L_{\epsilon, R}\) is compact.

(ii) By the residue theorem, if \(z \in \text{int } L\) and \(\Re z < R\), then we have

\[ \int_{\partial L_{\epsilon}(R)} h(w) \left(1 - \exp(z - w)\right)^{-1} dw = 2\pi i h(z), \]

where \(\partial L_{\epsilon}(R)\) denotes the boundary of the rectangle
Let us denote by $C_{\epsilon}(R)$ the boundary of $L_{\epsilon}\cap\{w; \Re w \geq R\}$. We have to show, for $z$ fixed in int $L_{\epsilon}$,

$$\int_{C_{\epsilon}(R)} h(w) \left(1 - \exp(z-w)\right)^{-1} dw$$

tends to 0 as $R \to \infty$.

As $h \in \mathcal{E}_{\epsilon}(L; k'+\epsilon')$, we have with some constant $C \geq 0$,

$$\left| \int_{C_{\epsilon}(R)} h(w) \left(1 - \exp(z-w)\right)^{-1} dw \right| \leq C \int_{C_{\epsilon}(R)} e^{-(k'+\epsilon')u}(1 - e^{z-R})^{-1} |dw| \to 0$$
as $R \to \infty$.

(iii) We have to show, putting $C'(R) = \partial L_{\epsilon} - \partial L_{\epsilon,R}$

$$\int_{C'(R)} h(w) \left(1 - \exp(z-w)\right)^{-1} dw$$
tends to 0 in the topology of $\mathcal{E}_{\epsilon}(L; k')$ as $R \to \infty$. Remark that $e^{-w}$ belongs to the closed angular domain $\overline{A}(-k_{1} + \epsilon, -k_{2} + 2\pi - \epsilon)$ if $w \in C'(R)$. Therefore as in the proof of Proposition 3, we can show

$$\inf_{z \in L_{\epsilon}/2} |e^{-z} - e^{-w}| \geq e^{-u} \sin(\epsilon/2)$$

and

$$\inf_{w \in C'(R)} |1 - e^{-we^{2}}| \geq \sin(\epsilon/2) \quad \text{for } w \in C'(R).$$

Therefore we have with some constants $C$ and $C' \geq 0$

$$\sup_{z \in L_{\epsilon}/2} \left| e^{(k'+\epsilon'/2)z} \int_{C'(R)} h(w) \left(1 - \exp(z-w)\right)^{-1} dw \right| = \sup_{z \in L_{\epsilon}/2} \left| \int_{C'(R)} h(w) (e^{-z} - e^{-w})^{-k'-\epsilon'/2}(1 - e^{-we^{2}})^{-1+k'+\epsilon'/2} dw \right|$$
\[
\leq C \int_{C, R} |h(\omega)| e^{((k'+\epsilon'/2)u)} \, |d\omega|
\leq C \int_{C, R} \exp \left( (1/2) \omega \right) |d\omega|.
\]

The last term converges to 0 as \( R \to \infty \).

THEOREM 4. (Inversion formula) Let \( \mu \in \mathcal{D}^\prime (L; k') \) and \( h \in \mathcal{D} (L; k') \) with \( 0 \leq k' < 1 \) and \( L = [a, \infty) + i[k_1, k_2] \), \( k_2 - k_1 < 2\pi \). Choose positive numbers \( \epsilon \) and \( \epsilon' \) so small that \( 0 < 2\epsilon < 2\pi + k_1 - k_2 \), \( 0 < \epsilon' < 1 - k' \) and that \( h \in \mathcal{D} (L; k + \epsilon') \). Then we have the inversion formula:

\[
\langle \mu, h \rangle = (2\pi i)^{-1} \int_{\partial L} G_{\mu}(e^{-w}) h(w) \, dw.
\]

PROOF. We have by Lemma 1 (i)

\[
\int_{sL, R} \langle \mu, (1-\exp(z-w))^{-1} \rangle h(w) \, dw
= \langle \mu, \int_{sL, R} (1-\exp(z-w))^{-1} h(w) \, dw \rangle
\]

for \( R > 0 \). By Lemma 1 [iii], the righthand side converges to \( \langle \mu, 2\pi i h(z) \rangle \). As the lefthand side converges because of Proposition 3, we obtain the inversion formula.

q. e. d.

THEOREM 5. Suppose \( 0 \leq k < 1 \) and \( L = [a, \infty) + i[k_1, k_2] \), \( k_2 - k_1 < 2\pi \). If the function \( F \in \text{Exp} ((-\infty, -k') + iR; L) \) satisfies the condition

\[
F(-n) = 0 \quad \text{for every } n = 1, 2, 3, \ldots,
\]

then the function \( F(\zeta) \) vanishes identically.

PROOF. By Theorem 3, there exists an analytic functional \( \mu \in \mathcal{D}^\prime (L; k') \) such that \( F(\zeta) = \tilde{\mu}(\zeta) \). By Proposition 2 (iii), we have the Laurent expansion:

\[
G_{\mu}(\zeta) = -\sum_{n=1}^{\infty} \zeta^{-n} \tilde{\mu}(-n) = -\sum_{n=1}^{\infty} \zeta^{-n} F(-n)
\]

for \( |\zeta| > e^{-a} \). By the assumption, \( G_{\mu}(\zeta) = 0 \). By Theorem 4, we conclude \( \mu = 0 \) and \( F(\zeta) = 0 \).

q. e. d.

Putting \(-k_1 = k_2 = k\), \( 0 \leq k < \pi \), we obtain Theorem 2 as a corollary.

§ 5. The image of the Avanissian-Gay transformation.

We determine in this section the image of the Avanissian-Gay transformation.

THEOREM 6. Suppose the width of \( L \) is less than \( 2\pi \) and \( 0 \leq k' < 1 \).
Then the Avanissian-Gay transformation $G$ is a linear topological isomorphism of $\mathcal{B}'(L; k')$ onto $\mathcal{B}_0(C \setminus \exp(-L); k')$.

**Proof.** We have proved the Avanissian-Gay transformation $G$ is a continuous linear mapping of $\mathcal{B}'(L; k')$ into $\mathcal{B}_0(C \setminus \exp(-L); k')$ in Proposition 4. If we can prove the bijectivity of $G$, the continuity of the inverse mapping results from the closed graph theorem for Fréchet spaces. The injectivity of $G$ is a consequence of the inversion formula (Theorem 4). Let us prove the surjectivity of $G$. Let $\varphi \in \mathcal{B}_0(C \setminus \exp(-L); k')$ be given. We put, for $h \in \mathcal{B}(L; k')$,

$$
\langle \mu(\varphi), h \rangle = \int_{r^1} \varphi(\tau) h(-\log \tau) \frac{d\tau}{\tau}
$$

(5.1)

where $\varepsilon > 0$ is a sufficiently small number and $\Gamma_{\varepsilon} = \Gamma_{\varepsilon}^1 + \Gamma_{\varepsilon}^2 + \Gamma_{\varepsilon}^3 = \exp(-\partial L_{\varepsilon})$ is the path in the $\tau$-plane depicted in the figure 3.

First we show the improper integral of the righthand side of (5.1) exists and is independent of sufficiently small $\varepsilon > 0$. If $0 < \varepsilon_1 < \varepsilon < \pi + (k_1 - k_2)/2$ and $0 < \varepsilon' < 1 - k'$, we have

$$
\sup_{\varepsilon \in L_{\varepsilon}^1} \left| \varphi(e^{-2}) e^{-(k' + \varepsilon')z} \right| < \infty.
$$

If $h \in \mathcal{B}_b(L_{\varepsilon_0}; k' + \varepsilon_0)$, then the righthand side of (5.1) converges clearly for $0 < \varepsilon < \min (\pi + (k_1 - k_2)/2, \varepsilon_0)$ and is independent of such $\varepsilon$ by the Cauchy integral theorem. Therefore $\langle \mu(\varphi), h \rangle$ is well defined by (5.1) and $\mu(\varphi)$ is continuous linear on the space $\mathcal{B}_b(L_{\varepsilon_0}; k' + \varepsilon_0)$ for any $\varepsilon_0 > 0$ and $\varepsilon_0' > 0$. By the definition of the inductive limit topology, $\mu(\varphi)$ is a continuous linear functional on $\mathcal{B}(L; k')$. 

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Fig. 3.
We shall compute the Avanissian-Gay transformation of the functional $\mu(\varphi)$. By the definition, we have
\[
G_{\mu,\varphi}(\zeta) = \langle \mu(\varphi), (1-\zeta e^{r})^{-1} \rangle \\
= \int_{r_{s}} \varphi(\tau) (1-\zeta \exp (-\log \tau))^{-1} d\tau/	au \\
= \int_{r_{s}} \varphi(\tau) (\tau-\zeta)^{-1} d\tau \\
= \lim_{\delta \to 0} \int_{r_{s},\delta} \varphi(\tau) (\tau-\zeta)^{-1} d\tau ,
\]
where $r_{s}=r \cap \{ \tau ; |\tau| \geq \delta \}$. For a sufficiently large number $R>0$ and sufficiently small number $\delta>0$, we put
\[
C_R = \{ \tau ; |\tau| = R \}
\]
and
\[
C_{\epsilon}(\epsilon) = \{ \tau ; |\tau| = \delta, -k_{1}+\epsilon \leq \arg \tau \leq -k_{2}+2\pi -\epsilon \}.
\]
By Cauchy's integral formula, we have
\[
\frac{1}{2\pi i} \int_{C_{R}+r_{s,}\delta} \varphi(\tau) (\tau-\zeta)^{-1} d\tau = \varphi(\zeta) .
\]
We will show the integral over the path $C_{R}$ tends to 0 as $R \to \infty$ and that the integral over the path $C'_{\epsilon}(\epsilon)$ tends to 0 as $\epsilon \to 0$. If $|\tau|=R$ and $R>|\zeta|$, we have $|\tau-\zeta| \geq |\tau|-|\zeta| = R-|\zeta| > 0$. Therefore
\[
\left| \int_{C_{R}} \varphi(\tau) (\tau-\zeta)^{-1} d\tau \right| \leq \int_{C_{R}} |\varphi(\tau)| |\tau-\zeta|^{-1} |d\tau| \\
\leq \sup_{|\tau|=R} |\varphi(\tau)| (R-|\zeta|)^{-1} 2\pi R \to 0 \text{ as } R \to \infty .
\]
If $|\zeta| > \delta$, then
\[
\left| \int_{C'_{\epsilon}(\epsilon)} \varphi(\tau) (\tau-\zeta)^{-1} d\tau \right| \leq \int_{C'_{\epsilon}(\epsilon)} |\varphi(\tau)| (|\zeta|-\delta)^{-1} |d\tau| \\
\leq C_{1} \delta^{-k'+\epsilon'} (|\zeta|-\delta)^{-1} 2\pi \delta \\
= C_{1} 2\pi (|\zeta|-\delta)^{-1} \delta^{1-(k'+\epsilon')} \to 0 \text{ as } \delta \to 0 ,
\]
because we may choose $\epsilon'$ so that $1-(k'+\epsilon')>0$. We have thus proved
\[
G_{\mu,\varphi} = \varphi
\]
and the surjectivity of the Avanissian-Gay transformation $G$. q.e.d.
References


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