Note on Miyashita-Ulbrich action and H-separable extension

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Abstract. The Miyashita-Ulbrich action is an action of a Hopf algebra \( A \) on the centralizer \( E^C \) associated to an \( A \)-Galois extension \( B/C \) with algebra homomorphism \( \alpha : B \to E \). Doi and Takeuchi [DT] ask when the action of a Hopf algebra \( A \) on the centralizer \( E^C \) of a ring extension \( E/C \) comes from such an \( A \)-Galois extension \( B/C \). They provide an affirmative answer for Azumaya algebra \( E \) with subalgebra \( C \) such that \( E_C \) is a progenerator. In this note we observe how their proof extends to an \( H \)-separable extension \( E/C \) with the same condition on \( E_C \). Similarly, we establish the converse: if \( E/C \) is an \( H \)-separable, right \( A \)-Galois extension, then \( E^C \) is a left \( A^* \)-Galois extension over the center \( Z(E) \).

Key words: Hopf-Galois extension, centralizer, Miyashita-Ulbrich action, \( H \)-separable extension, Azumaya algebra.

1. Introduction

We let \( R \) be a commutative ground ring, \( A \) be a Hopf algebra over \( R \) which is finite projective as an \( R \)-module, and \( A^* \) its dual Hopf algebra. If \( B \) is an associative unital algebra and \( M \) is a unitary \( B \)-bimodule, we let \( M^B \) denote the central elements of \( M : M^B = \{ m \in M \mid mb = bm, \forall b \in B \} \). Ulbrich [U] defines an action of \( A \) on the centralizer \( V := V_B(C) = B^C \) of an \( A \)-Galois extension \( B/C \). If \( \beta : B \otimes_C B \to B \otimes A \) denotes the Galois isomorphism given by \( \beta(x \otimes y) = xy(0) \otimes y(1) \), the action of an \( a \in A \) on \( x \) in the centralizer \( V \) is given by

\[
x \triangleleft a = \sum_i b_i x b'_i
\]

where \( \sum_i b_i \otimes b'_i = \beta^{-1}(1 \otimes a) \). It is easy to compute that this is a module action with invariant subalgebra \( Z(B) \), the center of \( B \). Indeed, it is a measuring action with \( V \) becoming a right \( A \)-module algebra [U, II].

Doi and Takeuchi [DT] extend this action to the centralizer \( E^C \) of an algebra extension \( E/C \) with algebra homomorphism \( \alpha : B \to E \) by means.
of a $\pi$-method. The mapping $\pi$ is the map induced from the isomorphism $\beta$ by applying the functor $\text{Hom}_B^f(-, E)$, where $E$ is a $B$-bimodule via $\alpha$. The $C$-bimodule isomorphism obtained induces

$$\pi : \text{Hom}(A, E^C) \cong \text{Hom}_{C\leftarrow C}(B, E), \quad \pi(f)(b) = b(0)f(b(1)). \quad (2)$$

For every $x \in E^C$, there is a map $x^{(\cdot)}$ in $\text{Hom}(A, E^C)$ defined as $x^{(\cdot)} = \pi^{-1}(\lambda_x)$ where $\lambda_x$ is left multiplication by $x$ on $B$. This obtains a right measuring action of $A$ on $E^C$ by

$$x \cdot a := x^a, \quad (3)$$

the value of $x^{(\cdot)}$ on $a$. This action extends Ulbrich’s action, is characterized by

$$xb = b(0)(x \cdot b(1)) \quad (4)$$

for every $b \in B$, $x \in E^C$, and is called the Miyashita-Ulbrich action $[\text{DT}]$. The authors of $[\text{DT}]$ ask whether a measuring action of a Hopf algebra $A$ on the centralizer $E^C$ of a ring extension $E/C$ is the Miyashita-Ulbrich action of some $A$-Galois extension $B/C$ with algebra homomorphism $\alpha : B \to E$ $[\text{DT},$ p.489]. They prove that this is the case for Azumaya algebra $E$ with subalgebra $C$ such that $E_C$ is a progenerator. Their method of proof is general and careful enough to invite an extension of their answer to the more general H-separable extensions.

An H-separable extension is a certain type of separable extension which generalizes Azumaya algebra $[\text{H1}]$. Given an Azumaya algebra $D$ and algebra $C$, the tensor algebra $C \otimes D$ is an H-separable extension of $C$ $[\text{H1},$ Prop. 3.1]. Another example is the algebra extension $B/C$ where $B$ is an Azumaya algebra with subalgebra $C$ such that $B_C$ is projective and $C$ is a direct summand of $B$ as $C$-bimodules (cf. $[\text{S1},$ Prop. 1.4] and $[\text{K},$ 2.9]). H-separable extensions have several well-known properties in common with Azumaya algebras which we review in Section 2. Certain H-separable extension are automatically Frobenius extensions $[\text{S1},$ $\text{S2}]$. If $B/C$ is an H-separable extension, then $B_C$ a projective module implies $B_C$ is finitely generated $[\text{T}]$.

In this paper we note that the problem of Doi and Takeuchi has an affirmative answer as well for an H-separable extension $E/C$ where $E_C$ is a projective generator $[\text{Theorem 3.2}]$. We moreover show that an H-separable right $A$-Galois extension $B/C$ induces a left $A^*$-Galois extension $V/Z(B)$
(Theorem 3.1). The proofs of these two theorems follows [DT] except for a modification required by the fact that the center $Z$ no longer is $R_1E$.

2. H-separable extension

Let $B$ be a ring and $M$ a $B$-bimodule. $M$ is centrally projective if $M$ is isomorphic to a $B$-bimodule direct summand of a finite direct sum of $B$ with itself:

$$M \oplus * \cong B(B \oplus \cdots \oplus B)_B.$$

Note that either the left or right $B$-dual $\text{Hom}_B(M, B)$ is then centrally projective. A ring extension $B/C$ is H-separable if the natural $B$-bimodule $B \otimes_C B$ is centrally projective [H1].

It follows from Hirata’s extension of Morita theory [H1] that the centralizer $V = V_B(C)$ is a finitely generated projective module over the center $Z$ of $B$, and there is a $B$-bimodule isomorphism,

$$B \otimes_C B \cong \text{Hom}_Z(V, B), \quad b \otimes b' \mapsto (v \mapsto vbv').$$

From this it follows that $B/C$ is a separable extension [H1, HS], as the name should indicate. Moreover, we obtain the characterization:

**Proposition 2.1 ([S1])** The ring extension $B/C$ is H-separable if and only if for every $B$-bimodule $M$, the multiplication mapping $V \otimes_Z M^B \to M^C$ is an isomorphism.

By letting $M = \text{End}_Z B$ and $\mathcal{E} := \text{End}(B_C)$ we obtain the next corollary, which is important to this note. Let $\rho_v \in \mathcal{E}$ denote right multiplication by $v \in V$.

**Corollary 2.2 ([H2])** If $B/C$ is H-separable, there is a ring isomorphism

$$B \otimes Z V^\text{op} \cong \mathcal{E}, \quad b \otimes v \mapsto \lambda_b \circ \rho_v$$

and a similarly defined ring isomorphism

$$V \otimes Z V^\text{op} \cong \text{End}_{C-C}(B).$$

Now we call a ring extension $B/C$ a (right) HS-separable extension if 1) it is an H-separable extension; and 2) $B_C$ is a projective generator (therefore $B_C$ is a progenerator [T]). [S2] has conditions where right and left extensions of this type are equivalent.
The characterization below of HS-separable extensions is key to this note. For any ring extension \( B/C \), the functor (of induction) \( B \otimes_C - \) from the category \( \mathcal{C} \) of left \( C \)-modules into the category \( \mathcal{D} \) of \( B\)-\( V \)-bimodules \( M \) such that \( M^Z = M \), has right adjoint \( G : \mathcal{D} \to \mathcal{C} \) given by \( G(M) = M^V \). The associated counit of adjunction is given by (8), though not necessarily an isomorphism.

**Lemma 2.3** A ring extension \( B/C \) is HS-separable if and only if there is a ring isomorphism (6), \( V \) is finitely generated projective over \( Z \), and \( G \) is an equivalence where multiplication leads to the isomorphism,

\[
B \otimes_C P^V \xrightarrow{\cong} P,
\]

for every \( B\)-\( V \)-bimodule \( P \in \mathcal{D} \).

**Proof.** (\( \Rightarrow \)) We have already noted that the isomorphism (6) and the condition on \( V \) hold for H-separable extensions. Since \( B_C \) is a progenerator, \( B \otimes_C - \) is a (Morita) equivalence of \( \mathcal{C} \) with the category of \( \text{End}(B_C) \cong B \otimes V^{\text{op}} \)-modules, i.e. the category \( \mathcal{D} \). By uniqueness of right adjoint, \( G \) is an inverse equivalence of induction and (8) is a natural isomorphism.

(\( \Leftarrow \)) Induction is clearly an equivalence of \( \mathcal{C} \) and \( \mathcal{D} \), so \( B_C \) is a progenerator by (6) and the Morita theorems. Since \( V \) is finitely generated projective over \( Z \), the assumption that \( \mathcal{E} \cong B \otimes Z V^{\text{op}} \) implies that \( \mathcal{E} \) is centrally projective as a \( B \)-bimodule [S2, Lemma 3]. Then \( B/C \) is H-separable, since \( B \otimes_C B \) is the left \( B \)-dual of \( \mathcal{E} \) [S3, Cor. 2.3]. \( \square \)

3. Hopf-Galois actions on the centralizer

We continue our notation \( V \) for the centralizer and \( Z \) for the center of the overalgebra; as before, \( A \) is a Hopf algebra, which is finitely generated projective over the commutative ground ring \( R \), with comultiplication denoted by \( \Delta \) and counit \( \epsilon \). The next theorem generalizes [DT, 5.2, (i) \( \Rightarrow \) (ii)] and part of [U, Satz 2.7].

**Theorem 3.1** Suppose \( B/C \) is an H-separable right \( A \)-Galois extension. Then \( V/Z \) is a left \( A^* \)-Galois extension.

**Proof.** The Miyashita-Ulbrich action \( V \otimes A \to V \) induces via duality a left \( A^* \)-comodule algebra structure on \( V \). The values of the coaction are denoted by \( x \mapsto x_{(-1)} \otimes x_{(0)} \in A^* \otimes V \), whence the dual action is given by
\[ V \otimes_{Z} V \cong \text{End}_{C-C}(B) \]
\[
\beta' \downarrow \quad \uparrow \pi
\]
\[ A^* \otimes V \cong \text{Hom}(A, V) \]

Fig. 1.

\[ \times_{(-1)}(a) \times_{(0)} = x \triangleleft a \] for every \( x \in V, \ a \in A \). The coinvariants are the same as the invariant subalgebra, namely \( Z = Z(B) \).

Now consider the commutative diagram (Fig. 1). The bottom horizontal arrow is a standard isomorphism due to the assumption that \( A \) is finite projective over \( R \). The top horizontal arrow is the isomorphism (7) as a left \( V \)-isomorphism. \( \beta' \) is the left Galois map given by

\[ x \otimes y \mapsto x_{(-1)} \otimes x_{(0)} y \]

for every \( x, y \in V \). \( \pi \) is the Doi-Takeuchi map \( \pi \) discussed above in the case \( E = B \). The diagram commutes since

\[ x \otimes y \mapsto x_{(-1)} \otimes x_{(0)} y \mapsto x^{(0)} y \mapsto \lambda_x \circ \rho_y, \]

which is the image of \( x \otimes y \in V \otimes V \) under the isomorphism (7). It follows that \( \beta' \) is an isomorphism and \( V/Z \) a left \( A^* \)-Galois extension.

The next result is a type of converse of the theorem above. Recall the category \( G^A_C \) in [DT, Section 6] consisting of all triples \( (B, i, \alpha) \) such that \( B \) is a right \( A \)-Galois extension with invariants isomorphic to \( C \) via \( i \) and algebra homomorphism \( \alpha : B \to E \). The morphisms in this category satisfy two obvious commuting triangles and are then isomorphisms [DT, p.509].

In the isomorphism class of \( (B, i, \alpha) \) is a canonical representative \( \hat{B} \subseteq E \otimes A \) with right \( A \)-comodule structure given by the restriction of \( \text{id}_E \otimes \Delta \) to \( \hat{B} \), which is the image of \( \hat{\alpha} : B \to E \otimes A \) given by \( \hat{\alpha}(b) = \alpha(b_{(0)}) \otimes b_{(1)} \) [DT, Theorem (6.12)]. This map is injective if \( E_C \) is faithfully flat. That \( B/C \) is a right \( A \)-Galois extension translates to the conditions [DT, (6.6)–(6.8)] for \( \hat{B} \) being right Galois over \( C = C \otimes 1_A \), including the condition

\[ \eta : E \otimes_C \hat{B} \isoarrow \xrightarrow{\sim} E \otimes A, \quad \eta(x \otimes \hat{b}) = (x \otimes 1) \hat{b} \]  \hspace{1cm} (9)

Each isomorphism class \( (B, i, \alpha) \) in \( G^A_C \) determines a measuring
(Miyashita-Ulbrich) action $E^C \otimes A \to E^C$ denoted by $\triangleleft_B$ and characterized by Eq. (4). Denote the set of right measuring actions of $A$ on $E^C$, where $Z$ is contained in the invariants $V^{(A)}$, by $\text{Meas}(E^C/Z, A)$.

**Theorem 3.2** Suppose $E/C$ is an HS-separable extension with right measuring action $\triangleleft$ of $A$ on its centralizer $V$ such that $V^{(A)} \supseteq Z$. Then there is a right $A$-Galois extension $B/C$ with algebra homomorphism $\alpha : B \to E$ such that $\triangleleft_B = \triangleleft$. Moreover, $\text{Meas}(E^C/Z, A)$ is in 1-1 correspondence with the isomorphism classes of $G^{A,E}_C$ via $\hat{B} \mapsto \triangleleft_{\hat{B}}$.

**Proof.** The main steps of the proof, following [DT, (6.13)-(6.20)], are to define an $E$-$V$-bimodule structure on $E \otimes A$ from the given action $\triangleleft$, define a right Galois subalgebra

$$\hat{B}_{\triangleleft} = (E \otimes A)^V$$

over $C$, and prove that $\triangleleft_{\hat{B}_{\triangleleft}} = \triangleleft$. The $E$-$V$-bimodule structure on $E \otimes A$ is indicated by

$$e(e' \otimes a)v = ee'(v \triangleleft a_{(1)}) \otimes a_{(2)} \quad (10)$$

Note that $(E \otimes A)^Z = E \otimes A$ since $Z \subset V^{(A)}$. That $\eta : E \otimes_C \hat{B}_{\triangleleft} \to E \otimes A$ is an isomorphism follows from our assumption that $E/C$ is HS-separable and Lemma 2.3. That $\hat{B}_{\triangleleft}$ is a subalgebra which is right Galois over $C$ follows from the proof of [DT, Lemma (6.17)]. The map $\alpha_{\triangleleft} : \hat{B}_{\triangleleft} \to E$ is $\text{id}_E \otimes e$. That the measuring action on $E^C$ associated with $\hat{B}_{\triangleleft}$ is $\triangleleft$ itself follows from the proof of [DT, Lemma (6.18)].

To finish the proof that $G^{A,E}_C \leftrightarrow \text{Meas}(E^C/Z, A)$ we must show that given $\hat{B} \in G^{A,E}_C$, the construction applied to the associated measuring action $\triangleleft_{\hat{B}}$ leads back to $\hat{B}$: i.e., $\hat{B}_{\triangleleft_{\hat{B}}} = \hat{B}$. Now $\hat{B} \subseteq \hat{B}_{\triangleleft_{\hat{B}}}$ by the computation in the proof of [DT, Lemma (6.19)]. A proper inclusion is impossible since $E \otimes_C \hat{B} = E \otimes_C \hat{B}_{\triangleleft_{\hat{B}}} = E \otimes A$ and $E_C$ a generator implies the existence of a right $C$-module projection $E_C \to C_C$. \hfill $\Box$

**References**


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