On the sharpness of Seeger-Sogge-Stein orders

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Abstract. We will extend the sharpness results on $L^p$- and $L^p - L^q$-continuity of Fourier integral operators for an arbitrary rank of the canonical projection. For the elliptic operators of small negative orders we will show that by a coordinate change they are equivalent to pseudo-differential operators.

Key words: Fourier integral operator, regularity, sharp estimates, pseudo-differential operator, Lagrangian manifold.

1. Introduction

Let $X$, $Y$ be smooth paracompact $n$-dimensional manifolds. Let $d\sigma_X$ and $d\sigma_Y$ be the standard symplectic forms on $T^*X$ and $T^*Y$ and let $\Lambda$ be a conic Lagrangian submanifold of $T^*X \setminus 0 \times T^*Y \setminus 0$, equipped with the symplectic form $d\sigma_X - d\sigma_Y$. We will assume that $\Lambda$ is a local graph of a symplectomorphism from $T^*Y \setminus 0$ to $T^*X \setminus 0$. Let $T \in \mathcal{I}^\mu(X, Y; \Lambda)$ be a Fourier integral operator with the canonical relation $\Lambda$. The distributional kernel $K \in \mathcal{D}'(X \times Y)$ of $T$ is a Lagrangian distribution of order $\mu$ whose wavefront set is contained in $\Lambda' = \{(x, \xi, y, \eta) : (x, \xi, y, -\eta) \in \Lambda\}$. The global theory of such operators can be found in [1]. Let $\pi_{X \times Y}$ be the natural projection from $T^*X \setminus 0 \times T^*Y \setminus 0$ to $X \times Y$. The deep result of Seeger, Sogge and Stein [5] states that for $1 < p < \infty$ and $\mu \leq -(n - 1)|1/p - 1/2|$ the operators $T \in \mathcal{I}^\mu(X, Y; \Lambda)$ are continuous from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$. This result is sharp if $T$ is elliptic and $d\pi_{X \times Y}|_{\Lambda}$ has full rank equal to $2n - 1$ anywhere, which follows from the stationary phase method as in [3]. Somewhat different approaches to this are in [6] and [7]. If the rank of the canonical projection on $\Lambda$ can be bounded from above by

$$\text{rank } d\pi_{X \times Y}|_{\Lambda} \leq 2n - k$$

with some $1 \leq k \leq n$, then under the so-called smooth factorization condition introduced in [5] the operators $T \in \mathcal{I}_\rho^\mu(X, Y; \Lambda)$, $1/2 \leq \rho \leq 1$, are continuous from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$ for $1 < p < \infty$ and $\mu \leq -(n - k\rho)|1/p - 1/2|$.

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In [4] the factorization condition is shown to be satisfied in a number of important cases, if a phase function of the operator is analytic.

Using analysis of some convolution operators in [8], it was shown in [5] that there exist conormal operators with constant rank $d\pi_{X \times Y} |_{\Lambda} \equiv 2n - k$, for which the estimate of the critical order $\mu$ is sharp. We want to show that for $\rho = 1$ this order is sharp for an arbitrary elliptic operator whose canonical relation satisfies inequality (1). The basic idea to test the $L^p$-continuity of an operator will be to investigate its behavior on the functions obtained from a $\delta$-distribution at some $y_0 \in Y$ after the application of elliptic pseudo-differential operators of sufficiently negative orders. The only singularities of such functions are at $y_0$, meanwhile the singularities of $T$ applied to them happen only in the directions transversal to some $(n - k)$-dimensional subset $\Sigma_{y_0}$ of $X$. Finally, this will be applied to the continuous Fourier integral operators of zero order.

It was pointed out in [7, p. 398], that in $\mathbb{R}^3$ the operator $T : f \mapsto \frac{\partial}{\partial x_j}(f * d\sigma)$ with $j = 1, 2, 3$, and $d\sigma$ the usual measure on the unit sphere $S^2 \subset \mathbb{R}^3$, is essentially a Fourier integral operator of order 0, which is not continuous in $L^p(\mathbb{R}^3)$, $1 < p < \infty$. We will show that this is not a single example and derive a structural formula for the continuous elliptic Fourier integral operators of order 0 (Theorem 2) and then generalize it for small negative orders and $L^p \to L^q$ continuity (Theorem 3).

2. Results

By the equivalence-of-phase-function theorem as in [1, Th. 2.3.4] and [5] it is sufficient to consider operators in $\mathbb{R}^n$ with kernel

$$K(x, y) = \int_{\mathbb{R}^n} e^{i[(x, \xi) - \phi(y, \xi)]} b(x, y, \xi) d\xi,$$

with some symbol $b \in S^\mu$ vanishing for $x, y$ outside a compact set and phase function satisfying

$$\det \phi_{y\xi}' \neq 0$$

on the support of $b$, which is equivalent to $\Lambda$ being a canonical graph. Locally $\Lambda$ is the set of the form $\{(\nabla_x \phi, \xi, y, \nabla_y \phi)\}$. We begin with the following

**Proposition 1** Let $T \in I^\mu(X, Y; \Lambda)$ be elliptic. Assume that the canon-
ical relation \( \Lambda \) is a local graph and rank \( d\pi_{X \times Y}|_{\Lambda} \equiv 2n - k \), \( 1 \leq k \leq n \). Then \( T \) is not bounded as a linear operator \( L^{p}_{comp}(Y) \rightarrow L^{p}_{loc}(X) \), if \( \mu > -(n - k)|1/p - 1/2|, 1 < p < \infty \).

**Proof.** By the above reduction it is sufficient to restrict ourselves to the case of \( \mathbb{R}^{n} \) and operators satisfying (2) and (3). Let \( P_{-s} \in \Psi^{-s}(Y) \) be an elliptic pseudo-differential operator in \( Y \) and consider \( f_{s}(y) = (P_{-s}\delta_{y_{0}})(y) \). Then by Schwartz kernel theorem \( f_{s}(y) = \int K_{-s}(y, z)\delta_{y_{0}}(z)dz = K_{-s}(y, y_{0}) \), and in view of the kernel estimates for pseudo-differential operators in, for example, [7, p. 241, 245], we have \( |K_{-s}(y, y_{0})| \leq C|y - y_{0}|^{-n+s} \) in some local coordinate system. It follows that \( f_{s} \in L^{p}_{loc} \) if and only if \( s > n(1 - 1/p) \).

We assume here \( 1 < p \leq 2 \), for the rest would follow by considering the adjoint operators.

Let \( \Sigma = \pi_{X \times Y}(\Lambda) \). Then in view of the assumption on the rank of \( \pi_{X \times Y} \), \( \Sigma \subset X \times Y \) is a smooth submanifold of codimension \( k \). Let \( \Sigma \) be given by the set of equations \( h_{j}(x, y) = 0, 1 \leq j \leq k \), in a neighborhood of \( y_{0} \), where \( \nabla h_{1}, \ldots, \nabla h_{k} \) are linearly independent. Then \( \Lambda \) is the conormal bundle of \( \Sigma \) and the phase function of \( T \) may be given by

\[
\psi(x, y, \lambda) = \sum_{j=1}^{k} \lambda_{j}h_{j}(x, y).
\]

Let \( T_{s} = T \circ P_{-s} \). Then \( T f_{s}(x) = T_{s}(\delta_{y_{0}})(x) \) and the canonical relations of \( T_{s} \) and \( T \) coincide, since a composition with a pseudo differential operator leaves it invariant. The operator \( T_{s} \) is of order \( \mu - s \) and in local coordinates it can be expressed as

\[
T f_{s}(x) = \int_{\mathbb{R}^{n}} \left( \int_{\mathbb{R}^{k}} e^{i\sum_{j=1}^{k} \lambda_{j}h_{j}(x, y)}a(x, \lambda)d\lambda \right)dy
= \int_{\mathbb{R}^{k}} e^{i\langle \lambda, \tilde{h}(x, y_{0}) \rangle}a(x, \lambda)d\lambda
= (2\pi)^{k}\tilde{a}(x, \tilde{h}(x, y_{0})),
\]

where \( \tilde{\lambda} \) and \( \tilde{h} \) are the vectors with the components \( \lambda_{j} \) and \( h_{j} \) respectively, and \( a \in S^{\mu-s+(n-k)/2}(\mathbb{R}^{k}) \) is a symbol of \( T_{s} \) after applying the stationary phase method and integrating away \( (n - k) \)-variables. Now, the inverse Fourier transform of \( a \) in the second variable is \( (2\pi)^{k}\tilde{a}(x, \zeta) = \int_{\mathbb{R}^{k}} e^{i\langle \lambda, \zeta \rangle}a(x, \lambda)\delta_{0}(\lambda)d\lambda = P_{0}\delta_{0}(\zeta) = K_{0}(\zeta, 0) \) and this is equivalent to \( |\zeta|^{-k - \text{ord}(a)} \), where \( P_{0} \in \Psi^{\text{ord}(a)}(\mathbb{R}^{k}) \) with symbol equal to \( a(x, \lambda) \) and \( K_{0} \).
is a distributional kernel of $P_0$. In view of $\text{dist}(x, \Sigma_{y_0}) \approx |\tilde{h}(x, y_0)|$ with $\Sigma_{y_0} = \{x : (x, y_0) \in \Sigma\}$ and formulas above, we have $(2\pi)^k \tilde{a}(x, \tilde{h}(x, y_0)) \sim \text{dist}(x, \Sigma_{y_0})^{-k-(\mu-s+(n-k)/2)}$, locally uniformly in $x$. Formula (4) implies that $T_{f_s}$ is smooth along $\Sigma_{y_0}$, so $T_{f_s} \notin L^p_{\text{loc}}(\mathbb{R}^n)$ if and only if $p(k+\mu-s+(n-k)/2) \geq k$, or, equivalently, $s \leq \mu+k(1-1/p)+(n-k)/2$. Together with condition on $f_s \in L^p_{\text{loc}}$ this implies that $T$ is not continuous in $L^p$-norms if such $s$ exists, i.e. when $\mu > -(n-k)|1/p-1/2|$. This completes the proof.

Assume now that the operator $T$ is not conormal and that (1) is satisfied with $2n-k$ at some point. Then the set $\Lambda_0 = \{\lambda \in \Lambda : \text{rank } \pi_{X \times Y}|_{\Lambda}(\lambda) = 2n-k\}$ is nonempty and open in $\Lambda$. Applying the equivalence of the phase function and the same argument as in Proposition 1 at some $\lambda_0 = (x_0, \xi_0, y_0, \eta_0) \in \Lambda_0$, we get

**Theorem 1** Let $T \in \mathcal{I}^\mu(X, Y; \Lambda)$ be elliptic. Assume that the canonical relation $\Lambda$ is a local graph and that $\text{rank } \pi_{X \times Y}|_{\Lambda} \leq 2n-k$, $1 \leq k \leq n$, equal to $2n-k$ at some point. Then $T$ is not bounded as a linear operator $L^p_{\text{comp}}(Y) \to L^p_{\text{loc}}(X)$, if $\mu > -(n-k)|1/p-1/2|$, $1 < p < \infty$.

The application of the arguments of [5] to Theorem 1 yields that an operator $T$ as in Theorem 1 is not bounded as a linear operator in Sobolev spaces $L^p_{\alpha} \to L^p_{\alpha-(n-k)|1/p-1/2|-\mu}$, $1 < p < \infty$.

It is well known ([2]) that pseudo-differential operators of zero order are continuous in $L^p$-spaces, $1 < p < \infty$. It turns out that all elliptic Fourier integral operators with this property can be obtained from pseudo-differential operators by a smooth coordinate change in one of the spaces $X$ or $Y$. For a smooth map $\kappa : X \to Y$ the pullback by $\kappa$ is a mapping $\kappa^* : C^\infty(Y) \to C^\infty(X)$ defined by $(\kappa^* f)(x) = f(\kappa(x))$. This pullback is a Fourier integral operator with the canonical relation corresponding to the phase function $(\kappa(x) - y, \eta)$ and given by the graph of the induced transformation $\tilde{\kappa} : T^*X\backslash 0 \to T^*Y\backslash 0$ with $\tilde{\kappa}(x, \xi) = (\kappa(x), -(t^iD\kappa_x)^{-1}(\xi))$. See [1, 2.4] for more detailed discussion.

**Theorem 2** Let $T \in \mathcal{I}^0(X, Y; \Lambda)$ be elliptic and assume $\Lambda$ to be a local graph, $1 < p < \infty$, $p \neq 2$. Then $T$ is continuous from $L^p_{\text{comp}}(Y)$ to $L^p_{\text{loc}}(X)$ if and only if there exist $P \in \Psi^0(X), Q \in \Psi^0(Y)$, such that $T = P \circ \kappa^*_+ = \kappa^*_- \circ Q$, where $\kappa^*_-$ and $\kappa^*_+$ are the pullbacks by smooth coordinate changes $X \to Y$. 
Proof. The operators $\kappa_+^*$ and $\kappa_-^*$ are clearly $L^p$ continuous, and this together with the continuity of pseudo-differential operators of order 0 imply the continuity of $T$. Conversely, let $k$ be a minimal codimension of $\Sigma = \pi_{X \times Y}(\Lambda)$ in $X \times Y$, i.e. $2n - k = \max_{\lambda \in \Lambda} \text{rank} d\pi_{X \times Y}|_{\Lambda}(\lambda)$. Then Theorem 1 together with our assumption of the continuity of $T$ imply $k = n$. This means that $\text{rank} d\pi_{X \times Y}|_{\Lambda} = n$ and $\Sigma$ is a smooth $n$-dimensional submanifold of $X \times Y$. The rank of $d\pi_X|_{\Sigma}$ of the projection $\pi_X : X \times Y \to X$ is equal to $n$ in view of the assumption on $\Lambda$ to be a local graph. The surjectivity of $d\pi_X|_{\Sigma}$ together with $\dim \Sigma = n$ imply that $\pi_X|_{\Sigma}$ is a diffeomorphism, and locally $\Sigma = \{(x, \sigma(x))\}$, $\sigma$ a diffeomorphism. The pullback operator $\kappa_+^* = \sigma^*$ has the canonical relation equal to the conormal bundle of $\Sigma$, which is $\Lambda$, implying that the operator $Q$ in $T = \kappa_+^* \circ Q$ is pseudodifferential. The same argument applies for $Y$ space to yield the second part of the Theorem. \qed

Finally we would like to make some remarks about $L^p(Y) \to L^q(X)$-continuity. Under the factorization assumptions of [5], the interpolation between $L^p \to L^p$ and $H^1 \to L^2$ for operators of order $-n/2$ ([7, Ch. 3.5.21]) yields that for $1 < p \leq q \leq 2$ and $2 \leq p \leq q < \infty$ the operators $T \in I^\mu(X, Y; \Lambda)$ are continuous from $L^p(Y)$ to $L^q(X)$ for $\mu \leq -n/p + k/q + (n - k)/2$. Note that for $k = 1$ we get the orders of [7, Ch. 9.6.15]. The technique of the proof of Proposition 1 can be applied to show that an elliptic operator $T \in I^\mu(X, Y; \Lambda)$ with maximal rank equal to $2n - k$ at some point is not continuous from $L^p(Y)$ to $L^q(X)$ if $\mu > (n - k)/2 - n/p + k/q$, which shows that the orders above are sharp. A straightforward generalization of Theorem 2 yields

**Theorem 3** Let $T \in I^\mu(X, Y; \Lambda)$ be elliptic and assume $\Lambda$ to be a local graph, $1 < p \leq q < 2$. Assume that $-n(1/p - 1/q) \geq \mu \geq -(1/q - 1/2) - n(1/p - 1/q)$. Then $T$ is continuous from $L^p_{\text{comp}}(Y)$ to $L^q_{\text{loc}}(X)$ if and only if there exist $P \in \Psi^\mu(X)$, $Q \in \Psi^\mu(Y)$, such that $T = P \circ \kappa_-^* = \kappa_+^* \circ Q$, where $\kappa_-^*$ and $\kappa_+^*$ are the pullbacks by smooth coordinate changes $X \to Y$.

The converse statement follows from $L^p \to L^q$-continuity of pseudo-differential operators of order $-n(1/p - 1/q)$, which can be obtained from [7, Ch. 9.6.15] by Hardy-Littlewood argument or by interpolation between $H^1 \to L^2$ and $L^p \to L^p$ for zero order operators. Note that the argument of Proposition 1 with $k = n$ implies that this order is also sharp. By duality...
the same conclusion holds for $2 < p \leq q < \infty$. Finally we would like to note that because the graphs of the transformations $\kappa^*_+$ and $\kappa^*_-$ in Theorems 2 and 3 are the same, it follows that $\kappa_+$ and $\kappa_-$ are equal.

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