THE HELMHOLTZ DECOMPOSITION OF SOME BANACH SPACES IN SPECIAL SECTORIAL DOMAINS

TSUYOSHI YONEDA

In this talk we give an explicit representation formula for the Helmholtz projection from some Banach space $V(\Omega)$ to its solenoidal subspace, where Ω is a direct product of special sectorial plains, half lines and whole lines. More precisely, suppose that Ω is a domain of the form

$$\Omega := \Omega_{m_1} \times \cdots \times \Omega_{m_{n_1}} \times (\mathbb{R}_+)^{n_2} \times \mathbb{R}^{n_3} \subset \mathbb{R}^{2n_1+n_2+n_3},$$

where

$$\Omega_m := \left\{ x = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : r > 0, 0 < \theta < \frac{\pi}{m} \right\} \quad \text{for} \quad m \in \mathbb{N}$$

and

$$\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}.$$

Let

$$V(\Omega) := \{f \in \mathcal{D}'(\Omega) : \textit{there is } g \in V(\mathbb{R}^n) \textit{ s.t. } g|_\Omega = f\},$$
$$\|f : V(\Omega)\| := \inf_{g\in V(\mathbb{R}^n), g|_\Omega = f} \|g : V(\mathbb{R}^n)\|.$$

Note that $V(\Omega)$ is also Banach space. Let us assume that $V(\Omega)$ and $V(\mathbb{R}^n)$ have the following properties.

(i) $\overline{C^\infty_{comp}(\Omega)}^{\|\cdot\|_{V(\Omega)}} = V(\Omega)$.

(ii) $\|\varphi : V(\mathbb{R}^n)\| = \|\varphi : V(\Omega)\|$ for $\varphi \in C^\infty_{comp}(\Omega)$.

(iii) Let the operator R_k be the Riesz operator of a kernel $\frac{x_k}{|x|^{n+1}}$, then

$$\|R_k f : V(\mathbb{R}^n)\| \le C\|f : V(\mathbb{R}^n)\|,$$

for $1 \le k \le n$.

(iv) Let $\eta x := Ax + b$, $A \in O(n)$, b be some vector and $O(n)$ be the set of $n \times n$ orthogonal matrices, then

$$C^{-1}\|f : V(\mathbb{R}^n)\| \le \|\eta f : V(\mathbb{R}^n)\| \le C\|f : V(\mathbb{R}^n)\|.$$

(v) Let $V^*(\Omega)$ be a dual space of $V(\Omega)$. The dual space $V^*(\Omega)$ also has properties (i)–(iv).

For example, some amalgam spaces and some Orlicz spaces (see [7] and [14]) satisfy these properties. (Note that the property of (iii) in amalgam space is proved by [12])

Let us define

$$X_V(\Omega) := \{\vec{u} \in C^\infty(\Omega) \cap C(\bar{\Omega}) \cap V(\Omega) : \text{ div } \vec{u} = 0, \ \vec{u} \cdot \vec{n}|_\Omega = 0\},$$

$\overline{X}_p(\Omega)$ is the closure of $X_V(\Omega)$ in $V(\Omega)$-norm,

$$Y_V(\Omega) := \{\nabla q : q \in C^\infty(\Omega),\ \nabla q \in V(\Omega) \cap C(\bar{\Omega})\}$$

and $\overline{Y}_V(\Omega)$ is the closure of $Y_V(\Omega)$ in $V(\Omega)$, where $\vec{n}$ is the unit exterior normal vector of $\partial\Omega$. The aim of this talk is to give an explicit integral kernel E^* for $\mathbf{P}$ of the form

$$\mathbf{P}(\vec{u})(x) := \vec{u} + \nabla_x \int_\Omega \nabla_z E^*(x,z) \cdot \vec{u}(z)dz,$$

where $\mathbf{P} : V(\Omega) \to \overline{X}_V(\Omega)$ is the Helmholtz projection which gives the direct sum decomposition:

$$V(\Omega) = \overline{X}_V(\Omega) \oplus \overline{Y}_V(\Omega). \tag{1}$$

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Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku Tokyo 153-8914, Japan

E-mail address: yoneda@ms.u-tokyo.ac.jp