Perfect braided crossed modules and their mod-\(q\) analogues

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Abstract. In this paper, we consider the extension theory of braided crossed modules. In particular, we prove the braided version of Norrie's theorem and its mod-\(q\) analogues.

Key words: crossed module, braided crossed module, mod-\(q\) non-Abelian tensor product.

1. Introduction

Crossed modules are known in many areas. For example, in non-Abelian homological algebra, crossed modules play the role of coefficients for degree two cohomology groups (see [1]). Alternatively, Brown and Spencer [8] obtained certain crossed modules as the fundamental groupoids of topological groups.

Higher dimensional groupoids are known too. For example, Brown and Higgins [5] defined the fundamental double groupoid of a pair of spaces, and Loday [16] developed the point of view to the fundamental \(\text{cat}^n\)-group \(\text{IX}\) of a \(n\)-cube of spaces \(X\). Among other results, he proved the equivalence between \(\text{cat}^2\)-groups and crossed squares, and braided crossed modules appeared as a special case of crossed squares. In the work of Bullejos and Cegarra [9], braided crossed modules were used as coefficients for certain degree three non-Abelian cohomology groups. More generally, Breen [1] considered, as the objects of degree three non-Abelian cohomology groups, the extensions of the form:

\[ 1 \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow k, \]

where \(\mathcal{G}, \mathcal{H}\) are crossed modules and \(k\) is a group. Thus it is quite natural to consider the case where \(k\) is also a crossed module, braided crossed module and so on.

By use of the Brown-Loday non-Abelian tensor product of groups, Norrie [18] determined the universal central extensions of perfect crossed modules.
modules. The Brown-Loday non-Abelian tensor product of groups was extended to mod-q tensor product by D. Conduché and C. Rodríguez-Fernández, and Doncel-Juárez and Grandjeán L.-Valcárcel used this to obtain the mod-q analogue of Norrie’s theorem.

In this paper, we shall consider the extension theory of braided crossed modules and prove the braided version of Norrie’s theorem and its mod-q analogues.

2. Preliminaries

We shall recall some definitions and properties of crossed modules and braidings on them.

**Definition 1** Let $N$ and $G$ be groups together with a homomorphism $\partial : N \to G$. This $\partial : N \to G$ is called a crossed module if $G$ acts on $N$ and satisfies the following conditions:

1. $\partial^g(n) = g\partial(n)g^{-1}, \quad g \in G, \quad n \in N,$
2. $\partial(n)n' = nn'n^{-1}, \quad n, n' \in N.$

**Example 1.** For a group $G$, the identity map $G \to G$ together with the action $g'g = gg'g^{-1}$ defines a crossed module.

**Definition 2** Let $(M, P, \partial), \ (N, G, \partial')$ be crossed modules. A crossed module morphism $(\varphi, \psi) : (M, P) \to (N, G)$, is a pair of group homomorphisms, $\varphi : M \to N$ and $\psi : P \to G$, such that

1. $\psi\partial = \partial'\varphi,$
2. $\varphi^g(n) = \psi^g(\varphi(n)), \quad g \in P, \quad n \in M.$

When $\varphi$ and $\psi$ are surjective, the morphism is called an extension.

**Definition 3** For a non-negative integer $q$, the $q$-center of a crossed module $N \to G$ is the crossed module

$$(N^G)^q \to Z(G)^q \cap St_G(N), \quad \text{where}$$

$$(N^G)^q = \{n \in N; n^q = 1, ^g n = n, \quad g \in G\}$$

$Z^q(G) = \{g \in Z(G); g^q = 1\}$$

In particular, we call the 0-center the center of $N \to G$.

**Definition 4** An extension of a crossed module is called $q$-central if the crossed module $\ker\varphi \to \ker\psi$ is contained in the $q$-center of the crossed
module $N \rightarrow G$. In particular, we call the 0-centeral extension the centeral extension.

**Definition 5** When $N \rightarrow G$ is a crossed module, the $q$-commutator crossed module is defined as a crossed module

$$D^q_G(N) \rightarrow [G, G]^q$$

where $D^q_G(N)$ is the subgroup of $N$ generated by

$$\{gn^{-1}r^q; g \in G, n, r \in N\}$$

and $[G, G]^q$ is the subgroup of $G$ generated by

$$\{[g, h]k^q; g, h, k \in G\}$$

In particular, we call the 0-commutator crossed module the commutator crossed module.

**Definition 6** A crossed module $N \rightarrow G$ is called $q$-perfect if it coincides with the $q$-commutator crossed module. In particular, we call the 0-perfect crossed module the perfect crossed module.

Based on the earlier works of Dennis [12] and Miller [17], Brown and Loday [6] defined the notion of non-Abelian tensor product $M \otimes N$ of two crossed modules. Later, the notion of mod-$q$ exterior product of groups, for a non-negative integer $q$, was introduced by Ellis [14], and Brown [3] defined the mod-$q$ non-Abelian tensor product $G \otimes^q G$ of group $G$.

The following definition of the mod-$q$ non-Abelian tensor product of crossed modules is due to Conduché and Rodríguez-Fernández [11].

**Definition 7** Let $(M, G, \partial), (N, G, \partial')$ be two crossed modules and $q$ a non-negative integer. Then the tensor product $M \otimes^q N$ is defined as a group generated by the symbols

$$a \otimes^q b (a \in M, b \in N) \quad \text{and} \quad \{k\} (k \in M \times_G N)$$

with the following relations:

1. $a \otimes^q bc = (a \otimes^q b)(b \otimes^q c)$,
2. $ab \otimes^q c = (ab \otimes^q a)(a \otimes^q c)$,
3. $\{k\}(a \otimes^q b)\{k\}^{-1} = a(k)^q \otimes^q \alpha(k)^q b$,
4. $[[k], \{h\}] = \pi_1(k)^q \otimes^q \pi_2(h)^q$. 
\[(5) \{kh\} = \{k\}(\Pi(\pi_1(k)^{-1} \otimes^q (\alpha(k)^{1-q+i}\pi_2(h)))\{h\}, \]
\[(6) \{(a^b a^{-1}, ab b^{-1})\} = (a \otimes^q b)^q\]

where \(\alpha = \partial \circ \pi_1\).

Note that the Brown-Loday non-Abelian tensor product \(M \otimes N\) can be regarded as the special case where the generators are just \(a \otimes^0 b\) \((a \in M, b \in N)\) and the relations are just \((1)\) and \((2)\). Besides, it was shown in [6] that, for a group \(G\), the following identities hold in \(G \otimes G\):
\[(a) \ (a \otimes b)(c \otimes d)(a \otimes b)^{-1} = [a, b]c \otimes [a, b]d,\]
\[(b) \ [a, b] \otimes c = (a \otimes b)(^c a \otimes cb),\]
\[(c) \ a \otimes [b, d] = (^a b \otimes ac)(b \otimes c)^{-1},\]
for all \(a, b, c \in G\), \([a, b] = aba^{-1}b^{-1}\).

We next consider braidings on crossed modules.

**Definition 8** A braiding on a crossed module \(\partial : N \rightarrow G\) is a map \(\{ , \} : G \times G \rightarrow N\) (bracket operation) satisfying the following conditions:
\[(1) \ \partial\{a, b\} = aba^{-1}b^{-1}\]
\[(2) \ \{\partial(n), b\} = n^b n^{-1}\]
\[(3) \ \{a, \partial(n)\} = ^a nn^{-1}\]
\[(4) \ \{a, bc\} = a^b \{a, c\}, a, b, c \in G, n \in N.\]
\[(5) \ \{ab, c\} = a\{b, c\}\{a, c\}, a, b, c \in G, n \in N.\]

**Example 2.** There are canonical braidings on the crossed modules id: \(G \rightarrow G\) and \(G \otimes G \rightarrow G\), \(a \otimes b \mapsto [a, b]\) by the following maps:
\[G \times G \rightarrow G, \ (a, b) \mapsto [a, b] = aba^{-1}b^{-1},\]
\[G \times G \rightarrow G \otimes G, \ (a, b) \mapsto a \otimes b.\]

**Definition 9** A morphism between two braided crossed modules is defined as a crossed module morphism which preserves the braiding structures. In particular, a \(q\)-central extension of a braided crossed module is a \(q\)-central extension of the underlying crossed module which preserves the braiding structures.

3. **Canonical braidings and their universalities**

To construct new braidings, we start from the following observation:

**Proposition 1** If a crossed module \(N \rightarrow G\) has a braiding \(\{ , \}\), then there is a group homomorphism \(G \otimes G \rightarrow N\), \(a \otimes b \mapsto \{a, b\}\).
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Proof. Let us check that \( f \) preserves the defining relations in \( G \otimes G \). By the definitions, we have

\[
\begin{align*}
  f(a \otimes bc) &= \{a, bc\} = \{a, b\}^b\{a, c\}, \\
  f(a \otimes b)f(b \otimes b c) &= \{a, b\}^b\{a, b c\}.
\end{align*}
\]

But by a result of Conduché [10], any braiding is equivariant (i.e., \( a\{b, c\} = \{^{a}b^{a},c\} \)), so that \( f(a \otimes bc) = f(a \otimes b)f(b \otimes b c) \). The other relation can be proved by the same computation. \( \square \)

We next consider the \( q \)-tensor analogues. The main difference is the existence of the elements \( \{k\} \), and to construct a well behaved map on \( G \otimes^{q} G \), we assume that crossed modules \( N \twoheadrightarrow G \) are \( q \)-central extensions of \( G \).

**Proposition 2** When a crossed module \( \partial : N \twoheadrightarrow G \) is a \( q \)-central extension of \( G \) and has a braiding \( \{ , \} \), there is a group homomorphism \( f : G \otimes^{q} G \twoheadrightarrow N, a \otimes b \longmapsto \{a, b\}, \{k\} \longmapsto s(k)^q \) (\( s \) is a section of \( \partial \)).

**Proof.** We have to check that \( f \) preserves the relations (3)-(6) in mod-q tensor product. We first consider the relation (3). Then we have

\[
  f(\{k\}(a \otimes^{q} b)\{k\}^{-1}) = s(k)^q\{a, b\}s(k)^{-q} = k^q\{a, b\} = f(k^q a \otimes^q k^q b).
\]

We next consider the relation (4). Then we have

\[
  f([\{k\}, \{h\}]) = [s(k)^q, s(h)^q] = s(k)^q s(h)^q (s(h)^q)^{-1} = k^q s(h)^q (s(h)^q)^{-1} = \{k^q, h^q\}.
\]

For the relation (5), we have

\[
  [s(k)^{-1}, (k^{-1}g^{-1}h^i)]s(h)^q = s(k)^q (\prod (k^{-1}g^{-1}h^i))s(h)^q.
\]

Finally, we consider the relation (6). Then we have

\[
  f(\{(k^h k^{-1}, k h h^{-1})\}) = s([k, h])^q,
\]

and because \( s([k, h]) \) and \( \{k, h\} \) have the same image under \( \partial \), \( s([k, h])^q \) coincides with \( \{k, h\}^q \). \( \square \)

We proceed to construct a canonical braiding on \( \rho : N \otimes G \twoheadrightarrow G \otimes G \) when \( N \twoheadrightarrow G \) is braided with a braiding \( \{ , \} \). Define \( \underline{\{ , \}} : G \otimes G \times G \otimes G \twoheadrightarrow N \otimes G \) by

\[
  \underline{\{ , \}} : (a \otimes b, c \otimes d) \longmapsto \{a, b\} \otimes [c, d].
\]

Then we have the following proposition:

**Proposition 3** \( \underline{\{ , \}} \) satisfies the braiding conditions.

**Proof.** The proof is by computations:
We first consider the identity (1). If we take $a = a \otimes b$, $b = c \otimes d$, we have $\rho(\{a \otimes b, c \otimes d\}) = \rho(\{a, b\} \otimes [c, d]) = \partial\{a, b\} \otimes [c, d] = [a, b] \otimes [c, d]$, so that we need the following identity:

$$(a \otimes b)(c \otimes d)(a \otimes b)^{-1}(c \otimes d)^{-1} = [a, b] \otimes [c, d],$$

but this is the product of (a) and (b) in page 4.

The identities (2) and (3) are proved by a result in Brown, Loday [6]. Alternatively, one can prove them using a technique which will be described in Lemma 2.

We next consider the identity (4). If we take $a = a \otimes b$ and $bc = (c \otimes d)(c' \otimes d')$, we have $\{a \otimes b, (c \otimes d)(c' \otimes d')\} = \{a, b\} \otimes [c, d][c', d']$. On the other hand, we have $\{a \otimes b, c \otimes d\}^{c \otimes d}\{a \otimes b, d' \otimes c'\} = (\{a, b\} \otimes [c, d])^{c \otimes d}\{a, b\} \otimes [c', d'] = \{a, b\} \otimes [c, d][c', d']$.

Finally, we consider the identity (5). If we take $ab = (a \otimes b)(a' \otimes b')$ and $c = c \otimes d$, we have $\{(a \otimes b)(a' \otimes b'), c \otimes d\} = \{a, b\}\{a', b'\} \otimes [c, d]$. On the other hand, $a^{b}a^{-1} \otimes [a', b'] = (a \otimes b)(a' \otimes b')^{-1} = \{a, b\} \otimes [a', b']$.

Remark 1. In (4), (5) the property $\partial(\{a, b\}) = [a, b]$ and $\partial(n)n' = nn'n^{-1}$ were used.

When a crossed modules $N \longrightarrow G$ is a $q$-central extension of $G$ and equipped with a braiding $\{,\}$, one can use Proposition 2 to define a canonical braiding $\{,\}^{q}$ on $N \otimes^{q} G \longrightarrow G \otimes^{q} G$.

Before checking the braiding conditions, we prove the next lemma.

**Lemma 1** In $N \otimes^{q} G$, the next identities hold:

(a) $a^{b}a^{-1} \otimes^{q} h^{q} = (a \otimes^{q} b)(h^{q} a \otimes^{q} h^{q} b)^{-1}$,

(b) $\{n\}^{q} \otimes^{q} [a, b] = \{n\}\{a, b\}^{-1}$,

(c) $n^{q} \otimes^{q} h^{q} = \{n\}^{h^{q} n^{-1}}$.

**Proof.** Recall that for two crossed modules $(M, G, \partial)$ and $(N, G, \partial')$, Doncel-Juárez and Grandjeán L.-Valcárcel constructed the following crossed module $\rho : M \otimes^{q} N \longrightarrow G \otimes^{q} G$:

$$\rho(m \otimes n) = \partial(m) \otimes \partial'(n), \rho(\{k\}) = \{\partial(\pi_{1}(k))\}$$
(a ⊗ b)(m ⊗ n) = [a, b]m ⊗ [a, b]n, (a ⊗ b){k} = {[a, b]k},
{h}(m ⊗ n) = h^q m ⊗ h^q n, {h}{k} = {h^q k},

([a, b] = aba^{-1}b^{-1}, a, b, h ∈ G, m ∈ M, n ∈ N, k ∈ M ×_G N, π_1 : M ×_G N → M), and proved that N ⊗^q G → G ⊗^q G becomes the universal central extension of a crossed module N → G.

To prove the identities (a) ∼ (c), we use the universality of N ⊗^q G, and show that, for any q-central extension (X_1, X_2, ∂') of (N, G, ∂), the unique map φ_1 : N ⊗^q G → X_1 defined by φ_1(n ⊗^q g) = s_1(n) s_2(g) s_1(n)^{-1}, φ_1({h}) = s_1(h)^q, where s_1 and s_2 are sections of ψ_1 : X_1 → N and ψ_2 : X_2 → G respectively, preserves the relations.

We first check the identity (a). By the definition, we have

φ_1(a^b a^{-1} ⊗^q h^q) = s_1(a^b a^{-1}) s_2(h^q) s_1(a^b a^{-1})^{-1}.

But because s_1(a^b a^{-1}) s_2(h^q) s_1(a^b a^{-1})^{-1} has a form x^y x^{-1} in X_1, we can change s_1(a^b a^{-1}) to s_1(a) s_2(b) s_1(a)^{-1}. Then we have

s_1(a^b a^{-1}) s_2(h^q) s_1(a^b a^{-1})^{-1} = (s_1(a) s_2(b) s_1(a)^{-1}) s_2(h^q) (s_1(a) s_2(b) s_1(a)^{-1})^{-1}.

On the other hand, we have

φ_1((a ⊗ b)(h^q a ⊗ h^q b)^{-1}) = (s_1(a) s_2(b) s_1(a)^{-1}) φ_1(h^q a ⊗ h^q b)^{-1} = (s_1(a) s_2(b) s_1(a)^{-1}) (s_1(h^q a) s_2(h^q b) s_1(h^q a)^{-1})^{-1}.

Hence we should prove the formula:

s_2(h^q) (s_1(a) s_2(b) s_1(a)^{-1})^{-1} = (s_1(h^q a) s_2(h^q b) s_1(h^q a)^{-1})^{-1},

but notice that the latter has the form (x^y x^{-1})^{-1}. Thus we can replace s_1(h^q a) by s_2(h^q) s_1(a) and s_2(h^q b) by s_2(h^q) s_2(b) s_2(h^q)^{-1}.

We next check the identity (b). By the definition, we have

φ_1({n} q ⊗ [a, b]) = s_1(n) s_2([a, b]) s_1(n)^{-1} = (s_1(n) q) s_2([a, b]) (s_1(n) q)^{-1}.

On the other hand, we have

φ_1({n} {[a, b]} n)^{-1} = s_1(n) q (s_1([a, b]) n)^{-1}.

But because $s_2([a,b])s_1(n)$ and $s_1([a,b], n)$ have the same image by $\psi_1 : X_1 \rightarrow N$, one can see that, by the property of $q$-central extensions of a crossed module, $s_2([a,b]) (s_1(n)^q)^{-1}$ coincides with $(s_1([a,b], n)^q)^{-1}$.

Finally, we check the identity (c). By the definition, we have

$$\varphi_1(n^q \otimes^q h^q) = s_1(n^q)s_2(h^q)s_1(n^q)^{-1} = s_1(n)^q(s_2(h^q)s_1(n))^{-1}.$$  

On the other hand, we have

$$\varphi_1(\{n\}\{h^q\}^{-1}) = s_1(n)^q(s_1(h^q))^{-1}.$$  

But one can easily see that $s_2(h^q)s_1(n)$ and $s_1(h^q)$ have the same image by $\psi_1$. Thus the result follows. \(\square\)

**Proposition 4** \{ , \}^q becomes a braiding on $N \otimes^q G \rightarrow G \otimes^q G$.

**Proof.** By the end of this proof, we denote \{ , \}^q by \{ , \}. When the elements \{k\} do not appear in the relations, they are derived from the results for \{ , \}. So we consider the case where the elements \{k\} are appearing in the relations.

We first consider the relation (1). If we take $a = \{k\}$ and $b = c \otimes^q d$, we have $\rho(\{k\}, c \otimes^q d) = \rho(s(k)^q \otimes^q \{c, d\}') = k^q \otimes^q [c, d]$. On the other hand, we have $\{k\}(c \otimes^q d)^{-1} = (k^q(c \otimes^q d)^{-1})$. Hence we need the identity:

$$k^q \otimes^q [c, d] = (k^q c \otimes^q k^q d)(c \otimes^q d)^{-1},$$

but this is the formula (c) applied to mod-$q$ tensor product with $a = k^q$, $b = c$, $c = d$.

We next consider the relation (2). If we take $n = a \otimes^q b$ and $b = \{h\}$, then by the definition we have $\{\partial(a) \otimes^q b, \{h\}\} = \{\partial(a), b\} \otimes^q h^q = a^b a^{-1} \otimes^q h^q$. On the other hand, we have $(a \otimes^q b)^{\{h\}}(a \otimes^q b)^{-1} = (a \otimes^q b)(^q a \otimes^q h^q b)^{-1}$. Thus by Lemma 1 (a), they coincide. If we take $n = \{n\}$ and $b = a \otimes^q b$, then we have $\{\rho(n), a \otimes^q b\} = n^q \otimes^q [a, b]$. On the other hand, we have $n^{a \otimes^q b} = n^{\{a, b\}}$. Thus by Lemma 1 (b), they coincide. If we take $n = \{n\}$ and $b = \{h\}$, we have $\{\rho(n), \{h\}\} = n^q \otimes^q h^q$. On the other hand, we have $\{n\}^{\{h\}} = n^{\{h\}}$. Thus by Lemma 1 (c), they coincide.

The relation (3) follows by the same computations.

We next consider the relation (4). If we take $a = \{k\}$ and $bc =
(a \otimes^q b)(c \otimes^q d), we have \{\{k\}, (a \otimes^q b)(c \otimes^q d)\} = s(k)^q \otimes^q [a, b][c, d]. On the other hand, we have \{\{k\}, a \otimes^q b\}^{(a \otimes b)(s(k)^q \otimes^q [a, b])} = (s(k)^q \otimes^q [a, b])(a \otimes b)(s(k)^q \otimes^q [a, b][c, d]) = s(k)^q \otimes^q [a, b][c, d]. If we take a = \{k\} and bc = \{h\}(c \otimes^q d), we have \{\{k\}, (c \otimes^q d)\} = s(k)^q \otimes^q s(h)^q[c, d]. On the other hand, we have \{(\{k\}, \{h\})\}^{(\{k\}, c \otimes^q d)} = (s(k)^q \otimes^q s(h)^q)(\{h\}(s(k)^q \otimes^q [c, d]) = (s(k)^q \otimes^q s(h)^q)(s(k)^q \otimes^q h^q[c, d]) = s(k)^q \otimes^q s(h)^q[c, d]. If we take a = \{k\}

If we take a = \{k\} and bc = (c \otimes^q d){\{h\}}, we have \{\{k\}, (c \otimes d){\{h\}}\} = s(k)^q \otimes^q [c, d](h)^q. On the other hand, we have \{\{k\}, c \otimes^q d\} = (c \otimes^q d)(s(k)^q \otimes^q [c, d]) = (s(k)^q \otimes^q [c, d])(s(k)^q \otimes [c, d]s(h)^q) = s(k)^q \otimes [c, d]s(h)^q.

(5) Omitted.

We have so far been concerned with constructing canonical braidings on the crossed modules \(N \otimes G \longrightarrow G \otimes G\) and \(N \otimes^q G \longrightarrow G \otimes^q G\). Since it is known that \(N \otimes G \longrightarrow G \otimes G\) (\(N \otimes^q G \longrightarrow G \otimes^q G\)) are the universal \((q\)-universal) central extensions of perfect \((q\)-perfect) crossed modules \(N \longrightarrow G\), it is quite natural to consider their braided version.

The next proposition shows that the canonical braidings \(\{\ , \\}\) on the crossed modules \(N \otimes G \longrightarrow G \otimes G\) are compatible with \(\{\ , \\}\).

**Proposition 5** The next diagram becomes commutative.

\[
\begin{array}{ccc}
(G \otimes G) \otimes (G \otimes G) & \longrightarrow & N \otimes G \\
\xi \times \xi & \downarrow & \lambda \\
G \otimes G & \longrightarrow & N
\end{array}
\]

**Proof.** It is enough to show that the next diagrams commute:

(1) \( (G \otimes G) \otimes (G \otimes G) \longrightarrow N \otimes G \) (2) \( N \otimes G \)

\[
\begin{array}{ccc}
G \otimes G & \longrightarrow & G \otimes G \\
\xi \times \xi & \downarrow & \lambda \\
G \otimes G & \longrightarrow & N
\end{array}
\]

The diagram (1) becomes commutative because of the braiding condition (1). The triangle (2) also becomes commutative by the braiding
condition (2) for \{ , \}. 

Thus we know that the braided crossed module \((N \otimes G \rightarrow G \otimes G, \{ , \})\) is an extension of \((N \rightarrow G, \{ , \})\). Furthermore, this braiding has a universal property.

**Theorem 1** If \((N \rightarrow G, \{ , \})\) is a perfect braided crossed module, and 
\((X_1 \overset{\Omega}{\rightarrow} X_2, \{ , \}')\) is a central extension of it with a compatible braiding, then the next diagram becomes commutative.

\[
\begin{array}{ccc}
(G \otimes G) \times (G \otimes G) & \rightarrow & N \otimes G \\
\downarrow & & \downarrow \\
X_2 \times X_2 & \rightarrow & X_1
\end{array}
\]

**Proof.** Define
\[r : G \otimes G \rightarrow X_1\] to be \(r = \{ , \}' \circ s_2\) (by choosing a section \(s_2 : G \rightarrow X_2\) and extending it on \(G \otimes G\)),
\[t : G \otimes G \rightarrow X_2, a \otimes b \mapsto [s_2(a), s_2(b)]\], by the same \(s_2\),
\[p = r \times t, q = \Omega \times id.\]

Let us consider the next diagram and show that each triangle commutes.

\[
\begin{array}{ccc}
(G \otimes G) \times (G \otimes G) & \rightarrow & N \otimes G \\
\downarrow & & \downarrow \\
X_2 \times X_2 & \rightarrow & X_1
\end{array}
\]

By the definitions, the diagram (1) becomes naturally commutative because the diagram (*) is commutative.
The next diagram (2) also becomes commutative because the diagram (**) is commutative by the braiding condition (2) and the choice of r.

Finally let us see the next diagram commutes.

It follows again by the braiding condition (2) and the constructions. □

Corollary 1 If \((N \to G, \{ , \})\) is a \(q\)-perfect braided crossed module with \(N\) being a \(q\)-central extension of \(G\), then \((N \otimes^q G \to G \otimes^q G, \{ , \}^q)\) becomes the universal \(q\)-central extension of it.

It follows because we can construct the similar maps by \(r(\{k\}) = (s_1 \circ s(k))^q\) and \(t(\{k\}) = (\omega \circ s_1 \circ s(k))^q\).

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