PSEUDO-DIFFERENTIAL OPERATORS WITH SYMBOLS IN $S^m_{\rho,\delta}$ ON MODULATION SPACES II

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In 1980's, the modulation spaces $M^{p,q}$ were introduced by Feichtinger. We first give the definition of modulation spaces. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp}\varphi$ is compact and

$$\sum_{k\in\mathbb{Z}^n} \varphi(\xi - k) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Then the modulation space $M^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M^{p,q}} = \left(\sum_{k\in\mathbb{Z}^n} \|\mathcal{F}^{-1}[\varphi(\cdot - k)\,\widehat{f}]\|_{L^p}^q \right)^{1/q} < \infty.$$

We remark that $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, and $M^{\infty,1}(\mathbb{R}^{2n})$ is called Sjöstrand's symbol class, because he proved that

$$\operatorname{Op}(M^{\infty,1}(\mathbb{R}^{2n})) \subset \mathcal{L}(L^2(\mathbb{R}^n)),$$

where $\operatorname{Op}(M^{\infty,1}(\mathbb{R}^{2n}))$ is the class of all pseudo-differential operators with symbols in $M^{\infty,1}(\mathbb{R}^{2n})$. For a symbol $\sigma(x,\xi)$, the pseudo-differential operator $\sigma(X,D)$ is defined by

$$\sigma(X,D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \sigma(x,\xi)\, \widehat{f}(\xi)\, d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. For the sake of simplicity, we assume that $1 < p,q < \infty$. As the generalization of Sjöstrand's boundedness, Gröchenig and Heil proved that

$$\operatorname{Op}(M^{\infty,1}(\mathbb{R}^{2n})) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)).$$

Then, since $S^0_{0,0} \subset M^{\infty,1}(\mathbb{R}^{2n})$, as a corollary, we have

$$\operatorname{Op}(S^0_{0,0}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)),$$

where $S^m_{\rho,\delta}$ consists of all $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for all multi-indices α, β. We remark that this boundedness was first proved by Tachizawa. On the other hand, Calderón and Vaillancourt proved that

$$\operatorname{Op}(S^0_{\delta,\delta}) \subset \mathcal{L}(L^2(\mathbb{R}^n)),$$

by reducing $\mathrm{Op}(S^0_{\delta,\delta})$ to $\mathrm{Op}(S^0_{0,0})$, where $0 \le \delta < 1$. Then, since we already have $\mathrm{Op}(S^0_{0,0}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n))$, by reducing $S^0_{\delta,\delta}$ to $S^0_{0,0}$ in the same way as the case of $L^2(\mathbb{R}^n)$, we can expect that

$$\mathrm{Op}(S^0_{\delta,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n)),$$

where $0 \le \delta < 1$. At least, $\mathrm{Op}(S^0_{\delta,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n))$ is true if $p = q = 2$, since $\mathrm{Op}(S^0_{\delta,\delta}) \subset \mathcal{L}(L^2(\mathbb{R}^n))$ and $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

In Harmonic Analysis and its Applications at Tokyo (2007), we considered the problem

"Are pseudo-differential operators with symbols in $S^0_{\delta,\delta}$ $M^{p,q}$-bounded?",

and stated

Theorem 1.1 ([1]). *Let $1 < p, q < \infty$, $m \in \mathbb{R}$ and $0 < \delta < 1$. If $m > -|1/q - 1/2|\delta n$, then there exists a symbol $\sigma \in S^m_{1,\delta}$ such that $\sigma(X, D)$ is not bounded on $M^{p,q}(\mathbb{R}^n)$.*

We note that, Theorem 1.1 with $q \neq 2$ says that there exists a Calderón-Zygmund operator which is not bounded on $M^{p,q}(\mathbb{R}^n)$, since pseudo-differential operators with symbols in $S^0_{1,\delta}$ are Calderón-Zygmund operators. Moreover, our counterexample satisfies $\sigma(X, D)(P) = \sigma(X, D)^*(P) = 0$ for all polynomials P, where $\sigma(X, D)^*$ is the transpose of $\sigma(X, D)$ ([2]).

Conversely, in this talk, we will consider the index m for the boundedness

$$\mathrm{Op}(S^m_{\rho,\delta}) \subset \mathcal{L}(M^{p,q}(\mathbb{R}^n))$$

to be true.

References

[1] M. Sugimoto and N. Tomita, A counterexample for boundedness of pseudo-differential operators on modulation spaces, Proc. Amer. Math. Soc., in press.

[2] M. Sugimoto and N. Tomita, Boundedness properties of pseudo-differential operators and Calderón-Zygmund operators on modulation spaces, J. Fourier Anal. Appl., in press.