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A note on an isometric imbedding of upper half-space into the anti de Sitter space

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Introduction. K. Nomizu [3] studied the upper half-space $U_n=\{(x_1, \cdots, x_n)\,;\ x_n>0,\ x_1, \cdots, x_{n-1}\in \boldsymbol{R}\}$ with the Lorentz metric

$$(1) \qquad ds_0^2=(dx_1{}^2+\cdots+dx_{n-1}{}^2-dx_n{}^2)/x_n{}^2$$

which has constant sectional curvature 1. U_n is diffeomorphic to the matrix group G_n consisting of all $n\times n$ matrices of the form

$$g=\begin{bmatrix} x_n & & x_1 \\ & \ddots & \\ & x_n & x_{n-1} \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } x_n>0,\ x_1, \cdots, x_{n-1}\in \boldsymbol{R}$$

by

$$g\in G_n \longrightarrow (x_1, \cdots, x_{n-1}, x_n)\in U_n\,.$$

The group G_n is of type $\mathfrak{S}$ in the sense of [2] and it admits a left-invariant Lorentz metric with any prescribed constant k as its constant sectional curvature (Theorem 1, [2]). The left translations on G_n

$$\begin{bmatrix} x_n & & x_1 \\ & \ddots & \vdots \\ & x_n & x_{n-1} \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} a & & b_1 \\ & \ddots & \vdots \\ & a & b_{n-1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_n & & x_1 \\ & \ddots & \vdots \\ & x_n & x_{n-1} \\ 0 & 0 & 1 \end{bmatrix}$$

correspond to the action of G_n on U_n by

$$(2) \qquad (x_1, \cdots, x_{n-1}, x_n) \longrightarrow (ax_1+b_1, \cdots, ax_{n-1}+b_{n-1}, ax_n)\,.$$

The Lorentz metric (1) on U_n is invariant by the action (2) of G_n and corresponds to a left-invariant Lorentz metric on the group G_n of constant sectional curvature 1.

In this note, we shall consider the upper half-space U_n with the Lorentz metric

$$(3) \qquad ds^2=(-dx_1{}^2+dx_2{}^2+\cdots+dx_n{}^2)/x_n{}^2$$

which corresponds to a left invariant Lorentz metric on G_n of constant sectional curvature -1. The metric (3) is not geodesically complete (see § 1), but there exists an isometric imbedding of U_n into the anti de Sitter space H_1^n and this imbedding is equivariant relative to an isomorphism of the largest connected isometry group of U_n into the largest connected group $SO(2, n-1)$ of isometries of H_1^n (see § 3).

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1. Geodesics of (U_n, ds^2). On the upper half-space $U_n=\{(x_1, \cdots, x_n);\ x_n>0,\ x_1, \cdots, x_{n-1}\in\boldsymbol{R}\}$, let X_i be $\partial/\partial x_i$ $(i=1, \cdots, n)$ and ∇ the Levi-Civita connection for the metric (3). Then we have easily

$$(4)\qquad \begin{aligned} &\nabla_{X_i} X_i = X_n/x_n && (i=2, \cdots, n-1)\\ &\nabla_{X_1} X_1 = -X_n/x_n && \\ &\nabla_{X_i} X_n = \nabla_{X_n} X_i = -X_i/x_n && (i=1, \cdots, n)\\ &\nabla_{X_i} X_j = 0 && (\text{otherwise}). \end{aligned}$$

From these we can calculate the curvature tensor R as follows

$$\begin{aligned} &R(X_1, X_j)\, X_1 = -X_j/x_n^2 && (j=2, \cdots, n)\\ &R(X_i, X_j)\, X_i = X_j/x_n^2 && (i\neq 1,\ i\neq j,\ j=1, \cdots, n)\\ &R(X_i, X_n)\, X_n = -X_i/x_n^2 && (i\neq n)\\ &R(X_i, X_j)\, X_k = 0 && (\text{otherwise}). \end{aligned}$$

Thus

$$R(X, Y)\, Z = -\big(\langle Y, Z\rangle X - \langle X, Z\rangle Y\big)$$

for any tangent vectors X, Y, Z (where $\langle\ ,\ \rangle$ denotes the inner product by the metric (3)). Hence (U_n, ds^2) has constant sectional curvature -1 ([1], Lemma 2).

When $n=2$, a geodesic $\gamma(t)=(x_1(t), x_2(t))$ with affine parameter t satisfies the differential equations

$$\begin{cases} d^2 x_1/dt^2 = 2(dx_1/dt)\,(dx_2/dt)/x_2 \\ d^2 x_2/dt^2 = \big\{(dx_1/dt)^2+(dx_2/dt)^2\big\}\big/x_2 \end{cases}$$

This equations appear in [3], and all the type of geodesics are determined. In our case, we may interprete time-like (resp. space-like) geodesics in [3] as space-like (resp. time-like) geodesics. From now on we assume $n\geqq 3$.

Now, let $\gamma(t)=(x_1(t), \cdots, x_n(t))$ be a geodesic with t as affine parameter. Then we get the differential equations for $\gamma(t)$ as follows;

$$(5)\qquad \begin{cases} \dfrac{d^2x_i}{dt^2}=2\left(\dfrac{dx_i}{dt}\right)\left(\dfrac{dx_n}{dt}\right)\Big/x_n \qquad (i=1,\cdots,n-1) \\ \dfrac{d^2x_n}{dt^2}=\left(\left(\dfrac{dx_1}{dt}\right)^2+\left(\dfrac{dx_n}{dt}\right)^2-\sum_{i=2}^{n-1}\left(\dfrac{dx_i}{dt}\right)^2\right)\Big/x_n\,. \end{cases}$$

Since an isometry group G_n acts transitively on U_n, we consider only the geodesics starting from $p_0=(0,\cdots,0,1)$. Let $\dot{\gamma}(0)=\sum_{i=1}^{n}c_iX_i(p_0)$ be the initial tangent vector of γ. By an appropriate rotation of the variables $x_2,\cdots,x_{n-1}$ (which is an isometry of the metric), we may assume that $c_3=\cdots=c_{n-1}=0$. From the equations (5), it follows that $x_3(t),\cdots,x_{n-1}(t)$ are constant in this case. Thus it is enough to study the geodesic behaviors of U_n in the case $n=3$.

For $n=3$, we write x, y, z instead of x_1, x_2, x_3. The equations (5) are

$$(5')\qquad \begin{cases} \dfrac{d^2x}{dt^2}=2\left(\dfrac{dx}{dt}\right)\left(\dfrac{dz}{dt}\right)\Big/z \\ \dfrac{d^2y}{dt^2}=2\left(\dfrac{dy}{dt}\right)\left(\dfrac{dz}{dt}\right)\Big/z \\ \dfrac{d^2z}{dt^2}=\left\{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dz}{dt}\right)^2-\left(\dfrac{dy}{dt}\right)^2\right\}\Big/z\,. \end{cases}$$

We shall find the solutions of (5′) with initial conditions $(x(0), y(0), z(0))=(0,0,1)$ and $(x'(0), y'(0), z'(0))=(a,b,c)$. Here and hereafter, we denotes d/dt by prime ′. Then we get $(x'/z^2)'=(y'/z^2)'=0$ so that

$$(6)\qquad x'=az^2\,,\qquad y'=bz^2\,.$$

We get also $(z'/z)'=(z''z-(z')^2)/z^2=((x')^2-(y')^2)/z^2$ so that

$$(7)\qquad z'=z(ax-by+c)\,.$$

From (6) and the initial condition,

$$(8)\qquad ay=bx\,.$$

Case I: $a\neq 0$. From (6), (7) and (8), we get

$$azz'=(a^2-b^2)\,xx'/a+cx'$$

so that

$$(9)\qquad z^2=(a^2-b^2)\,x^2/a^2+2cx/a+1\,.$$

Subcase I-(i): $-a^2+b^2+c^2=0$. In the case $c=0$, we get $\gamma(t)=(at, bt, 1)$. This null geodesic is complete in both directions. In the case $c\neq 0$, we get $\gamma(t)=(at/(1-ct), bt/(1-ct), 1/(1-ct))$. This null geodesic is complete in one direction and incomplete in the other direction.

Subcase I-(ii): $-a^2+b^2+c^2<0$. The curve satisfying (9) and (8) is a branch of hyperbola ($z>0$) in the plane P_{ab} spanned by the position vectors $(0, 0, 1)$ and $(a, b, 0)$. We may parametrize it by

$$x(u)=\frac{a}{\alpha}\left(\frac{\alpha^2-c^2}{\alpha^2}\right)^{\frac{1}{2}}\sinh u-\frac{ac}{\alpha^2}$$

$$y(u)=\frac{b}{\alpha}\left(\frac{\alpha^2-c^2}{\alpha^2}\right)^{\frac{1}{2}}\sinh u-\frac{bc}{\alpha^2}$$

$$z(u)=\left(\frac{\alpha^2-c^2}{\alpha^2}\right)^{\frac{1}{2}}\cosh u$$

where $\alpha=(a^2-b^2)^{\frac{1}{2}}$. The tangent vector $(dx/du, dy/du, dz/du)$ is time-like with length $1/\cosh u$. The proper time parameter t measured from u_0 is given by

$$t(u)=\int_{u_0}^{u} du/\cosh u=\sin^{-1}(\tanh u)-\sin^{-1}(\tanh u_0)$$

where $\sinh u_0=c/(\alpha^2-c^2)^{\frac{1}{2}}$.

This time-like geodesic is incomplete in both directions, because $t(u)\to \pm\pi/2-\sin^{-1}(\tanh u_0)$ as $u\to\pm\infty$.

Subcase I-(iii): $c^2>a^2-b^2>0$. The curves satisfying (9) and (8) are two half-branches of hyperbolas ($z>0$) in the plane P_{ab}. We may parametrize them by

$$x(u)=\pm\left(a(c^2-\alpha^2)^{\frac{1}{2}}/\alpha^2\right)\cosh u-ac/\alpha^2$$

$$y(u)=\pm\left(b(c^2-\alpha^2)^{\frac{1}{2}}/\alpha^2\right)\cosh u-bc/\alpha^2$$

$$z(u)=\left((c^2-\alpha^2)^{\frac{1}{2}}/\alpha\right)\sinh u$$

where $\alpha=(a^2-b^2)^{\frac{1}{2}}$ and $u>0$. The tangent vector $(dx/du, dy/du, dz/du)$ has length $1/\sinh u$ and so the arc-length parameter t measured from $p_0=(0, 0, 1)$ is given by

$$t(u)=\int_{u_0}^{u} du/\sinh u=\log(\tanh u/2)-\log(\tanh u_0/2)$$

where $\sinh u_0=\alpha/(c^2-\alpha^2)^{\frac{1}{2}}$. This space-like geodesic is complete as it approaches the $x-y$ plane and incomplete in the other direction, since $t(u)\to -\infty$ as $u\to 0$ and $t(u)\to-\log(\tanh u_0/2)$ as $u\to\infty$.

Subcase I-(iv): $c^2>a^2-b^2=0$. The curve satisfying (9) and (8) is a half-branch of parabola ($z>0$) in the plane P_{ab}. We may parametrize it by

$$x(u)=au\,,\qquad y(u)=bu\,,\qquad z(u)=(2cu+1)^{\frac{1}{2}}\,.$$

The tangent vector $(dx/du,\ dy/du,\ dz/du)$ has length $|c/(2cu+1)|$ and the arc-length parameter t measured from p_0 is given by

$$t(u)=\frac{1}{2}\int_0^u du\Big/|(u+1/2c)|=\frac{1}{2}\Big\{\log|(u+1/2c)|-\log|1/2c|\Big\}\,.$$

This space-like geodesic is complete in both directions.

Subcase I-(v): $a^2-b^2<0$. The curve satisfying (9) and (8) is an upper half of ellipse $(z>0)$ in the plane P_{ab}. We may parametrize it by

$$x(u)=a\Big((\alpha^2+c^2)^{\frac{1}{2}}/\alpha^2\Big)\cos u+ac/\alpha^2$$

$$y(u)=b\Big((\alpha^2+c^2)^{\frac{1}{2}}/\alpha^2\Big)\cos u+bc/\alpha^2$$

$$z(u)=\Big((\alpha^2+c^2)^{\frac{1}{2}}/\alpha\Big)\sin u\,,\qquad 0<u<\pi$$

where $\alpha=(b^2-a^2)^{\frac{1}{2}}$. The tangent vector $(dx/du,\ dy/du,\ dz/du)$ has length $1/\sin u$ and the arc-length parameter t measured from p_0 is given by

$$t(u)=\int_{u_0}^u du/\sin u=\log(\tan u/2)-\log(\tan u_0/2)$$

where $u_0=\cos^{-1}(c/(\alpha^2+c^2)^{\frac{1}{2}})$. This space-like geodesic is complete in both directions, since $t(u)\to+\infty$ (as $u\to\pi$) and $t(u)\to-\infty$ (as $u\to 0$).

Case II: $a=0$. When $b=0$, we have $\gamma(t)=(0,0,e^{ct})$. This space-like geodesic is complete in both directions. When $b\neq 0$, we have easily

$$z^2+(y-c/b)^2=c^2/b^2(z>0)\,,\qquad x=0\,.$$

We may parametrize it by

$$x=0,\ y(u)=(c^2/b^2+1)^{\frac{1}{2}}\cos u,\ z(u)=(c^2/b^2+1)^{\frac{1}{2}}\,,\qquad 0<u<\pi\,.$$

The tangent vector $(dx/du,\ dy/du,\ dz/du)$ has length $1/\sin u$ and the arc-length parameter t measured from p_0 is given by

$$t(u)=\int_{u_0}^u du/\sin u=\log(\tan u/2)-\log(\tan u_0/2)$$

where $u_0=\cos^{-1}(-c/(b^2+c^2)^{\frac{1}{2}})$. This space-like geodesic is complete in both directions as in the subcase I-(v).

2. Full isometry group. We determine the full isometry group $I(U_n)$ of the space with metric (3). $I(U_n)$ acts transitively on U_n because of the transitivity of the group G_n. In the first, we find the isotropy group at $p_0=(0, \cdots, 0, 1)$. When $n=2$, it is verified by the same argument as [3] that the isotropy group at p_0 consists of matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

From now on, we assume $n \geqq 3$. Suppose that g is an isometry of U_n fixing p_0. The differential dg at p_0 is described as

$$dg(X_j)_{p_0} = \sum_{i=1}^{n} A_{ij}(X_i)_{p_0} \qquad (j=1, \cdots, n)$$
$$\text{with } (A_{ij}) \in O(1, n-1)\,.$$

Let $\gamma(t)$ be the null geodesic starting at p_0 with the initial tangent vector

$$\dot{\gamma}(0) = \sum_{j=1}^{n} a_j(X_j)_{p_0} \qquad (-a_1^2+a_2^2+\cdots+a_n^2=0)\,.$$

Considering an appropriciate rotation of the variable $x_2, \cdots, x_{n-1}$ and the subcase I-(i), we get

$$\gamma(t) = \left(a_1 t/(1-a_n t), \cdots, a_{n-1}/(1-a_n t), 1/(1-a_n t)\right).$$

The null geodesic $\tilde{\gamma}(t)=g\gamma(t)$ has the initial vector

$$\dot{\tilde{\gamma}}(0) = \sum_{j,i=1}^{n} A_{ji} a_i (X_j)_{p_0}$$

so that we get

$$\tilde{\gamma}(t) = \Big((\textstyle\sum_{j=1}^{n} A_{1j} a_j)\Big)\Big/\Big(1-(\textstyle\sum_{j=1}^{n} A_{nj} a_j)\, t, \cdots,$$
$$(\textstyle\sum_{j=1}^{n} A_{n-1j} a_j)\, t\Big)\Big/\Big(1-(\textstyle\sum_{j=1}^{n} A_{nj} a_j)\, t,\ 1\Big/\Big(1-(\textstyle\sum_{j=1}^{n} A_{nj} a_j)\, t\Big)\Big).$$

When $a_n=0$, $\gamma(t)$ is complete in both directions, so is $\tilde{\gamma}(t)$. Therefore

$\sum_{j=1}^{n-1} A_{nj} a_j = 0$ for any $a_1, \cdots, a_{n-1}$ such that $a_1^2 = a_2^2 + \cdots + a_{n-1}^2$.

Then we get easily

$$A_{n1} = A_{n2} = \cdots = A_{n\, n-1} = 0\,.$$

By considering domains of γ and $\tilde{\gamma}$ in the case of $a_n \neq 0$, we can see $A_{nn}=1$. $(A_{ij}) \in O(1, n-1)$ implies ${}^t(A_{ij}) \in O(1, n-1)$ so that we get

$$(A_{ij}) = \begin{bmatrix} A_{11} & A_{1\, n-1} & 0 \\ & & \\ A_{n-1\, 1} & A_{n-1\, n-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We define the map g_A of U_n which is an isometry, by

$$g_A: (x_1, \cdots, x_{n-1}, x_n) \longrightarrow (y_1, \cdots, y_{n-1}, x_n)$$

where

$$y_i = \textstyle\sum_{j=1}^{n-1} A_{ij} x_j \qquad (i=1, \cdots, n-1).$$

Then $g_A(p_0)=p_0$ and $dg_A=dg$ at p_0. Therefore g coincides with g_A.

Thus the full isometry group $I(U_n)$ consists of all matrices of the form

$$\begin{bmatrix} aA & \begin{matrix} b_1 \\ \vdots \\ b_{n-1} \end{matrix} \\ 0 \cdots 0 & 1 \end{bmatrix} \quad \text{with} \quad A \in O(1, n-2), \ a>0, \ b_1, \cdots, b_{n-1} \in \boldsymbol{R}$$

acting on U_n in the natural fashion. The identity component $I^0(U_n)$ consists of all such matrices with $A \in SO^+(1, n-2)$.

3. Isometric imbedding of U_n into H_1^n. Let H_1^n be the anti de Sitter space which is the hypersurface

$$\left\{u=(u_0, u_1, \cdots, u_n)\,;\ \langle u, u\rangle := -u_0^2 - u_1^2 + \cdots + u_n^2 = -1\right\}$$

in the indefinite Euclidean space $\boldsymbol{R}_2^{n+1}$ with its induced Lorentz metric of constant sectional curvature -1([4], p. 334).

We define $f: U_n \rightarrow H_1^n$ by

$$f(x_1, \cdots, x_n) = (u_0, \cdots, u_n)$$

where

$$\begin{cases} u_0 = (1 - x_1^2 + x_2^2 + \cdots + x_n^2)/2x_n \\ u_i = -x_i/x_n\,, \qquad i=1, \cdots, n-1 \\ u_n = (1 + x_1^2 - x_2^2 - \cdots - x_n^2)/2x_n\,. \end{cases}$$

Then f is an isometric imbedding of U_n into H_1^n and the image $f(U_n)$ is the open submanifold

$$\left\{u=(u_0, \cdots, u_n) \in H_1^n\,;\ u_0 + u_n > 0\right\}.$$

Now, we define an isomorphism h of the group G_n into the identity component $SO(2, n-1)$ of the full isometry group of H_1^n. In the first, we define an isomorphism of the Lie algebra $\mathfrak{g}$ of G_n into the Lie algebra $\mathfrak{o}(2, n-1)$ of $SO(2, n-1)$. In the Lie algebra $\mathfrak{g}$, let

$$X_i=\begin{bmatrix} & & 0\\ \text{\huge 0} & & 1\\ & & 0\\ 0\cdots 0 & & 0\end{bmatrix}<i \qquad (i=1,\cdots,n-1)\quad X_n=\begin{bmatrix}-1 & & 0\\ & \ddots & \vdots\\ & -1 & 0\\ 0\ \cdots & 0 & 0\end{bmatrix}$$

Then $X_1,\cdots,X_n$ form a basis of $\mathfrak{g}$ such that

$$[X_i,X_j]=0 \qquad \text{for}\quad 1\leqq i,\ j\leqq n-1$$
$$[X_i,X_n]=X_i \qquad \text{for}\quad 1\leqq i\leqq n-1\,.$$

In the Lie algebra $\mathfrak{o}(2,n-1)$, let

$$Y_1=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ -1 & 0 & 0\cdots 0 & -1\\ 0 & 0 & & \\ & & \text{\huge 0} & \\ 0 & -1 & & \end{bmatrix}\qquad Y_n=\begin{bmatrix}0 & \cdots\ 0 & 1\\ \vdots & & \\ 0 & \text{\huge 0} & \\ 1 & & \end{bmatrix}$$

$$Y_i=\begin{bmatrix} & \overset{\overset{i}{\vee}}{-1} & \\ & & \\ -1 & & -1\\ & & \\ & 1 & \end{bmatrix}<i \qquad \text{for}\quad 2\leqq i\leqq n-1\,.$$

Then $Y_1,\cdots,Y_n$ satisfy

$$[Y_i,Y_j]=0 \qquad \text{for}\quad 1\leqq i,\ j\leqq n-1$$
$$[Y_i,Y_n]=Y_i \qquad \text{for}\quad 1\leqq i\leqq n-1\,,$$

and generate a Lie subalgebra of $\mathfrak{o}(2,n-1)$ which is isomorphic to $\mathfrak{g}$ by $dh(X_i)=Y_i$ $(i=1,\cdots,n)$. Since G_n is simply connected, the isomorphism dh gives rise to a homomorphism h of G_n into $SO(2,n-1)$ which maps

$$\exp(sX_1)=\begin{bmatrix}1 & & & s\\ & \ddots & & 0\\ & & 1 & 0\\ 0 & & 0 & 1\end{bmatrix}\ \text{into}\ \exp(sY_1)=\begin{bmatrix}1-t^2/2, & -t, & 0 & 0, & t^2/2\\ -t, & 1, & 0 & 0, & -t\\ 0 & & \ddots & & 0\\ \vdots & & & & \vdots\\ 0 & & & 1 & 0\\ t^2/2, & -t, & 0\ \cdots & 0, & 1+t^2/2\end{bmatrix}$$

and

$$\exp(sX_n)=\begin{bmatrix} e^{-s} & & 0 \\ & \ddots & \\ & e^{-s} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ into } \exp(sY_n)=\begin{bmatrix} \cosh s, 0, \cdots, 0, \sinh s \\ 0 \quad 1 \qquad\quad 0 \\ \ddots \\ 0 \qquad\quad 1 \quad 0 \\ \sinh s, 0, \cdots, 0, \cosh s \end{bmatrix}$$

and

$$\exp(sX_i)=\begin{bmatrix} 1 & & 0 \\ & \ddots & s \\ & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} < i$$

$$\text{into } \exp(sY_i)=\begin{bmatrix} 1+s^2/2, 0, \cdots, 0, \overset{i}{\vee}{-s}, 0, \cdots, 0, s^2/2 \\ 0 \quad 1 \qquad\qquad\qquad 0 \\ \vdots \qquad \ddots \qquad\qquad \vdots \\ 0 \qquad\qquad\qquad\qquad 0 \\ -s \qquad\qquad \ddots \qquad -s \\ 0 \qquad\qquad\qquad\qquad 0 \\ \vdots \qquad\qquad\qquad \ddots \quad \vdots \\ 0 \qquad\qquad\qquad 1, \quad 0 \\ -s^2/2, 0, \cdots, 0, \underset{\hat{i}}{s}, 0, \cdots, 0, 1-s^2/2 \end{bmatrix} < i$$

for each i, $1 \leqq i \leqq n-1$.

It is verified by the same method as [3] that the imbedding $f\colon U_n \to H_1^n$ is equivariant relative to $h\colon G_n \to SO(2, n-1)$, that is,

$$f(gp)=h(g)f(p) \quad \text{for all} \quad g \in G_n \quad \text{and} \quad p \in U_n\,,$$

and h is an isomorphism.

We can extend h to an isomorphism of the largest connected isometry group $I^0(U_n)$ into $S(2, n-1)$ in such a way that f remains equivariant. To do this, it is sufficient to define

$$h(g)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SO(2, n-1) \text{ for } g=\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \in I^0(U_n)$$

where $A \in SO^+(1, n-2)$.

Thus we have

THOREM. *There exists an isometric imbedding of the upper half-space* U_n *with the metric* (3) *into the anti de Sitter space* H_1^n *which is equivariant relative to an isomorphism of the largest connected isometry group* $I^0(U_n)$ *into the largest connected isometry group* $SO(2, n-1)$ *of* H_1^n.

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