L^p estimates for singular Radon transforms and extrapolation

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Let

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y))K(y)\, dy = \lim_{\epsilon\to 0} \int_{|y|>\epsilon} f(x - P(y))K(y)\, dy,$$

for an appropriate function f on $\mathbb{R}^d$, where $P(y) = (P_1(y), P_2(y), \dots, P_d(y))$ is a polynomial mapping and $K(y) = h(|y|)\Omega(y')|y|^{-n}$, $y' = |y|^{-1}y$, $n \geq 2$. We assume $P(-y) = -P(y)$, $P \neq 0$, and

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0.$$

Then

Theorem 1. *Let* $\Omega \in L^q(S^{n-1})$, $q \in (1, 2]$ *and*

$$\|h\|_{\Delta_s} = \sup_{j\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s\, dt/t \right)^{1/s} < \infty,$$

$s \in (1, 2]$. *Then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p(q-1)^{-1}(s-1)^{-1}\|\Omega\|_{L^q(S^{n-1})}\|h\|_{\Delta_s}\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, *where the constant* C_p *is independent of* q, s, Ω, h *and polynomials* P_j *if we fix* $\deg(P_j)$ $(j = 1, 2, \dots, d)$.

For $a > 0$ and a function h on $\mathbb{R}_+$, let

$$N_a(h) = \sum_{m\geq 1} m^a 2^m d_m(h),$$

where $d_m(h) = \sup_{k\in\mathbb{Z}} 2^{-k}|E(k, m)|$, $E(k, m) = \{r \in (2^k, 2^{k+1}] : 2^{m-1} < |h(r)| \leq 2^m\}$ for $m \geq 2$, $E(k, 1) = \{r \in (2^k, 2^{k+1}] : 0 < |h(r)| \leq 2\}$. Let $\mathcal{N}_a$ be the class of all measurable functions h on $\mathbb{R}_+$ satisfying $N_a(h) < \infty$. Then, by Theorem 1 and Yano's extrapolation we have

Theorem 2. *Suppose* $\Omega \in L\log L(S^{n-1})$ *and* $h \in \mathcal{N}_1$. *Then*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, *where* C_p *is independent of polynomials* P_j *if the polynomials are of fixed degree.*