Sapporo Lectures
on Representations in Lie Theory

– An Introduction and the Beyond –

Time Table (as of 20 July)

	Mon 27	Tue 28	Wed 29	Thu 30	Fri 31
09:00-10:00	—	Oda(1)	Kato(2)	Ishi(3)	Oda(3)
10:30-11:30	—	Kato(1)	Vogan(2)	Vogan(3)	Vogan(4)
14:00-15:00	Vogan(1)	Ishi(2)	—	Oda(2)	—
15:30-16:30	Ishi(1)	—	—	Kato(3)	—
17:00-18:00	Yamashita*	—	—	Chuah	—
17:30-18:45	—	Vogan	—	—	—
(*tentative)	—	dinner	—	—	—

Dates: 27–31 August 2007

Venue: Department of Mathematics, Hokkaido Univeristy
Science building # 4, Room 508 (Lectures and Seminar Talks)
Science building # 7, Room 310 (Colloquium Talk on Tue 28)

URL: http://coe.math.sci.hokudai.ac.jp/sympo/070827/

Organizers: Kyo Nishiyama (Kyoto University), Hiroyuki Ochiai (Nagoya University), Hiroshi Yamashita (Hokkaido University)

Colloquium Talk(Public lecture): The 21st Century COE Program "Mathematics of Nonlinear Structures via Singularities", Department of Mathematics of Hokkaido University, Sapporo International Communication Plaza Foundation, Communicators in Science and Technology Education Program, Hokkaido University

Seminar Talks: Seminar on Representation Theory, Hokkaido University

Secretary: Ms. M. Sasamori
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Related Workshop: Tambara Workshop 2007: Geometry and Representations in Lie Theory, 20–24 August 2007, Tambara Institute of Mathematical Sciences, The University of Tokyo

URL: http://rtweb.math.kyoto-u.ac.jp/workshop/tambara2007/

1. Lectures

Lecturer: David Vogan (Massachusetts Institute of Technology)

Title: Representations of simple Lie groups

Abstract: The most powerful idea for studying representations of a group is to study restriction to a normal subgroup. A simple Lie group G has (almost) no normal subgroups, so one needs to modify this idea. The next best thing is to find some large subgroup H of G, and to study both the restriction of the representation to H, and the collection of all conjugates of H in G.

Two examples of such subgroups are a maximal compact subgroup K, and a parabolic subgroup P. I will emphasize the first: how to study representations of G using their restriction to K. But parabolic subgroups inevitably appear along the way.

One goal of the subject is to describe irreducible unitary representations of G in geometric terms. I will describe some of what is known about that, and what still remains to be understood.

Schedule:

Lecture 1:	14:00-15:00	Mon	27 Aug.
Lecture 2:	10:30-11:30	Wed	29 Aug.
Lecture 3:	10:30-11:30	Thu	30 Aug.
Lecture 4:	10:30-11:30	Fri	31 Aug.

Lecturer: Hideyuki Ishi (Nagoya University)

Title: Invariant Hilbert spaces of holomorphic functions on homogeneous Kähler manifolds

Abstract: Let M be a Kähler manifold on which a Lie group G acts transitively as Kähler automorphisms. When the Kähler form on M is integral, we have a canonical holomorphic Hermitian line bundle L, called *quantization bundle*, on M. Then the group G acts on L naturally, so that a continuous representation ρ of G is defined on the space $\Gamma_{hol}(L)$ of holomorphic sections of L, where $\Gamma_{hol}(L)$ is regarded as a topological vector space with the compact-open topology. We shall consider the condition that ρ is unitarizable, that is, there exists a Hilbert space $\mathcal{H}(L) \neq \{0\}$ continuously imbedded into $\Gamma_{hol}(L)$ such that $(\rho, \mathcal{H}(L))$ is a unitary representation of G. If the space $\Gamma^2_{hol}(L)$ of square integrable holomorphic sections of L is non-trivial, then we can take $\Gamma^2_{hol}(L)$ as the Hilbert space $\mathcal{H}(L)$. On the other hand, even if $\Gamma^2_{hol}(L) = \{0\}$, there may exist a non-trivial $\mathcal{H}(L)$ giving us a unitary representation of G. When G is non-compact and reductive, then $(\rho, \Gamma^2_{hol}(L))$ is nothing but the holomorphic discrete series representation, while $(\rho, \mathcal{H}(L))$ is a realization of the unitarizable highest weight representation.

The fundamental theorem of homogeneous Kähler manifold states that M has a structure of holomorphic fiber bundle over a homogeneous bounded domain in which the fiber is the product of a flat homogeneous Kähler manifold and a compact simply connected homogeneous Käler manifold. At the same time, the Lie algebra of G has a specific structure called *Kähler algebra*, which is finely studied by several authors. It is natural to try

to apply these geometric and algebraic results to the study of the representation $(\rho, \mathcal{H}(L))$ of G. In this lecture, I present a general framework of the problem, and give results for the case that M is a homogeneous bounded domain.

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Lecture 3:	9:00-10:00	Thu	30 Aug.

Lecturer: Hiroshi Oda (Kakushoku University)

Title: Cherednik operators and radial parts of non-symmetric elements

Abstract: Let G be a non-compact real semisimple Lie group, $G = KAN$ its Iwasawa decomposition, and $W = W(A, G)$ the Weyl group. A *hypergeometric function* associated to the root system (Heckman-Opdam's hypergeometric function) is a W-invariant solution for a certain system of W-invariant differential-difference equations constructed via Cherednik operators. It equals the restriction of a spherical function on G/K when the multiplicity parameter corresponds to G. On the other hand, the solution space for the above equation system contains many non-symmetric elements such as *non-symmetric hypergeometric functions* [2]. In this talk we exhibit a direct connection of general non-symmetric solutions with the harmonic analysis of G/K by the following result:

Let $U(\mathfrak{g})$ be the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}$ of G. For any $D \in U(\mathfrak{g})$ let $\gamma(D)$ denote its image under the Harish-Chandra homomorphism and $T(\gamma(D))$ the corresponding Cherednik operator. It is well-known that for $D \in U(\mathfrak{g})^K$ and $f \in C^\infty(G/K)^K$

$$(Df)|_A = T(\gamma(D))(f|_A). \qquad (\star)$$

This formula holds for more general combinations of $D \in U(\mathfrak{g})$ and $f \in C^\infty(G/K)$. Let S be the set of *single-petaled* K-types (cf. [1]) and put

$$\alpha = \sum_{\sigma \in S} \deg \sigma (\text{character of } \sigma) \in C^\infty(K).$$

Then $(\star)$ holds for (D, f) such that $\mathrm{Ad}_{U(\mathfrak{g})}(\alpha)D = D$ and $f \in C^\infty(G/K)^K$, and in addition, for (D, f) such that $D \in U(\mathfrak{g})^K$ and $\alpha *_K f = f$.

A similar result also holds for the Cartan motion group and rational Dunkl operators.

[1] H. Oda, *Generalization of Harish-Chandra's basic theorem for Riemannian symmetric spaces of non-compact type*, Adv. Math., **208** (2007), 549–596

[2] E. M. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math., **175** (1995), 75–121.

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Lecturer: Syu Kato (RIMS, Kyoto University)

Title: Introduction to Joseph's theory on orbital varieties

Abstract: For a semi-simple Lie algebra $\mathfrak{g}$ (over the field of complex numbers), we have its enveloping algebra $U(\mathfrak{g})$. Since $U(\mathfrak{g})$ is a non-commutative ring, we have classification problems of nice $U(\mathfrak{g})$-ideals arising from the context of general theory of non-commutative rings.

One important class of such $U(\mathfrak{g})$-ideals is primitive ideals. It is defined by purely ring-theoretic way, but has a concrete description in terms of highest weight modules of $U(\mathfrak{g})$ (Duflo's surjectivity theorem). This enables us to treat primitive ideals in terms of the character of $Z(\mathfrak{g})$ (the center of $U(\mathfrak{g})$) and the Weyl group W of $\mathfrak{g}$.

A primitive ideal of $U(\mathfrak{g})$ carries an invariant called its Goldie rank, which measures the size of the quotient ring in some (non-commutative) way. A general method in the study of semi-simple Lie algebras (translation principle) allows us to divide the set of primitive ideals into infinite families in which the Goldie ranks behave as polynomials (depending on characters of $Z(\mathfrak{g})$).

Associated to $\mathfrak{g}$, we have its subvariety $\mathcal{N}$ given by the set of nilpotent elements. We call $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$. (Here we regard $\mathfrak{g}$ as an algebraic variety isomorphic to $\mathbb{C}^{\dim \mathfrak{g}}$.) Joseph defined certain collection of subvarieties of $\mathcal{N}$, which is usually called orbital varieties.

An orbital variety, together with its embedding into $\mathfrak{g}$, defines its characteristic polynomials (usually called the Joseph polynomials). The span of Joseph polynomials admits W-action, which is a reincanation of the Springer representation of W.

The (first) main result of Joseph's theory is the identification of some span of Joseph polynomials and that of Goldie rank polynomials. (It further gives an equality of special Joseph polynomials and special Goldie rank polynomials up to scalar when $\mathfrak{g}$ is of type A.) Moreover, there is a close relationship between the inclusion relation of primitive ideals and the inclusion relation of orbital varieties (at least when $\mathfrak{g}$ is of type A), which is still a subject of on-going studies. (For detailed exposition, see Joseph [Perspectives in Math. **17** 53–99, Academic Press, 1997] §1–4 and Hinich-Joseph [Selecta Math. **11** 9–36 (2005)] §1 and 6.)

We explain some introductory part of this story as follows:

1. We introduce the basic notion from general theories, including equivariant K-theory, characteristic polynomials, and nilpotent cones;
2. We define orbital varieties and Joseph polynomials and see why it admits an action of the Weyl group;
3. We explain
   (a) the statement of Dulfo's surjectivity theorem;
   (b) the definition of the Goldie rank polynomials;
   (c) relation of the Joseph polynomials and the Goldie rank polynomials;

We might omit some definitions if it is already familiar in the other lectures.

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2. Colloquium Talk (Public Lecture)

Lecturer: David Vogan (Massachusetts Institute of Technology)

Title: The character table for E_8, or how we wrote down a 453060×453060 matrix and found happiness

Abstract: This is a story about what happens when pure mathematicians, proud of their inability to add, try to do a really large calculation. You'll learn why, if you ask your computer to count to 1,181,642,979, you should not write a counter to the screen.

Schedule: 17:30-18:45 Tue 28 Aug. (bld. # 7, Room 310)

URL: http://coe.math.sci.hokudai.ac.jp/sympo/070828/index.html (Japanese)

3. Seminar Talks "Seminar on Representation Theory"

Lecturer: Hiroshi Yamashita (Hokkaido University) (tentative)

Title: Isotropy representations and theta correspondence (tentative)

Abstract: We discuss the isotropy representations due to Vogan, for irreducible Harish-Chandra modules with irreducible associated varieties. Among others, the theta correspondence (Howe duality correspondence) for compact dual pairs will be reproduced by using the isotropy representations. (tentative)

Schedule: 17:00-18:00 Mon 27 Aug.

Lecturer: Meng-Kiat Chuah (National Tsing Hua University)

Title: Vogan Diagrams

Abstract: A Vogan diagram is a Dynkin diagram with a diagram involution, such that the vertices fixed by the diagram involution are painted white or black. The Vogan diagrams represent real forms of complex simple Lie algebras. We discuss some generalizations of Vogan diagrams and their applications, including symmetric spaces and finite order automorphisms.

Schedule: 17:00-18:00 Thu 30 Aug.